Random walks in semigroups: stability and sensitivity

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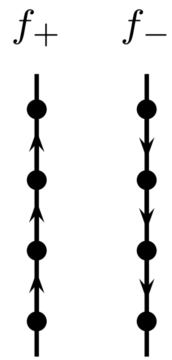
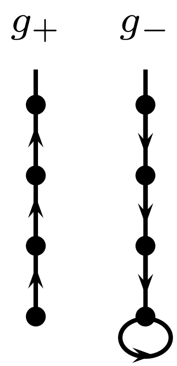
A talk on RW July 2007 Durham

Example 1 (trivial). $f_-, f_+ : \mathbb{Z} \to \mathbb{Z}$,

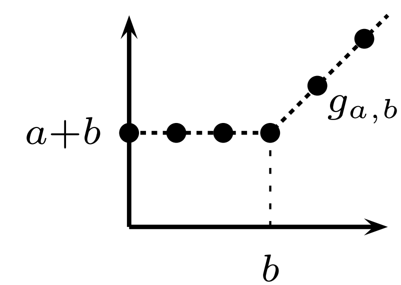


$$f_-(x) = x - 1\,, \quad f_+(x) = x + 1$$
$$f_a(x) = x + a$$

Example 2. $g_-, g_+ : \mathbb{Z}_+ \to \mathbb{Z}_+$,



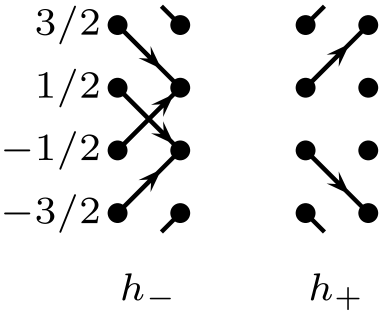
$$g_+(x) = x + 1\,, \quad g_-(x) = \max(0, x - 1)\,,$$
$$g_{a,b}(x) = a + \max(x, b)$$
$$\text{for } a, b \in \mathbb{Z},\ b \geq 0,\ a + b \geq 0\,.$$



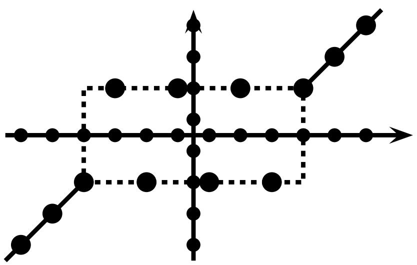
Example 3. $h_-, h_+ : \mathbb{Z} + \frac{1}{2} \to \mathbb{Z} + \frac{1}{2}$,

$$\left.\begin{aligned} h_-(x) &= x - 1 \\ h_+(x) &= x + 1 \end{aligned}\right\} \quad \text{for } x \in \left(\mathbb{Z} + \tfrac{1}{2}\right) \cap (0, \infty)\,,$$

$$h_-(-x) = -h_-(x)\,, \quad h_+(-x) = -h_+(x)\,.$$



$$h_{a,b}(x) = \begin{cases} x + a & \text{for } x \ge b, \\ x - a & \text{for } x \le -b, \\ (-1)^{b-x}(a+b) & \text{for } -b \le x \le b; \end{cases}$$



$$b, a + b \in \left(\mathbb{Z} + \tfrac{1}{2}\right) \cap (0, \infty) = \left\{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \dots\right\}.$$

In fact, $|h_{a,b}(x)| - \frac{1}{2} = g_{a,b}(|x| - \frac{1}{2})$.

Algebraically:

semigroup	generators	relations
G_1	f_-, f_+	$f_-f_+ = 1 = f_+f_-$
G_2	g_-, g_+	$g_+g_- = 1$
G_3	h_-, h_+	$h_+h_- = 1$

$f_a = f_+^a$ or f_-^{-a} for $a \in \mathbb{Z}$;

$g_{a,b} = g_-^b g_+^{a+b}$ for $a, b \in \mathbb{Z}$, $b \geq 0$, $a + b \geq 0$;

the same for $h_{a,b}$.

G_1 is commutative; G_2 and G_3 are isomorphic, noncommutative.

