# The geometry of shell membranes: The Lamé equation

by

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Fundamental forms in terms of curvature coordinates ($\kappa_i$ = principal curvatures):

$$\mathrm{I} = H^2dx^2 + K^2dy^2, \qquad \mathrm{II} = \kappa_1 H^2dx^2 + \kappa_2 K^2dy^2$$

'Gauß-Weingarten equations' for the frame $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{N})$:

$$\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix}_x = \begin{pmatrix} 0 & -p & -H_\circ \\ p & 0 & 0 \\ H_\circ & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix}_y = \begin{pmatrix} 0 & q & 0 \\ -q & 0 & -K_\circ \\ 0 & K_\circ & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix}$$

with

$$\boxed{H_y = pK \quad K_x = qH,} \quad H_\circ = -\kappa_1 H, \quad K_\circ = -\kappa_2 K$$

These are compatible modulo the 'Gauß-Mainardi-Codazzi equations' for the coefficients $H_\circ, K_\circ$ and $p, q$:

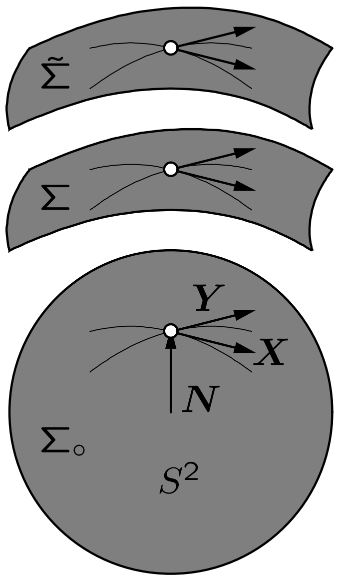
$$\boxed{p_y + q_x + H_\circ K_\circ = 0, \quad H_{\circ y} = pK_\circ, \quad K_{\circ x} = qH_\circ}$$

Combescure transforms:

$$\boldsymbol{r}_x = H\boldsymbol{X}, \qquad \boldsymbol{r}_y = K\boldsymbol{Y} \qquad \text{(surface)}$$

$$\boldsymbol{N}_x = H_\circ\boldsymbol{X}, \qquad \boldsymbol{N}_y = K_\circ\boldsymbol{Y} \qquad \text{(Gauß map)}$$

$$\tilde{\boldsymbol{r}}_x = \tilde{H}\boldsymbol{X}, \qquad \tilde{\boldsymbol{r}}_y = \tilde{K}\boldsymbol{Y}, \qquad \text{(Combescure transform)}$$




where

$$\tilde{H}_y = p\tilde{K}, \quad \tilde{K}_x = q\tilde{H}.$$

Soliton-theoretic interpretation:

- $\boldsymbol{X}, \boldsymbol{Y}$: eigenfunctions [Note that $\boldsymbol{X}_y = q\boldsymbol{Y}, \quad \boldsymbol{Y}_x = p\boldsymbol{X}$ (2d AKNS!)]
- $H, K, H_\circ, K_\circ, \tilde{H}, \tilde{K}$: adjoint eigenfunctions
- $\boldsymbol{r}, \boldsymbol{N}, \tilde{\boldsymbol{r}}$: 'squared eigenfunctions'

# 3. Guichard surfaces (1900)

C. Guichard, *Comptes Rendus de l'Académie des Sciences* (1900):

> *"Il existe une surface* $(\mathsf{N}')$ *ayant même image sphérique de ses lignes de courbure que la surface* $(\mathsf{N})$ *et telle que si* $\mathsf{R}_1$ *et* $\mathsf{R}_2$ *sont les rayons de courbure principaux de* $(\mathsf{N})$*,* $\mathsf{R}_1'$ *et* $\mathsf{R}_2'$ *les rayons correspondants de* $(\mathsf{N}')$*, on ait*
>
> $$\mathsf{R}_1\mathsf{R}_2' + \mathsf{R}_2\mathsf{R}_1' = \mathrm{const.},$$
>
> *la constante n'étant pas nulle."*

