Comments on Yang-Mills theory with Twistorial Overtones

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Comments

- A way of looking at some nonperturbative features of YM theory, mostly in 2+1 dimensions
- This talk is a set of comments on questions related to the mass gap
  - Hamiltonian approach, vacuum wave function
  - A gauge-invariant mass term
  - General comments about configuration space for $3 + 1$ Yang-Mills

Matrix variables, volume element

- Choose $A_0 = 0$, this leaves $A_i,\ i = 1, 2$. Gauge transformations act as

$$A_i^g = g\ A_i\ g^{-1}\ -\ \partial_i g\ g^{-1}$$

Wave functions are gauge-invariant (This is equivalent to imposing Gauss law)

- Choose complex coordinates, $z = x_1 - ix_2,\ \ \bar{z} = x_1 + ix_2$

$$A \equiv A_z = \tfrac{1}{2}(A_1 + iA_2), \qquad \bar{A} = \tfrac{1}{2}(A_1 - iA_2)$$

- Parametrize A as

$$A = -\partial M\ M^{-1} \qquad \bar{A} = M^{\dagger -1}\bar{\partial} M^{\dagger}$$

- $G = SU(N) \implies M \in SL(N, \mathbf{C}) = SU(N)^{\mathbf{C}}$ (Generally $G \to G^{\mathbf{C}}$)

Matrix variables, volume element (cont'd.)

- Under a gauge transformation

$$A \to A_i^g = g\, A_i\, g^{-1} - \partial_i g\, g^{-1} \Longrightarrow M \to M^g = g\, M$$

- $H = M^\dagger M$ is gauge-invariant
- Calculation of volume element of the configuration space

$$\begin{aligned} ds^2_{\mathcal{A}} &= \int d^2x\ \mathrm{Tr}(\delta A \delta \bar{A}) \\ &= \int \mathrm{Tr}\left[(M^{\dagger -1} \delta M^\dagger)(-\bar{D} D)(\delta M M^{-1}) \right] \\ ds^2_{SL(N,\mathbf{C})} &= \int \mathrm{Tr}(M^{\dagger -1} \delta M^\dagger\ \delta M M^{-1}) \\ d\mu_{\mathcal{A}} &= \det(-\bar{D} D)\ \underbrace{d\mu(M, M^\dagger)} \end{aligned}$$

Haar measure for *SL(N,* **C***)*

Matrix variables, volume element (cont'd.)

- We can split the $SL(N, \mathbf{C})$ volume element as

$$d\mu(M, M^\dagger) = \underbrace{d\mu(H)}_{\text{Haar for } SL(N,\mathbf{C})/SU(N)} \qquad \underbrace{d\mu(U)}_{\text{Haar for } SU(N)}$$

- The volume element is now

$$d\mu_{\mathcal{A}} = \det(-\bar{D}D)\, d\mu(H)\;\; d\mu(U)$$

- For the gauge-invariant configuration space

$$d\mu(\mathcal{C}) = \det(-\bar{D}D)\, d\mu(H)$$
$$= d\mu(H)\;\; \exp\left[2\, c_A\, S_{wzw}(H)\right]$$

- $S_{wzw}(H)$ is the Wess-Zumino-Witten (WZW) action,

$$S_{wzw}(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) - \frac{i}{12\pi}\int \mathrm{Tr}(H^{-1} dH)^3$$

The inner product and current

- The inner product is now given as

$$\langle 1|2\rangle = \int d\mu(H)\ \exp\left[2\, c_A\ S_{wzw}(H)\right]\ \ \Psi_1^*\ \Psi_2$$

- The Wilson loop operator is given by

$$W(C) = \mathrm{Tr}\,\mathcal{P}e^{-\oint A} = \mathrm{Tr}\,\mathcal{P}\exp\left(\frac{\pi}{c_A}\oint J\right)$$

$$J = \frac{c_A}{\pi}\,\partial H\ H^{-1}$$

All gauge-invariant quantities can be made from J.

Construction of $\mathcal{H}$

- The Hamiltonian is given by

$$\mathcal{H} = \underbrace{\frac{e^2}{2}\int E^a E^a}_{\equiv \quad T} + \underbrace{\frac{1}{2e^2}\int B^a B^a}_{+ \quad V}$$

- The kinetic term is simplified via the chain rule

$$T\,\Psi = -\frac{e^2}{2}\int_x \frac{\delta^2}{\delta A(x)\delta \bar{A}(x)}\,\Psi$$

$$= -\frac{e^2}{2}\left[\int \underbrace{\frac{\delta J(u)}{\delta A(x)}\,\frac{\delta J(v)}{\delta \bar{A}(x)}}_{\Omega}\,\frac{\delta^2 \Psi}{\delta J(u)\delta J(v)} + \int \underbrace{\frac{\delta^2 J(u)}{\delta A(x)\delta \bar{A}(x)}}_{\omega}\,\frac{\delta \Psi}{\delta J(u)}\right]$$

$$= \int \Omega_{ab}(u,v)\frac{\delta^2 \Psi}{\delta J^a(u)\delta J^b(v)} + \int \omega^a(u)\frac{\delta \Psi}{\delta J^a(u)}$$

Construction of $\mathcal{H}$ (cont'd.)

