$\mathcal{N} = 8$ Self-Dual Supergravity

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Twistors, Strings and Scattering Amplitudes
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MW, arXiv:0705.1422 [hep-th]
L. J. Mason and MW, arXiv:0706.1941 [hep-th]

Contents

Twistor Space

- To begin with, consider complexified Minkowski 4-space $\mathbb{M} \cong \mathbb{C}^4$ coordinatised by $x^\mu \sim x^{\alpha\dot{\alpha}}$.
- A twistor is an object $Z = (\omega^\alpha, \pi_{\dot{\alpha}}) \in \mathbb{T}$ satisfying the relation

  $$\omega^\alpha \; = \; x^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}.$$

  Clearly, it's defined up to scalings. Thus twistors associated with $\mathbb{M}$ are just points on $\mathbb{P} = \mathbb{P}^3 \setminus \mathbb{P}^1$ the latter being termed projective twistor space.

Twistor Space

- What do they describe geometrically?
- In fact, by virtue of $\omega^\alpha = x^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$ we find

$$z = (\omega^\alpha, \pi_{\dot{\alpha}}) \in \mathbb{P} \quad \Longleftrightarrow \quad \text{isotropic 2-plane } \mathbb{C}^2_z \hookrightarrow \mathbb{M}: \; x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \rho^\alpha \pi^{\dot{\alpha}}$$

$$\mathbb{P}^1_x \hookrightarrow \mathbb{P} \quad \Longleftrightarrow \quad x = (x^{\alpha\dot{\alpha}}) \in \mathbb{M}$$

- What's the meaning of these spheres $\mathbb{P}^1_x \hookrightarrow \mathbb{P}$?
- $\mathbb{P}^1_x$ corresponds to the projective null cone or the celestial sphere at x in real Minkowski space.

Penrose Transform

- What is twistor space good for?
- Let's consider the following contour integral:

$$\phi(x) \;=\; \frac{1}{2\pi\mathrm{i}}\oint_{S^1} \mathrm{d}\,\pi^{\dot\alpha}\pi_{\dot\alpha}\; f(x^{\beta\dot\beta}\pi_{\dot\beta},\pi_{\dot\beta})$$

- Clearly, for this to be well-defined f must be homogeneous of degree -2 in $\pi_{\dot\alpha}$.
- It then follows that ϕ is a Klein-Gordon field

$$\Box\phi \;=\; \partial_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha}\phi \;=\; 0.$$

- Being more precise, one has an isomorphism:

$$H^1(U',\mathcal{O}(-2)) \;\cong\; \{\ker\Box \text{ on } U\subset\mathbb{M}\},\quad \text{where}\quad U'\subset\mathbb{P}$$

Penrose Transform

- Generally speaking, one can show that

$$H^1(U', \mathcal{O}(2h-2)) \cong H^0(U, \mathcal{Z}_h),$$

  where $\mathcal{Z}_h$ denotes the sheaf of solutions to the helicity h zero rest mass (z.r.m.) field equations.

  This is called the Penrose transform.

- Thus, any solution to z.r.m. field equations can be represented by certain holomorphic "functions" on twistor space which are free of differential constraints.

Generalisation

- So far, we've discussed linear field equations. Can we use the above ideas to learn more about non-linear equations?
- Yes, but how?
  - Replace space-time as background manifold by a twistor space and
  - try to reinterpret the physical theory in question on that space

such that:

free analytic data on twistor space	$\Longleftrightarrow$	solutions to the field equations on space-time

Double Fibrations

- The central objects of our discussion are double fibrations of the form

$$\begin{array}{ccccc} & & F & & \\ & \stackrel{\pi_2}{\swarrow} & & \stackrel{\pi_1}{\searrow} & \\ P & & & & M \end{array}$$

where M, F and P are complex manifolds:
  - M space-time
  - F correspondence space
  - P twistor space

Twistor Correspondence: $P \overset{\pi_2}{\longleftarrow} F \overset{\pi_1}{\longrightarrow} M$

- Then we've a correspondence between P and M, i.e. between points in one space and subspaces of the other:

$$z \in P \quad \Longleftrightarrow \quad \pi_1(\pi_2^{-1}(z)) \hookrightarrow M$$
$$\pi_2(\pi_1^{-1}(x)) \hookrightarrow P \quad \Longleftrightarrow \quad x \in M$$