In other words, the surfaces $\Sigma$ and $\Sigma'$ are Combescure-related and constrained by

$$HK' + H'K = cH_0K_0$$

or, equivalently, by the 'orthogonality' condition

$$\boxed{\mathsf{H}^{\mathsf{T}}\Lambda\mathsf{K} = 0,} \qquad \mathsf{H} = \begin{pmatrix} H \\ H' \\ H_\circ \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} K \\ K' \\ K_\circ \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -c \end{pmatrix}.$$

## 4. O surfaces (Schief & Konopelchenko 2003)

Definition. A surface $\Sigma$ constitutes an O surface if there exist $n-1$ Combescure-related surfaces $\Sigma_k$ and a symmetric constant matrix $\Lambda$ such that the orthogonality condition

$$\boxed{\mathsf{H}^{\mathsf{T}}\Lambda\mathsf{K} = 0,} \qquad \mathsf{H} = \begin{pmatrix} H \\ H_1 \\ \vdots \\ H_{n-1} \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} K \\ K_1 \\ \vdots \\ K_{n-1} \end{pmatrix}$$

holds.

Theorem. O surfaces are integrable!

- Lax pair = extended Gauß-Weingarten equations:

$$\begin{pmatrix} X \\ Y \\ N \\ \mathsf{R} \end{pmatrix}_x = \begin{pmatrix} 0 & -p & -H_\circ & m\mathsf{H}^{\mathsf{T}}\Lambda \\ p & 0 & 0 & 0 \\ H_\circ & 0 & 0 & 0 \\ \mathsf{H} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ N \\ \mathsf{R} \end{pmatrix}, \quad \begin{pmatrix} X \\ Y \\ N \\ \mathsf{R} \end{pmatrix}_y = \cdots$$

- Bäcklund transformation = constrained Ribaucour transformation

Examples:

- $n = 2$: Surfaces of constant Gaußian and mean curvature; isothermic, minimal and linear Weingarten surfaces

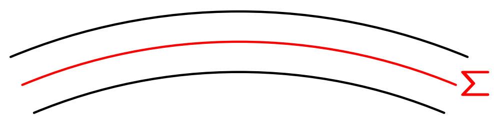
- $n = 3$: Guichard and Petot surfaces

In short: all (I think..) special O nets discussed by Eisenhart in his monograph Transformations of surfaces and more!

## 5. The equilibrium equations of classical shell membrane theory

- Lamé and Clapeyron (1831): Symmetric loading of shells of revolution
- Lecornu (1880) and Beltrami (1882): Governing equations of membrane theory
- Love (1888; 1892, 1893): Theory of thin shells
- By now well-established branch of structural mechanics

Idea (see Novozhilov (1964)): Replace the three-dimensional stress tensor $\sigma_{ik}$ of elasticity theory defined throughout a thin shell by statically equivalent internal forces $\mathsf{T}_{ab}$, $\mathsf{N}^a$ and moments $\mathsf{M}_{ab}$ acting on its mid-surface $\Sigma$.




Vanishing of total force and total moment $\longrightarrow$ equilibrium equations

Definition of (shell) membranes: $\mathsf{M}_{ab} = 0$

## 6. Vanishing 'shear stress' and constant 'normal loading'

Assumptions:
- lines of principal stress = lines of curvature
- additional (external) constant normal loading: $\bar{p} = \text{const}$

Equilibrium equations:

$$\begin{aligned} T_{1x} + (\ln K)_x (T_1 - T_2) &= 0 \\ T_{2y} + (\ln H)_y (T_2 - T_1) &= 0 \\ \kappa_1 T_1 + \kappa_2 T_2 + \bar{p} &= 0 \end{aligned}$$

Gauß-Mainardi-Codazzi equations:

$$\begin{gathered} \kappa_{2x} + (\ln K)_x (\kappa_2 - \kappa_1) = 0 \\ \kappa_{1y} + (\ln H)_y (\kappa_1 - \kappa_2) = 0, \\ \left(\frac{K_x}{H}\right)_x + \left(\frac{H_y}{K}\right)_y + HK\kappa_1\kappa_2 = 0. \end{gathered}$$

The above system is coupled and nonlinear. Only privileged membrane geometries are possible.

Claim: The above system is integrable!