- $\omega^a(u)$ needs regularization

$$\omega^a = -\frac{e^2}{2}\int_x \frac{\delta^2 J^a(u)}{\delta A^b(x)\delta \bar{A}^b(x)}$$
$$= \left(e^2 c_A/2\pi\right) M^\dagger_{am}(x)\ \mathrm{Tr}\left[t^m \bar{D}^{-1}_{reg}(y,x)\right]_{y\to x}$$
$$= m\ J^a$$

$m = e^2 c_A/2\pi$ (This is the basic mass scale of the theory.)

- The kinetic energy is thus given by

$$T = m\left[\int J^a \frac{\delta}{\delta J^a} + \int \Omega_{ab}(u,v)\frac{\delta^2}{\delta J^a(u)\delta J^b(v)}\right]$$

$$\Omega_{ab}(u,v) = \frac{c_A}{\pi^2}\frac{\delta_{ab}}{(u-v)^2} - i\frac{f_{abc}J^c(v)}{u-v} + \mathcal{O}(\epsilon)$$

Can be rechecked, particularly the term $\int J\frac{\delta}{\delta J}$, by self-adjointness of T.

Back to the Hamiltonian $\mathcal{H}$ and vacuum wave function

- The potential energy is easy to simplify

$$V = \frac{1}{2e^2} \int B^a B^a = \frac{\pi}{m c_A} \int :\bar{\partial} J \, \bar{\partial} J:$$

- The regularization for T and for V have to agree (in the choice of the parameter λ) so that $\mathcal{H}$ transforms correctly under Lorentz boosts.
- The summed-up result is

$$P = -\frac{2}{e^2} \left[\frac{\pi^2}{c_A^2} \int \bar{\partial} J^a(x) \, K(x,y) \, \bar{\partial} J^a(y) + f_{abc} \int J^a(x) J^b(y) J^c(z) f(x,y,z) + \ldots \right]$$

$$K(x,y) = \left[\frac{1}{m + \sqrt{m^2 - \nabla^2}} \right]_{x,y}$$

Vacuum wave function (cont'd.)

- The vacuum wave function leads to a value for string tension which agrees well with lattice simulations.
- The high k limit agrees with perturbation theory.
- There are a couple of independent checks of this wave function.
- One is based on Lorentz invariance, another is as follows.

Vacuum wave function: A different argument

- Absorb $\exp(2c_A S_{wzw})$ from the inner product into the wave function by $\Psi = e^{-c_A S_{wzw}(H)}\Phi$. The Hamiltonian acting on Φ is

$$\mathcal{H} \rightarrow e^{-c_A S_{wzw}(H)}\, \mathcal{H}\, e^{-c_A S_{wzw}(H)}$$

- Consider $H = e^{t^a \varphi^a} \approx 1 + t^a \varphi^a + \cdots$, a small φ limit appropriate for a (resummed) perturbation theory. The new Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int \left[-\frac{\delta^2}{\delta \phi^2} + \phi(-\nabla^2 + m^2)\phi + \cdots \right]$$

where $\phi_a(\vec{k}) = \sqrt{c_A k \bar{k}/(2\pi m)}\, \varphi_a(\vec{k})$.

- The vacuum wave function is

$$\Phi_0 \approx \exp\left[-\frac{1}{2} \int \phi^a \sqrt{m^2 - \nabla^2}\, \phi^a \right]$$

Vacuum wave function: A different argument (cont'd.)

- Transforming back to Ψ,

$$\Psi_0 \approx \exp\left[-\frac{c_A}{\pi m}\int (\bar{\partial}\partial\varphi^a)\left[\frac{1}{m+\sqrt{m^2-\nabla^2}}\right](\bar{\partial}\partial\varphi^a) + \cdots\right]$$

- The full wave function must be a functional of J. The only form consistent with the above is

$$\Psi_0 = \exp\left[-\frac{2\pi^2}{e^2 c_A^2}\int \bar{\partial}J^a(x)\left[\frac{1}{m+\sqrt{m^2-\nabla^2}}\right]_{x,y}\bar{\partial}J^a(y) \; + \cdots\right]$$

since $J \approx (c_A/\pi)\partial\varphi + \mathcal{O}(\varphi^2)$.