Example 1: a representation of G_1; Example 2: a representation of G_2;
Example 3: a two-sheeted representation of G_2? What about three-sheeted?

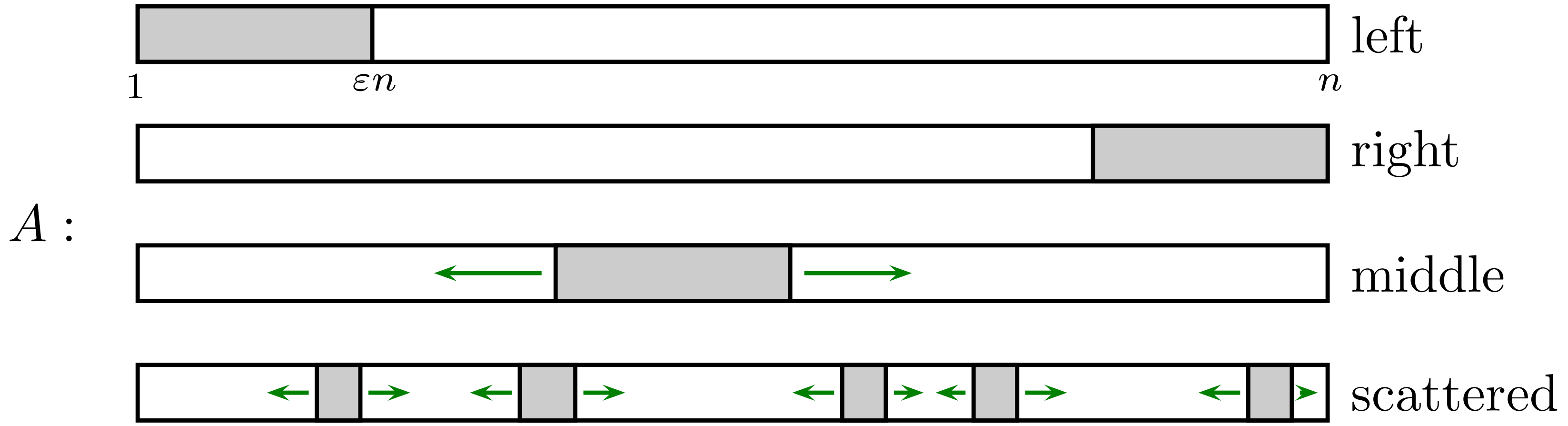
Random walk in a semigroup G with given generators x_-, x_+:

$\xi_n(\omega_1, \omega_2, \dots) = x_{\omega_1} \dots x_{\omega_n}$ for $\omega_1, \omega_2, \dots = \pm 1$.

Perturbation:

$\xi_n = \xi_n(\omega_1, \dots), \quad \xi'_n = \xi_n(\omega'_1, \dots),$

$$\mathbb{E}\omega_k \omega'_k = \begin{cases} +1 & \text{for } k \notin A, \\ 0 & \text{for } k \in A; \end{cases}; \quad |A| = \varepsilon n. \qquad \mathbb{E}\omega_k \omega'_l = 0 \text{ for } k \neq l.$$



Equivalent in the commutative case.

DEF. A function $\varphi : G \to \mathbb{R}$ is n-stable if $\mathbb{E}|\varphi(\xi_n) - \varphi(\xi'_n)|^2 \le \varepsilon$ for all $\varepsilon \in \{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$.

DEF. Metric ρ_n on G (possibly $+\infty$):

$$\rho_n(x,y) = \sup\{|\varphi(x) - \varphi(y)| : \varphi \text{ is } n\text{-stable}\} .$$

Depends on A (left, ..., scattered), unless G is commutative.

EXAMPLE 1 (commutative; $G_1 = \{f_a : a \in \mathbb{Z}\} \cong \mathbb{Z}$)

$\rho_n(f_a, f_{a+2}) \to 0$ as $n \to \infty$;

$\rho_n(f_a, f_{a+1}) = \infty$.

$\varphi(x) = \text{const} \cdot \frac{x}{\sqrt{n}}$ is n-stable.