# 7. Classical and novel integrable reductions

- 'Homogeneous' stress distribution $T_1 = T_2 = c = \text{const}$:

$$\boxed{\mathcal{H} = \frac{\kappa_1 + \kappa_2}{2} = -\frac{\bar{p}}{2c}}$$ (Young 1805; Laplace 1806; integrable)

Constant mean curvature/minimal surfaces (modelling thin films ('soap bubbles')).

- Identification $T_1 = c\kappa_2$, $T_2 = c\kappa_1$:

$$\boxed{\mathcal{K} = \kappa_1\kappa_2 = -\frac{\bar{p}}{2c}}$$ (integrable)

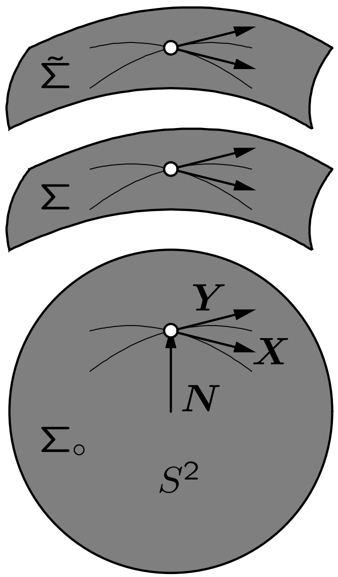
Surfaces of constant Gaußian curvature governed by $\omega_{xx} \pm \omega_{yy} + \sin(\mathrm{h})\,\omega = 0$.

- Superposition $2T_1 = \lambda\kappa_2 + \mu$, $2T_2 = \lambda\kappa_1 + \mu$:

$$\boxed{\lambda\mathcal{K} + \mu\mathcal{H} + \bar{p} = 0}$$ (integrable)

Classical linear Weingarten surfaces.

# 8. A theorem (Rogers & Schief 2003)




## Change of variables

$$\tilde{H} = T_2 H, \quad \tilde{K} = T_1 K, \quad H_\circ = -\kappa_1 H, \quad K_\circ = -\kappa_2 K$$

so that

$$H_y = pK, \qquad K_x = qH \qquad \text{(definitions of } p, q\text{)}$$

$$H_{\circ y} = pK_\circ, \qquad K_{\circ x} = qH_\circ \qquad \text{(Mainardi-Codazzi eqs)}$$

$$\tilde{H}_y = p\tilde{K}, \qquad \tilde{K}_x = q\tilde{H} \qquad \text{(2 equilibrium eqs)}$$

Theorem. A shell membrane $\Sigma$ with vanishing 'shear' stress and constant purely normal loading is in equilibrium if and only if there exists a Combescure transform $\tilde{\Sigma}$ such that the orthogonality condition

$$\boxed{\mathsf{H}^{\mathsf{T}}\Lambda\mathsf{K} = 0} \qquad \text{(3rd equilibrium eq)}$$

is satisfied, where

$$\mathsf{H} = \begin{pmatrix} H_\circ \\ H \\ \tilde{H} \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} K_\circ \\ K \\ \tilde{K} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\bar{p} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

# 9. 'Almost' geometric characterisation

Any O surface admits the first integrals

$$\mathsf{H}^T \Lambda \mathsf{H} = -f(x), \quad \mathsf{K}^T \Lambda \mathsf{K} = -g(y).$$

The latter may be used to eliminate the stresses $T_1$ and $T_2$ (i.e. $\tilde{H}$ and $\tilde{K}$).

Theorem. The geometry of the membranes (considered here) is characterised by the constraint

$$\boxed{\frac{g(y)}{K^2}\kappa_1^2 + \frac{f(x)}{H^2}\kappa_2^2 = \bar{p}(\kappa_1 - \kappa_2)^2} \tag{1}$$

on the Gauß-Mainardi-Codazzi equations. The stress components $T_1$ and $T_2$ are determined by

$$2\kappa_2 T_1 + \bar{p} = \frac{g(y)}{K^2}, \quad 2\kappa_1 T_2 + \bar{p} = \frac{f(x)}{H^2}$$

for any given admissible geometry and $f(x)$, $g(y)$.