- To jump ahead of our story a bit, what's the relation between M and P in later applications?
- Let $\hat{x} := \pi_2\pi_1^{-1}(x) \hookrightarrow P$ be compact for any $x \in M$. Assume further that $H^1(\hat{x}, N_{\hat{x}|P}) = 0$. Then M is taken to be the $h^0(\hat{x}, N_{\hat{x}|P})$-dimensional family of deformations of $\hat{x}$ inside P (which exists due to Kodaira's theorem).

Twistor Transform: $P \overset{\pi_2}{\longleftarrow} F \overset{\pi_1}{\longrightarrow} M$

- Using the correspondence, we can transfer data given on P to data on M and vice versa.
- Take some analytic object Ob_P on P and transform it to an object Ob_M on M which will be constrained by some PDEs since $\pi_2^*\mathrm{Ob}_P$ has to be constant along the fibers of $\pi_2 \,:\, F \to P$.
- Under suitable topological conditions, the maps

$$\mathrm{Ob}_P \;\mapsto\; \mathrm{Ob}_M \qquad \text{and} \qquad \mathrm{Ob}_M \;\mapsto\; \mathrm{Ob}_P$$

define a bijection between $[\mathrm{Ob}_P]$ and $[\mathrm{Ob}_M]$ (the objects in question will only be defined up to equivalence).

Example A: Gauge Theory

Penrose-Ward Transform

Consider the flat case again

$$\begin{array}{ccc} & \mathbb{F} = \mathbb{C}^4 \times \mathbb{P}^1 & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ \mathbb{P} = \mathbb{P}^3 \setminus \mathbb{P}^1 & & \mathbb{M} = \mathbb{C}^4 \end{array}$$

Then there is a 1-1 correspondence between:

- holomorphic vector bundles $E_{\mathbb{P}}$ on $\mathbb{P}$ holomorphically trivial on any $\mathbb{P}^1_x \hookrightarrow \mathbb{P}$,
- holomorphic vector bundles $E_{\mathbb{M}}$ on $\mathbb{M}$ equipped with a connection $\nabla = \mathrm{d} + A$ flat on each $\mathbb{C}^2_z \hookrightarrow \mathbb{M}$ for $z \in \mathbb{P}$, which implies $\nabla^2 = F = F^+$, i.e. self-dual YM fields.

Example B: Gravity

Penrose's Non-Linear Graviton

Consider a complex spin 4-manifold $(M, [g])$ equipped with a conformal structure $[g] = \{g' \sim g \,|\, g' = e^{\phi}\, g\}$. Hence $TM \cong S_+ \otimes S_-$

$$\begin{array}{ccccc} & & P(S_-^*) & & \\ & \stackrel{\pi_2}{\swarrow} & & \stackrel{\pi_1}{\searrow} & \\ P & & & & M \end{array}$$

Then there is a 1-1 correspondence between complex 3-manifolds P and self-dual vacuum metrics on M such that:

- There is a holomorphic fibration $\pi : P \to \mathbb{P}^1$.
- P has a 4-parameter family of sections each with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.
- There is a symplectic structure on the fibres of π with values in $\pi^* \mathcal{O}(2)$.

Supertwistors

- To discuss supersymmetry, we need supermanifolds.
- A supermanifold is a pair $(|M|, \mathcal{O}_M)$ such that:
    - $|M|$ is a topological space and
    - $\mathcal{O}_M$ is the structure sheaf describing the "superfunctions" on M with $\mathcal{O}_M \cong \Lambda^\bullet \mathcal{E}$ (locally).
- For the time being, simply think of them as manifolds coordinatised by a set of $\mathbb{Z}_2$-graded coordinates (x^a, η^i) and superfunctions are of the form

$$f(x, \eta) \;=\; f_0(x) + f_i(x)\eta^i + \cdots + f_{1\ldots m}(x)\eta^1 \cdots \eta^m.$$