- This indicates the robustness of the Gaussian term in Ψ_0, since this argument only presumes
  1. Existence of a regulator, so that the transformation $\Psi \Leftrightarrow \Phi$ can be carried out
  2. The two-dimensional anomaly calculation

Mass term in resummed perturbation theory

- Since $T = m\left[\int J\frac{\delta}{\delta J} + \int \Omega \frac{\delta}{\delta J}\frac{\delta}{\delta J}\right]$,

$$T\, J^a = m\, J^a$$

- Including the potential energy,

$$(T+V)\, J^a \Psi_0 = \sqrt{k^2+m^2}\, J^a \Psi_0 \; + \cdots$$

J^a is a "gauge-invariant" definition of a gluon.

- This is brought out more clearly by $\Psi = e^{-c_A S_{wzw}(H)} \Phi$

$$\mathcal{H} = \frac{1}{2}\int\left[-\frac{\delta^2}{\delta\phi^2} + \phi(-\nabla^2+m^2)\phi + \cdots\right]$$

- At the propagator level, we must get

$$\frac{1}{k^2-m^2}$$

Mass term (cont'd.)

- This must appear in *resummed* perturbation theory, because $m = e^2 c_A / 2\pi$.

$$\frac{1}{k_0^2 - \vec{k}^2 - m^2} = \frac{1}{k^2} + \frac{1}{k^2}\, m^2\, \frac{1}{k^2} + \frac{1}{k^2}\, m^2\, \frac{1}{k^2}\, m^2\, \frac{1}{k^2} + \cdots$$

- A strategy for seeing this explicitly.
- Write the action as

$$S_{YM} = \underbrace{S_{YM} + \mu^2 S_{mass}} \; - \; l\, \mu^2 S_{mass}$$

- S_{mass} is a gauge-invariant mass term for the YM field. $l = 1$ eventually.
- Use the first two terms to calculate Γ to, say, one-loop order. It has the form

$$\Gamma \;=\; S_{YM} + \mu^2 S_{mass} + \underbrace{\Gamma^{(1)} - l\, \mu^2 S_{mass}}_{=0} + \cdots$$

This gives an equation determining μ.

Mass term (cont'd.)

- Different choices of S_{mass} correspond to resummations of different sets of diagrams.
- What do we choose for S_{mass}?
- For many choices, for example,

$$S_{mass} = \int \mathrm{Tr}\left[F \frac{1}{(-D^2)} F\right]$$

the calculated $\Gamma^{(1)}$ has threshold singularities at $k^2 = 0$. Zero mass particles must reappear in external lines by unitarity.

- There is one mass term for which this is avoided. It is like $S_{wzw}(H)$ we can write in 3 dimensions. Define complex null vectors n_i, $\bar{n}_i$ in 3 dimensions with

$$n \cdot n \;=\; \bar{n} \cdot \bar{n} = 0, \qquad n \cdot \bar{n} = 2$$

Mass term (cont'd.)

- Now define

$$\tfrac{1}{2}n \cdot \nabla = \partial, \qquad \tfrac{1}{2}\bar{n} \cdot \nabla = \bar{\partial}, \qquad \tfrac{1}{2}n \cdot A = A, \qquad \tfrac{1}{2}\bar{n} \cdot A = \bar{A}$$

- We can now construct

$$S_{mass}(A) = \int d\Omega \, dx^T \; S_{wzw}(R^\dagger R)$$

  where R is defined by $A = -\partial R R^{-1}, \quad \bar{A} = R^{\dagger -1}\bar{\partial} R^\dagger$.
  x^T is the direction orthogonal to $n, \ \bar{n}$.

- This has many of the properties of the WZW action.
  - It becomes the usual S_{wzw} in two dimensions
  - It has full $3d$ Euclidean invariance
  - Allows for a certain holomorphic splitting, PW property

Mass term (cont'd.)

- The calculation leads to a long expression for $\Gamma^{(1)}$ with no threshold singularities and a value for $\mu \approx 1.2m$.
- The mass term can be written as

$$S_{mass} = \int d\mu \;\; \Delta(u,v) \; \frac{\mathrm{Tr}\log(-\bar{D}D)}{(u\cdot v)(\bar{u}\cdot\bar{v})}$$

where $D = \frac{1}{2}u^A D_{AA'}\bar{v}^{A'}$, $\bar{D} = \frac{1}{2}v^A \bar{D}_{AA'}\bar{u}^{A'}$ and

$$\begin{aligned} d\mu &= \frac{u\cdot du \;\; \bar{u}\cdot d\bar{u} \;\; v\cdot dv \;\; \bar{v}\cdot d\bar{v}}{(u\cdot v)^2(\bar{u}\cdot\bar{v})^2} \\ \Delta(u,v) &= (u\cdot v)(\bar{u}\cdot\bar{v})\;\delta(v(\eta\cdot e)\bar{u})\;\delta(u(\eta\cdot e)\bar{v}) \end{aligned}$$

$\eta = (1,0,0,0)$, $e_\mu = (1, \sigma_i)$.