# 10. Multiplicity of stress distributions

Problem: Under what circumstances does the geometry of a membrane determine the stress distribution?

Theorem. For $\bar{p} \neq 0$, the geometry of a membrane determines the stress distribution uniquely unless

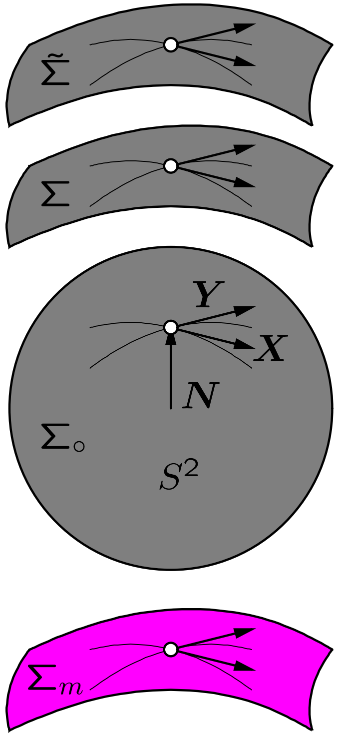
$$\left[ \ln \left( \frac{H_0}{K_0} \right) \right]_{xy} = 0.$$

In the latter case, there exists a one parameter ($\epsilon$) family of stress components $T_1$ and $T_2$ generated by the invariance

$$f(x) \to f(x) + \epsilon A^2(x), \quad g(y) \to g(y) - \epsilon B^2(y)$$

of the constraint (1), where

$$\frac{H_0}{K_0} = \frac{A(x)}{B(y)}.$$




Since the metric of the spherical representation is given by

$$d\boldsymbol{N}^2 = H_\circ^2 dx^2 + K_\circ^2 dy^2,$$

the spherical representation is conformally flat modulo an appropriate reparametrisation of the lines of curvature.

Corollary. For $\bar{p} \neq 0$, the geometry of a membrane determines the stress distribution uniquely unless there exists a Combescure transform $\Sigma_m$ of the membrane which is minimal.

Fact: The Gauß-Mainardi Codazzi equations and the constraint (1) reduce to

$$\theta_{xx} + \theta_{yy} = -e^{2\theta} \quad \text{(Liouville eq)}$$

$$(e^{-\theta})_{xy} = -\frac{f'g'}{4(f+g)^2} e^{-\theta} \quad \text{(Moutard eq)}$$

$$(H_\circ = K_\circ = e^\theta)$$

'Separable' solutions:

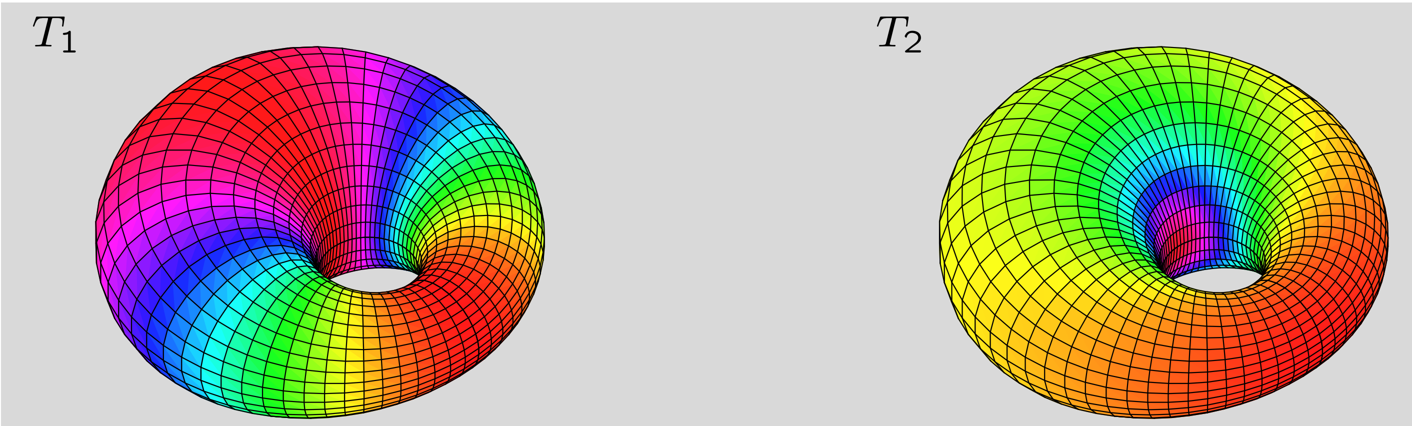
$$e^{-\theta} = a(x) + b(y) \quad \Leftrightarrow \quad f'g' = 0$$