Example

- Consider $\mathbb{C}^{m+1|n} = (\mathbb{C}^{m+1}, \Lambda^\bullet\mathbb{C}^n)$ coordinatised by $(Z^I) = (z^a, \eta^i)$ and introduce the equivalence relation
$$Z^I \sim t\,Z^I \qquad \text{for} \qquad t \in \mathbb{C}^*.$$
This gives the projective superspace
$$\mathbb{P}^{m|n} = (\mathbb{P}^m, \Lambda^\bullet(\mathbb{C}^n \otimes \mathcal{O}(-1)).$$
- On $\mathbb{C}^{m+1|n}$, we may put
$$\Omega_0 = D(\mathrm{d}Z^I) \in H^0(\mathbb{C}^{m+1|n}, \mathrm{Ber}\,\Omega^1\mathbb{C}^{m+1|n}),$$
i.e. when $\mathrm{d}Z^I \mapsto \mathrm{d}Z^J T_J{}^I$ we have $\Omega_0 \mapsto \Omega_0 \mathrm{Ber}(T)$.
- Thus, it scales under $Z^I \mapsto t\,Z^I \Rightarrow \mathrm{d}Z^I \mapsto t\,\mathrm{d}Z^I$ as
$$\Omega_0 \mapsto t^{m+1-n}\Omega_0.$$

Example

- For $m+1=n$, it's scale invariant and thus descending to a holomorphic volume form Ω on $\mathbb{P}^{n|n+1}$, i.e.

$$\mathrm{Ber}(\mathbb{P}^{n|n+1}) := \mathrm{Ber}\,\Omega^1\mathbb{P}^{n|n+1} \cong \mathcal{O}.$$

- This leads us to the notion of formal Calabi-Yau supermanifolds. They are complex supermanifolds fulfilling the following equivalent statements:
  - existence of a nowhere vanishing holomorphic volume form,
  - having a trivial holomorphic Berezinian bundle,
  - having vanishing first Chern class.

Supersymmetric Twistor Correspondence (Flat Case)

- Consider right-chiral complexified Minkowski superspace $\mathbb{M} \cong \mathbb{C}^{4|2\mathcal{N}}$ coordinatised by $(x^{A\dot{\alpha}}) = (x^{\alpha\dot{\alpha}}, \eta^{i\dot{\alpha}})$. A supertwistor is an object $Z = (\omega^A, \pi_{\dot{\alpha}}) \in \mathbb{T}$ satisfying the relation

$$\omega^A = x^{A\dot{\alpha}}\pi_{\dot{\alpha}}.$$

- Thus, $\mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}}$ and

$$\begin{array}{ccc} & \mathbb{F} = \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{P}^1 & \\ & {\scriptstyle \pi_2} \swarrow \quad \searrow {\scriptstyle \pi_1} & \\ \mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}} & & \mathbb{M} = \mathbb{C}^{4|2\mathcal{N}} \end{array}$$

Supersymmetric Penrose-Ward Transform

- Then there is a 1-1 correspondence between:
  - holomorphic vector bundles $E_{\mathbb{P}}$ on $\mathbb{P}$ holomorphically trivial on any $\mathbb{P}^1_x \hookrightarrow \mathbb{P}$ for $x \in \mathbb{M}$,
  - holomorphic vector bundles $E_{\mathbb{M}}$ on $\mathbb{M}$ equipped with a connection ∇ which is flat on each $\mathbb{C}^{2|\mathcal{N}}_z \hookrightarrow \mathbb{M}$ for $z \in \mathbb{P}$: In this case we obtain $\mathcal{N}$-extended self-dual SYM theory.
- Holomorphic vector bundles can be described within the Čech or Dolbeaut approaches. In the latter approach, we have $(0,1)$-connection $\nabla^{0,1} : E \to E \otimes \Omega^{0,1}$ and E is holomorphic iff

$$F^{0,2} = (\nabla^{0,1})^2 = \bar{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0.$$

Supersymmetric Penrose-Ward Transform

- For $E_{\mathbb{P}}$ on $\mathbb{P} = \mathbb{P}^{3|4} \setminus \mathbb{P}^{1|4}$, the field equation $F^{0,2} = 0$ follows from varying

$$S[A^{0,1}] \;=\; \int \Omega \wedge \mathrm{tr}\left(A^{0,1} \wedge \bar{\partial} A^{0,1} + \tfrac{2}{3}\, A^{0,1} \wedge A^{0,1} \wedge A^{0,1}\right).$$