YM(3+1) configuration space

- The configuration space $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$ (gauge orbit space) in two spatial dimensions has the volume element

$$d\mu(\mathcal{C}) = d\mu(H) \; \exp\left[2\, c_A \, S_{wzw}(H)\right]$$

$$S_{wzw}(H) = \frac{1}{2\pi} \int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) - \frac{i}{12\pi} \int \mathrm{Tr}(H^{-1} dH)^3$$

- This leads to a “finite” volume for $\mathcal{C}$,

$$\int d\mu(\mathcal{C}) < \infty$$

  (Some regularization needed; the point is the contrast with Abelian theory for which $c_A = 0$.)

- There are configurations which are separated by an infinite distance (spikes). This result shows that they have zero transverse measure.

YM(3+1) configuration space (cont'd.)

- This property is crucial for mass gap because $S_{wzw}(H)$ provides a cut-off for low momentum modes.
- Can one have a similar result for $3d$ gauge fields, relevant for YM_{3+1}?
- We focus on the volume measure, defining it as

$$d\mu(\mathcal{C})_{3d} = \left[\frac{[dA]}{vol(\mathcal{G}_*)} \exp\left[-\int \frac{F^2}{4M} \right] \right]_{M\to\infty}$$

where M is a parameter with the dimensions of mass.

- The right hand side $\approx$ Euclidean functional integral of a $(2+1) \to 3$ dimensional Yang-Mills theory.
- We can evaluate the rhs by Hamiltonian techniques in $2+1$ dimensions, using $\langle 0|e^{-\beta\mathcal{H}}|0\rangle$.
- We use Euclidean evolution operator, and further $\beta \to \infty$ since the third direction has infinite extent.

YM(3+1) configuration space (cont'd.)

- This gives

$$\int d\mu(\mathcal{C})_{3d} = \int \frac{[dA]}{vol(\mathcal{G}_*)} \;\exp\left[-\int \frac{F^2}{4M}\right] \Bigg]_{M\to\infty}$$

$$= \langle 0|\; e^{-\beta\mathcal{H}} \;|0\rangle \Big]_{\beta,M\to\infty}$$

$$= \int d\mu(\mathcal{C})_{2d}\; \Psi_0^*\; \Psi_0$$

- We know the large M $(= e_{3d}^2)$ limit of the $2d$ wave function, so

$$\int d\mu(\mathcal{C})_{3d} = \{2 - \text{dim. YM partition function for } e_{2d}^2 = M^2 c_A/2\pi\}$$

$$= \{\text{WZW partition function as } M \to \infty\}$$

$$< \infty$$

YM(3+1) configuration space (cont'd.)

- The volume of the configuration space for 3d YM is "finite". How is it possible?
- Define the distance and energy functionals as

$$L^2(A,B) = \mathrm{Inf}_g \int d^3x \, \mathrm{Tr}(A^g - B)^2, \qquad \mathcal{E}(A) = \int d^3x \, F^2/4\mu$$

- Consider orbits of $A_i(x)$ and $A_i^{(s)} = sA_i(sx)$. Then

$$L^2(A^{(s)}) = \frac{1}{s} L^2(A), \qquad \mathcal{E}(A^{(s)}) = s\,\mathcal{E}(A)$$

- As $s \to 0$, we scale up distances in $\mathcal{C}$, yet there is no cut-off imposed by $\mathcal{E}$ since it goes to zero (Orland).
- How do we square this with $\int d\mu(\mathcal{C}) < \infty$?

YM(3+1) configuration space (cont'd.)

- The solution has to do with dynamical generation of mass in 3 dimensions.
- In strong coupling, this is related to the generation of mass in the Hamiltonian analysis.
- Also seen by resummation in a 3d-covariant approach; or integrate out modes of high momenta to get an RG change

$$\int F^2/4M \Longrightarrow \int F^2/4M + \ \mu^2 \, S_{mass}, \qquad \mu \approx (1.2 \, c_A/2\pi)\, M$$

- $S_{mass}(A^{(s)}) = (1/s)\, S_m(A)$; this explains why small s values are cut-off and we get $\int d\mu < \infty$.
- Eventually, we expect $\frac{1}{M_{new}} = \frac{1}{M} + \frac{1}{\Lambda}$
- Can one understand better how such S_{mass} can arise from twistor space?