This implies that

$$\kappa_{1x}\kappa_{2y} = 0$$

so that the membranes constitute particular canal surfaces. These include all Dupin cyclides corresponding to

$$\kappa_{1x} = \kappa_{2y} = 0.$$



Compatibility condition for the existence of $\theta$:

$$q_{xx} + q_{yy} = 0,$$

where

$$q = -\frac{f'g'}{4(f+g)^2}.$$

Solution:

$$\begin{aligned} f'^2 &= \ c_4 f^4 + c_3 f^3 + c_2 f^2 + c1 f + c_0, \\ g'^2 &= \ -c_4 g^4 + c_3 g^3 - c_2 g^2 + c1 g - c_0 \end{aligned}$$

Problem: How does one find $\theta$?

Key: Note that q may be regarded as the general solution of another Liouville equation, namely

$$(\ln q)_{xy} = -8q.$$

# 13. Reformulation of the problem

Set $z = x + \mathrm{i}y$. Solve

$$
\begin{aligned}
4\theta_{z\bar{z}} &= -e^{2\theta} && (2) \\
\mathrm{i}(\partial_z^2 - \partial_{\bar{z}}^2)e^{-\theta} &= qe^{-\theta} && (3) \\
\mathrm{i}(\partial_z^2 - \partial_{\bar{z}}^2)\ln q &= -8q && (4)
\end{aligned}
$$

Step 1: The solution of (2) is given by

$$
e^{\theta} = \frac{2}{|\Phi_1|^2 + |\Phi_2|^2},
$$

where $\Phi_1$ and $\Phi_2$ are two solutions of the Schrödinger equation

$$
\Phi''(z) + U(z)\Phi(z) = 0
$$

related by $\mathcal{W}(\Phi_1, \Phi_2) = 1$.

Step 2: Eliminate $q$ between (3) and (4) and solve the ODE for $U$.

Result:
$$U = \frac{1}{4}\mathcal{P}(z) + c,$$

where $\mathcal{P}$ constitutes the Weierstrass $\mathcal{P}$ function obeying

$$\mathcal{P}'^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3.$$

Thus, it is required to solve the classical Lamé equation

$$\boxed{\Phi'' - n(n+1)\mathcal{P}\Phi = -c\Phi}$$

for

$$\boxed{n = -\frac{1}{2}.}$$

# 14. The Lamé equation

Origin: Separation of variables in Laplace's equation (ellipsoidal harmonics)

Solutions:

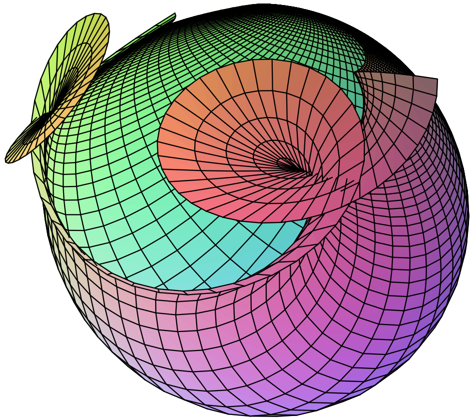
- $n \in \mathbb{Z}$: Lamé polynomials for any c
- $2n$ odd: 'algebraic' Lamé functions for particular values of $c$
- $n = -\frac{1}{2}$: ??????? Comments?

Remark (in hindsight):

The Moutard equation is the analogue of the linear equation obtained by Wangerin (1875) in the case of the axi-symmetric Laplace equation!

The Liouville equation then selects all separable solutions!

- $\mathcal{P} = \frac{1}{z^2}$: Bessel functions



- $\mathcal{P} = \frac{1}{\sin^2 z} - \frac{1}{3}$: Legendre functions