- In addition, we have

$$A^{0,1} \;=\; a + \eta^i \psi_i + \eta^i \eta^j \phi_{ij} + \eta^i \eta^j \eta^k \epsilon_{ijkl} \psi^l + \eta^1 \eta^2 \eta^3 \eta^4 b,$$

which leads upon inserting into $F^{0,2} = 0$ to the $\mathcal{N} = 4$ gauge multiplet $(A, \Psi_i, \Phi_{ij}, \Psi^i, B)$ and the e.o.m. of $\mathcal{N} = 4$ self-dual SYM theory in 4d.

- Recall that the above action can be shown to be equivalent to the Siegel action.

Self-Dual Supergravity

So what about the gravity sector?

In the following, I consider:

- Penrose's non-linear graviton construction from Ward's point of view,
- the supersymmetry extension thereof and
- the off-shell extension in the $\mathcal{N} = 8$ case.

Ward's Theorem

There is a 1-1 correspondence between:

- complex 4-manifolds M with holomorphic metric g such that the trace-free Ricci tensor and anti-self-dual Weyl curvature vanish, but the scalar curvature is non-vanishing and
- complex 3-manifolds P with non-degenerate holomorphic contact structure containing a $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

A non-degenerate holomorphic contact structure is a maximally non-integrable rank-2 subbundle D in $T^{1,0}P$. Think of D being defined as $D = \ker \tau$ with $\tau \in \Omega^{1,0}P$.

$$\text{non-degeneracy} \quad \Longleftrightarrow \quad \tau \wedge \mathrm{d}\tau \neq 0.$$

How can one extend this to supergravity?

Recap about Conformal Structures

Let M be a complex spin 4-manifold.

- A holomorphic conformal structure on M is an equivalence class $[g]$ of metrics defined by $[g] = \{g' \sim g \,|\, g' = e^{\phi} g\}$.
- Putting it differently, it's a line subbundle L in $\Omega^1 M \odot \Omega^1 M$.
- Since M is spin, we have $TM \cong S_+ \otimes S_-$. This isomorphism canonically yields a line subbundle $\Lambda^2 S_+^* \otimes \Lambda^2 S_-^*$ in $\Omega^1 M \odot \Omega^1 M$.
- Actually, L and $\Lambda^2 S_+^* \otimes \Lambda^2 S_-^*$ may be identified, which thus yields an equivalent definition of a conformal structure.

Supersymmetric Extensions: The Setting

Let M be a right-chiral complex supermanifold of dimension $4|2\mathcal{N}$ coordinatised by $(x^{M\dot{\mu}}) = (x^{\mu\dot{\mu}}, \eta^{m\dot{\mu}})$.

- Suppose that M is split, i.e. $M = (|M|, \Lambda^\bullet\mathcal{E})$.
- Suppose also that $TM \cong H \otimes S$, where H is of rank $2|\mathcal{N}$ and S of rank $2|0$, respectively.
- As usual, let's introduce viel-beine

$$E^{A\dot{\alpha}} = \mathrm{d}x^{M\dot{\mu}} E_{M\dot{\mu}}{}^{A\dot{\alpha}}, \qquad E_{A\dot{\alpha}} = E_{A\dot{\alpha}}{}^{M\dot{\mu}} \partial_{M\dot{\mu}}.$$

- Let's also introduce a torsion-free connection ∇ on TM by

$$\nabla V^{A\dot{\alpha}} = \mathrm{d}V^{A\dot{\alpha}} + V^{B\dot{\alpha}} \omega_B{}^A + V^{A\dot{\beta}} \omega_{\dot{\beta}}{}^{\dot{\alpha}}$$

which preserves $TM \cong H \otimes S$.