Roughly, an n-stable function of $f_a \in G_1$ is a continuous function of $a/\sqrt{n}$ and $(-1)^a$.

And no wonder; $\xi_n = f_a$ with $a + n \in 2\mathbb{Z}$ always.

EXAMPLE 2 (noncommutative; $G_2 = \{g_{a,b} : a, b \in \mathbb{Z}, b \geq 0, a + b \geq 0\}$)

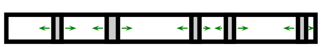
$\xi_n = g_{a,b}$ with $n + a \in \mathbb{Z}$ always (but b can be of any parity).

$A =$  : an n-stable function of $g_{a,b} \in G_2$ is a continuous function of $a/\sqrt{n}$ and $(-1)^a$, but arbitrary function of b.

$A =$  : ... continuous function of $(a-b)/\sqrt{n}$ and $(-1)^{a-b}$, but arbitrary function of $a + b$.

$A =$  : continuous function of $a/\sqrt{n}$, $b/\sqrt{n}$, $(-1)^a$ and $(-1)^b$.

$A =$  : the same.

$A =$  : continuous function of $a/\sqrt{n}$, $b/\sqrt{n}$ and $(-1)^a$.

Just two-sheeted! (representation of G_2)

Different modes of perturbation lead to different scaling limits.

discrete time $\xrightarrow[\text{limit}]{\text{scaling}}$ continuous time

n-stable?

yes	yes	classical
yes	no	nonclassical
no		not at all

G_1	classical (Brownian motion, white noise)
G_2	nonclassical (Warren's noise of splitting)
$\{-1, +1\}$	not at all

Homomorphism $G_2 \to G_1$,

$$g_- \mapsto f_- \,, \qquad g_{a,b} = g_-^b g_+^{a+b} \mapsto f_-^b f_+^{a+b} = f_a \,.$$
$$g_+ \mapsto f_+ \,,$$

Random walk in G_2,

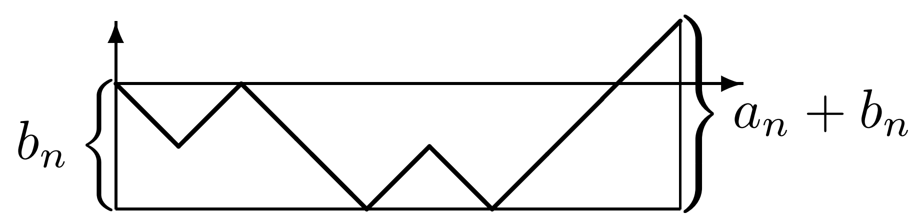
$$g_{\omega_1} \ldots g_{\omega_n} = g_{a_n, b_n} \,,$$

and in G_1,

$$f_{\omega_1} \ldots f_{\omega_n} = f_{a_n} \,, \qquad a_n = \omega_1 + \cdots + \omega_n \,,$$

related:

$$b_n = 0 - \min(a_0, \ldots, a_n) \,,$$
$$a_n + b_n = a_n - \min(a_0, \ldots, a_n) \,.$$



Scaling limit:

$$\frac{a_k}{\sqrt{n}} \to w\Big(\frac{k}{n}\Big)\,, \qquad w = \text{Brownian motion},$$

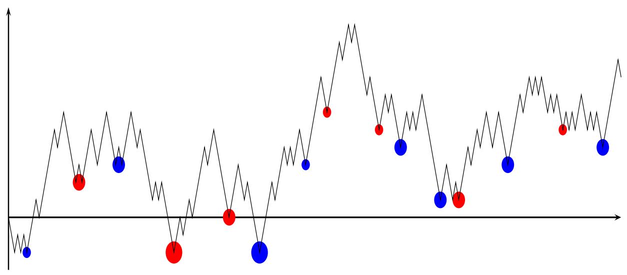
$$\frac{b_k}{\sqrt{n}} \to -\min_{[0,k/n]} w(\cdot)\,,$$

$$(-1)^{b_k} = (-1)^{\sqrt{n}\,\min_{[0,k/n]} w(\cdot)} \to ?$$

Warren's noise of splitting:

Brownian paths with independent

random signs attached to local minima.



(countable, dense)