Alexander Givental

Picard — Lefschetz Periods

and Integrable Hierarchies

The KP Hirota Equation:

$$\mathrm{Res}_{\zeta=\infty} d\zeta$$

$$\exp\left\{-2\sum_{k=1}^{\infty} \zeta^{-k} \frac{y_k}{\sqrt{\hbar}}\right\}$$

$$\exp\left\{\sum_{k=1}^{\infty} \frac{\zeta^k}{k} \sqrt{\hbar} \frac{\partial}{\partial y_k}\right\}$$

$$\mathcal{D}(x+y)\mathcal{D}(x-y) = 0$$

KdV as KP reduced mod-2

Put

$$\Gamma_{\pm\alpha}(\lambda) := \text{quantization of}$$

$$\exp\left\{\sum_{k\in\mathbf{Z}}(-z)^k\left(\frac{d}{d\lambda}\right)^k\frac{1}{\pm\sqrt{2\lambda}}\right\},$$

where

$$\frac{\zeta^2}{2} = \lambda.$$

Then

$$\sum_{\pm}\frac{d\lambda}{\pm\sqrt{\lambda}}\Gamma_{\pm\alpha}(\lambda)\otimes\Gamma_{\mp\alpha}(\lambda)\,(\mathcal{D}_{pt}\otimes\mathcal{D}_{pt})$$

is analytic in λ.

KdV as KW for A_1

Put

$$\Gamma_{\pm 2\alpha}(\lambda) := \text{quantization of}$$

$$\exp\left\{\sum_{k\in\mathbf{Z}}(-z)^k\left(\frac{d}{d\lambda}\right)^k\frac{2}{\pm\sqrt{2\lambda}}\right\}.$$

Then $\operatorname{Res}_{\lambda=\infty}\frac{d\lambda}{\lambda}$

$$\left[\sum_{\pm}\Gamma_{\pm 2\alpha}(\lambda)\otimes\Gamma_{\mp 2\alpha}(\lambda)\right](\mathcal{D}_{pt}\otimes\mathcal{D}_{pt})$$

$$= 16\ (l+\frac{1}{8})\ (\mathcal{D}_{pt}\otimes\mathcal{D}_{pt}).$$

$$l := \sum_{k\geq 0}\frac{2k+1}{2}(q_k\otimes 1 - 1\otimes q_k)\ (\partial_{q_k}\otimes 1 - 1\otimes\partial_{q_k})$$

***KW for ADE root system* R:**

$$\mathrm{Res}_{\lambda=\infty}\ \frac{d\lambda}{\lambda}$$

$$\sum_{\phi\in R} c_\phi\ \Gamma_\phi(\lambda,0)\otimes\Gamma_{-\phi}(\lambda,0)\ (\mathcal{D}\otimes\mathcal{D})$$

$$=\left[l_0+\frac{n(h+1)}{12h}\right]\ (\mathcal{D}\otimes\mathcal{D}),$$

$$l_0:=\sum_{k\geq 0}\sum_a(\frac{m_a}{h}+k)(q_k^a\otimes 1-1\otimes q_k^a)(\frac{\partial}{\partial q_k^a}\otimes 1-1\otimes\frac{\partial}{\partial q_k^a}).$$

$\Gamma_\phi(\lambda,\tau)$:= quantization of

$$\exp\left\{\sum_{k\in\mathbf{Z}}(-z)^k\left(\frac{d}{d\lambda}\right)^k I_\phi(\lambda,\tau)\right\},$$

where $I_\phi(\lambda,\tau)$ are period vectors.