Supersymmetric Extensions: The Setting

- Due to $TM \cong H \otimes S$, the curvature R of ∇ decomposes as

$$R_{A\dot{\alpha}}{}^{B\dot{\beta}} \;=\; \delta_{\dot{\alpha}}{}^{\dot{\beta}} R_A{}^B + \delta_A{}^B R_{\dot{\alpha}}{}^{\dot{\beta}}$$

- The Ricci identity is then

$$[\nabla_{A\dot{\alpha}}, \nabla_{B\dot{\beta}}\} V^{D\dot{\delta}} \;=\; (-)^{p_C(p_A+p_B)} V^{C\dot{\delta}} R_{A\dot{\alpha}B\dot{\beta}C}{}^D + \\ + (-)^{p_D(p_A+p_B)} V^{D\dot{\gamma}} R_{A\dot{\alpha}B\dot{\beta}\dot{\gamma}}{}^{\dot{\delta}}.$$

- The self-dual supergravity equations are given by

$$R_{A\dot{\alpha}B\dot{\beta}C}{}^D \;=\; \epsilon_{\dot{\alpha}\dot{\beta}} \left\{ C_{ABC}{}^D - 2(-)^{p_C(p_A+p_B)} \Lambda_{C\{A} \delta_{B]}{}^C \right\},$$

$$R_{A\dot{\alpha}B\dot{\beta}\dot{\gamma}}{}^{\dot{\delta}} \;=\; 2\Lambda_{AB} \delta_{(\dot{\alpha}}{}^{\dot{\delta}} \epsilon_{\dot{\beta})\dot{\gamma}}.$$

A Theorem

There is a 1-1 correspondence between:

- holomorphic complex solutions to $\mathcal{N}$-extended self-dual supergravity with non-degenerate cosmological constant Λ_{AB} and
- complex supermanifolds P of dimension $3|\mathcal{N}$ with a non-degenerate (even) contact structure τ and an embedded $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2|\mathcal{N}}$.

[MW, arXiv:0705.1422]
[L. J. Mason and MW, arXiv:0706.1941]

Sketch of Proof

We want to establish $P \stackrel{\pi_2}{\leftarrow} F \stackrel{\pi_1}{\rightarrow} M$. To do that, consider $F = P(S^*)$ over M with fibre coordinates $\pi_{\dot\alpha}$. We define the twistor distribution to be the rank-$2|\mathcal{N}$ distribution D_F on F given by

$$D_F \;:=\; \left\langle \pi^{\dot\alpha} E_{A\dot\alpha} + \pi^{\dot\alpha}\pi_{\dot\gamma}\omega_{A\dot\alpha\dot\beta}{}^{\dot\gamma}\frac{\partial}{\partial\pi_{\dot\beta}} \right\rangle .$$

Then

$$[D_F, D_F] \;\subset\; D_F \qquad \Longleftrightarrow \qquad R_{A(\dot\alpha B\dot\beta\dot\gamma\dot\delta)} \;=\; 0$$

and if so, we get $\pi_2 : F \to P$ with $\dim P = 3|\mathcal{N}$.

Sketch of Proof

Since F is a $\mathbb{P}^1$-bundle over M, we find:

$$\begin{aligned} \mathbb{P}^1_x \hookrightarrow P \quad &\Longleftrightarrow \quad x \in M \\ z \in P \quad &\Longleftrightarrow \quad \text{isotropic } 2|\mathcal{N}\text{-supermanifold in } M. \end{aligned}$$

The normal bundle of each $\mathbb{P}^1_x$ is $\mathcal{O}(1) \otimes \mathbb{C}^{2|\mathcal{N}}$. The inverse construction uses a supersymmetric extension of Kodaira's deformation theory. Let's skip it ...

... instead, let's move on towards the self-dual Einstein condition.

Sketch of Proof

So far, we have established $P \stackrel{\pi_2}{\longleftarrow} F \stackrel{\pi_1}{\longrightarrow} M$. Let's introduce a 1-form $\widetilde{\tau}$ on F by

$$\widetilde{\tau} \;=\; \pi^{\dot\alpha}\nabla\pi_{\dot\alpha} \;=\; \pi^{\dot\alpha}\,\mathrm{d}\pi_{\dot\alpha} - \omega_{\dot\alpha}{}^{\dot\beta}\pi^{\dot\alpha}\pi_{\dot\beta}.$$

Clearly, $\widetilde{\tau}$ is of homogeneity 2. It descents down to P, i.e. $\widetilde{\tau} = \pi_2^*\tau$, iff

$$D_F \lrcorner \widetilde{\tau} \;=\; 0 \;=\; D_F \lrcorner \mathrm{d}\widetilde{\tau}.$$

The first condition is always fulfilled while the second one only iff the self-dual supergravity equations are satisfied. Furthermore, non-degeneracy of the contact structure corresponds to non-degeneracy of Λ_{AB}. □

From Ward to Penrose

So in summary, non-degenerate holomorphic contact structures on supertwistor spaces correspond to solutions to the self-dual supergravity equations with non-zero cosmological constant.

