Ergodic theory for a class of non-Markovian processes

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Aim of the project

Study the long-time behaviour of stochastic processes without relying on the Markov property.

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There is extensive literature on the ergodic theory of Markov processes, but most results rely extremely heavily on the Markov property.

We will focus on extrinsic memory.

Basic setup

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Question: What sort of conditions should one impose on the ξ_n so that one has similar criteria to the Markovian case?

Example: Behaviour of queue known under the assumption that arriving customers are Poisson. What if this is changed?

A powerful criteria for Markov processes

Theorem: (Doob-Khas′minskii) If $\mathcal{P}$ is topologically irreducible ($\mathcal{P}(x, A) > 0$ for every x and every open set A) and strong Feller (maps bounded functions into continuous functions), then $\mathcal{P}$ has at most one invariant probability measure.

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Remark: Both conditions can often be verified by very similar techniques but neither of them is sufficient for the conclusion to hold.

A collection of useful objects

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Skew-product with Φ gives a Markov operator $\mathcal{Q}$ over $\mathcal{X} \times \mathcal{W}$ and a 'solution map' $\mathcal{S}\colon \mathcal{X} \times \mathcal{W} \to P_1(\mathcal{X}^{\mathbf{N}})$.

Topological irreducibility

Natural generalisation given by:

Definition A skew-product as before is topologically irreducible if $\mathcal{Q}(x, \xi; A \times \mathcal{W}) > 0$ for every $(x, \xi) \in \mathcal{X} \times \mathcal{W}$ and every open set $A \subset \mathcal{X}$.

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Can be checked in many situations by a controllability argument in exactly the same way as for the Markovian case.

About the strong Feller property

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Suggests the following generalisation.

Definition A skew-product as above is strong Feller if there exists a continuous map $\ell \colon \mathcal{X} \times \mathcal{X} \times \mathcal{W} \to \mathbf{R}$ such that $\ell(x, x, \xi) = 0$ and such that

$$\|\mathcal{S}\mathcal{Q}(x, \xi; \cdot) - \mathcal{S}\mathcal{Q}(y, \xi; \cdot)\|_{\mathrm{TV}} \leq \ell(x, y, \xi) \;.$$

Something more is required

Consider the following (discrete time) example with $\mathcal{X} = \{0,1\}$ and $\mathcal{W} = \{0,1\}^{\mathbf{Z}_-}$:

$$\Phi(x,\xi) = \begin{cases} x & \text{if } \xi_0 = \xi_{-1} \\ 1-x & \text{otherwise} \end{cases}$$

$$\begin{array}{l|cccccccccccl} \mathcal{X} & & & & & & & & & 1 & {\color{green}0} & & \\ \hline \mathcal{W} & \cdots & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & {\color{red}1} & \longleftarrow\ p = \frac{1}{2} \end{array}$$

There are two distinct stationary states, even though this example is both strong Feller and irreducible !

A quasi-Markovian property

Define

$$\mathcal{N} = \{(\xi, \xi') \,|\, \mathcal{S}\mathcal{Q}(x, \xi; \cdot) \approx \mathcal{S}\mathcal{Q}(x, \xi'; \cdot) \, \forall x\} \ .$$

Here, $\approx$ denotes equivalence of measures.

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Definition A skew-product as above is quasi-Markovian if, for almost every $w \in \mathcal{W}$ and for any two open sets $U, V \subset \mathcal{W}$ such that $\mathcal{P}(w, U)\mathcal{P}(w, V) > 0$, there is a coupling between $\mathcal{P}(w, \cdot \cap U)$ and $\mathcal{P}(w, \cdot \cap V)$ which gives positive mass to $\mathcal{N}$.

Main result

Theorem: If a skew-product is quasi-Markovian, then the strong Feller property and topological irreducibility imply the existence of at most one stationary state (up to equivalence).

Gaussian driving noise (discrete)

Discrete time: Quasi-Markov $\Leftrightarrow$ the a.c. part f of the spectral density satisfies $\int \frac{1}{f(\nu)}\,d\nu < \infty \Leftrightarrow \xi_0$ is not a deterministic function of $\{\xi_n\}_{n\neq 0}$.

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Counterexample: Take y_n a sequence of i.i.d. Gaussians, set $\xi_n = y_{n+1} - y_n$ and take $x_{n+1} = x_n + \xi_n$ with $x_n \in \mathbf{R}/\mathbf{Z}$. In this case, $x_n = (x_0 - y_0) + y_n$, so there are many distinct stationary solutions. However, $f(\nu) = 1 - \cos\nu$.

Gaussian driving noise (continuous)

Diffusions driven by fractional Brownian motion with $H > 1/2$ behave essentially like the corresponding diffusion driven by Brownian motion.

Joint work with A. Ohashi (ergodicity for ‘elliptic’ diffusions with multiplicative noise) and F. Baudoin (smooth densities under Hörmander condition).

Some open problems

- Characterise continuous-time noises that have the quasi-Markovian property.
- Which noises lead to Hörmander-type theorems? (cf. P. Friz)
- Incorporate (infinite) memory into the drift as well.
- Make the proofs more constructive to extract convergence rates. (A la Harris.)

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