What about the zero case?

In order for that to happen, one considers maximally degenerate contact structures on P. In particular, that means that $[D, D] \subset D$ for $D = \ker \tau$, i.e. D is integrable. Thus, we get a rank-$2|\mathcal{N}$ foliation with base space $\mathbb{P}^1$, i.e.

$$\pi \,:\, P \;\to\; \mathbb{P}^1 .$$

Finite Deformations of $\mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}}$

We want to describe a twistorial off-shell formulation of $\mathcal{N} = 8$ self-dual supergravity.

But how?

In the following, I choose as background

- flat supertwistor space $\mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}}$ with the standard $\bar{\partial}_0$-operator and holomorphic contact structure τ_0 and
- consider finite complex and contact structure deformations on $\mathbb{P}$ to get P with $\bar{\partial}$ and τ.

This will give a way to obtain an off-shell formulation for $\mathcal{N} = 8$.

Finite Deformations of $\mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}}$

Consider $\mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}}$ with coordinates $(z^I) = (z^a, \eta^i) = (\omega^\alpha, \pi_{\dot\alpha}, \eta^i)$. Let's introduce a Poisson structure on homogeneous functions f and g

$$[f, g\} \ := \ (-)^{p_I} \partial_I f \, \omega^{IJ} \, \partial_J g.$$

Let's also introduce a $(0,1)$-form $h = \mathrm{d}\bar{z}^{\bar{a}} h_{\bar{a}}$ of homogeneous degree 2 in z^I and 0 in $\bar{z}^{\bar{I}}$ holomorphic in η^i.

Finite Deformations of $\mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}}$

We then define the distribution $T^{0,1}P$ of anti-holomorphic tangent vectors on P by

$$T^{0,1}P \ := \ \langle \bar{D}_{\bar{I}} \rangle \ := \ \left\langle \frac{\partial}{\partial \bar{z}^{\bar{a}}} + (-)^{p_I} \frac{\partial h_{\bar{a}}}{\partial z^I} \omega^{IJ} \frac{\partial}{\partial z^J} \, , \frac{\partial}{\partial \bar{\eta}^{\bar{i}}} \right\rangle$$

This is to be understood as a finite perturbation of the standard complex structure on $\mathbb{P}$ with $\bar{\partial}_0 = \mathrm{d}\bar{z}^{\bar{I}} \bar{\partial}_{\bar{I}}$.

Since the $(1,0)$-forms are $Dz^I = \mathrm{d}z^I + \omega^{IJ} \partial_J h$ and $\tau_0 = \mathrm{d}z^I z^J \omega_{JI}$, the deformed contact structure τ is

$$\tau = Dz^I z^J \omega_{JI} = \mathrm{d}z^I z^J \omega_{JI} + z^J \underbrace{(-)^{p_I} \omega_{JI} \omega^{IK}}_{= \delta_J{}^K} \partial_K h = \tau_0 + 2h.$$

Finite Deformations of $\mathbb{P} = \mathbb{P}^{3|\mathcal{N}} \setminus \mathbb{P}^{1|\mathcal{N}}$

Then

$$[T^{0,1}P, T^{0,1}P] \subset T^{0,1}P \iff \omega^{IJ}\partial_J\left(\bar\partial_0 h + \tfrac{1}{2}[h,h\}\right) = 0,$$
$$\bar\partial\tau = 0 \iff \bar\partial_0 h + \tfrac{1}{2}[h,h\} = 0,$$

Thus, integrability of the complex structure follows from the holomorphy of the contact structure.

For $\mathcal{N} = 8$, $\bar\partial_0 h + \frac{1}{2}[h,h\} = 0$ follows from

$$S[h] = \int \Omega\wedge\left(h\wedge\bar\partial_0 h + \tfrac{1}{3}h\wedge[h,h\}\right),$$

since the homogeneity of h is 2, of $[\cdot,\cdot\}$ is -2 and of Ω is -4

Action Principle for $\mathcal{N} = 8$ Self-Dual Supergravity

So we have

$$S[h] = \int \Omega \wedge \left(h \wedge \bar{\partial}_0 h + \tfrac{1}{3} h \wedge [h, h\}\right).$$

Depending on the degeneracy of the Poisson structure ω, one can describe different self-dual supergravities:

- $\text{rank}\,\omega = 4|\mathcal{N} \iff$ R-symmetry maximally gauged to $SO(\mathcal{N})$ & non-zero cosmological constant
- $\text{rank}\,\omega = 4|r \iff$ R-symmetry gauged to some $H \subset SO(\mathcal{N})$ & non-zero cosmological constant
- $\text{rank}\,\omega = 2|r \iff$ R-symmetry gauged to some $H \subset SO(\mathcal{N})$ & zero cosmological constant

Note that since a CS action depends on a chosen background, diffeomorphism invariance (on supertwistor space) is explicitly broken.

Can one find a covariant formulation making diffeomorphism invariance manifest?

Covariant Approach for $\mathcal{N} = 0$: A Theorem

Suppose that on a (smooth) manifold P of dimension $4n + 2$ we are given a complex line bundle $L^* \subset \mathbb{C}T^*P$, represented by a complex 1-form τ defined up to complex rescalings. Suppose further that

- $\tau \wedge (\mathrm{d}\tau)^{n+1} = 0$ and $\tau \wedge (\mathrm{d}\tau)^n \neq 0$ and
- $\ker\{\tau \wedge (\mathrm{d}\tau)^n\} \cap \overline{\ker\{\tau \wedge (\mathrm{d}\tau)^n\}} = \{0\}$.

Then there is a unique integrable almost complex structure for which τ is proportional to a non-degenerate holomorphic contact structure.

[L. J. Mason and MW, arXiv:0706.1941]

Covariant Approach for $\mathcal{N} = 0$: An Action

In the twistor context, $n = 1$ and P is topologically $\mathbb{R}^4 \times S^2$ and $c_1(L) = 2$. Then, the field equation $\tau \wedge (\mathrm{d}\tau)^2 = 0$ follows from

$$S[b, \tau] \;=\; \int b \wedge \tau \wedge (\mathrm{d}\tau)^2,$$

where $b \in \Omega^1 P \otimes (L^*)^3$ is a Lagrange multiplier.

On-shell, L becomes $\mathcal{O}(2)$ and $b \in H^1(P, \mathcal{O}(-6))$.

Via the Penrose transform b corresponds to a helicity -2 field propagating in the 4-dimensional self-dual background determined by τ.

Conclusions and Outlook

What we've got:

- We saw how Ward's extension of Penrose's non-linear graviton construction needs to be extended to self-dual supergravity.
- For $\mathcal{N} = 8$, we found a Chern-Simons-like off-shell formulation of the theory on supertwistor space which was dependent on a choice of background.
- We made progress towards an invariant off-shell formulation for $\mathcal{N} = 0$.

[MW, arXiv:0705.1422]
[L. J. Mason and MW, arXiv:0706.1941]

Conclusions and Outlook

What's next:

- Invariant off-shell formulation of the $\mathcal{N}=8$ theory?
- Notice that we have an invariant on-shell formulation:
  - Let τ be a complex 1-form with only holomorphic dependence on η^i.
  - Assume that $\text{rank}\,\{\tau\wedge\text{d}\tau\}=3|\mathcal{N}$ and

$$\ker\{\tau\wedge\text{d}\tau\}\cap\overline{\ker\{\tau\wedge\text{d}\tau\}}=\{0\}.$$

  - Then define $T^{0,1}P:=\ker\{\tau\wedge\text{d}\tau\}$ and apply the theorem.
- What about the full $\mathcal{N}=8$ Einstein supergravity and the relation to twistor strings?
- ...

Thank you very much for your attention!