Continuum limits of Gaussian Markov random fields : resolving the conflict with geostatistics

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Agenda

- **Hidden Markov random fields (MRF's)**.
- **Geostatistical** versus **MRF** approach to **spatial data**.
- Describe simplest **Gaussian intrinsic autoregression** on 2–d **rectangular array**.
- Provide its **exact** and **asymptotic variograms**.
- Reconcile **geostatistics** and **Gaussian MRF's** via **regional averages**.
- **Generalizations** and **wrap–up**.

For general theory and some applications of Gaussian MRF's, see

H. Rue & L. Held (2005), *Gaussian Markov Random Fields*, Chapman & Hall.

For intrinsic autoregressions and the limiting de Wijs process, see

J. Besag & C. Kooperberg (1995), *Biometrika*, **82**, 733–746.

J. Besag & D. Mondal (2005), *Biometrika*, **92**, 909–920.

Hidden Markov random fields for spatial data

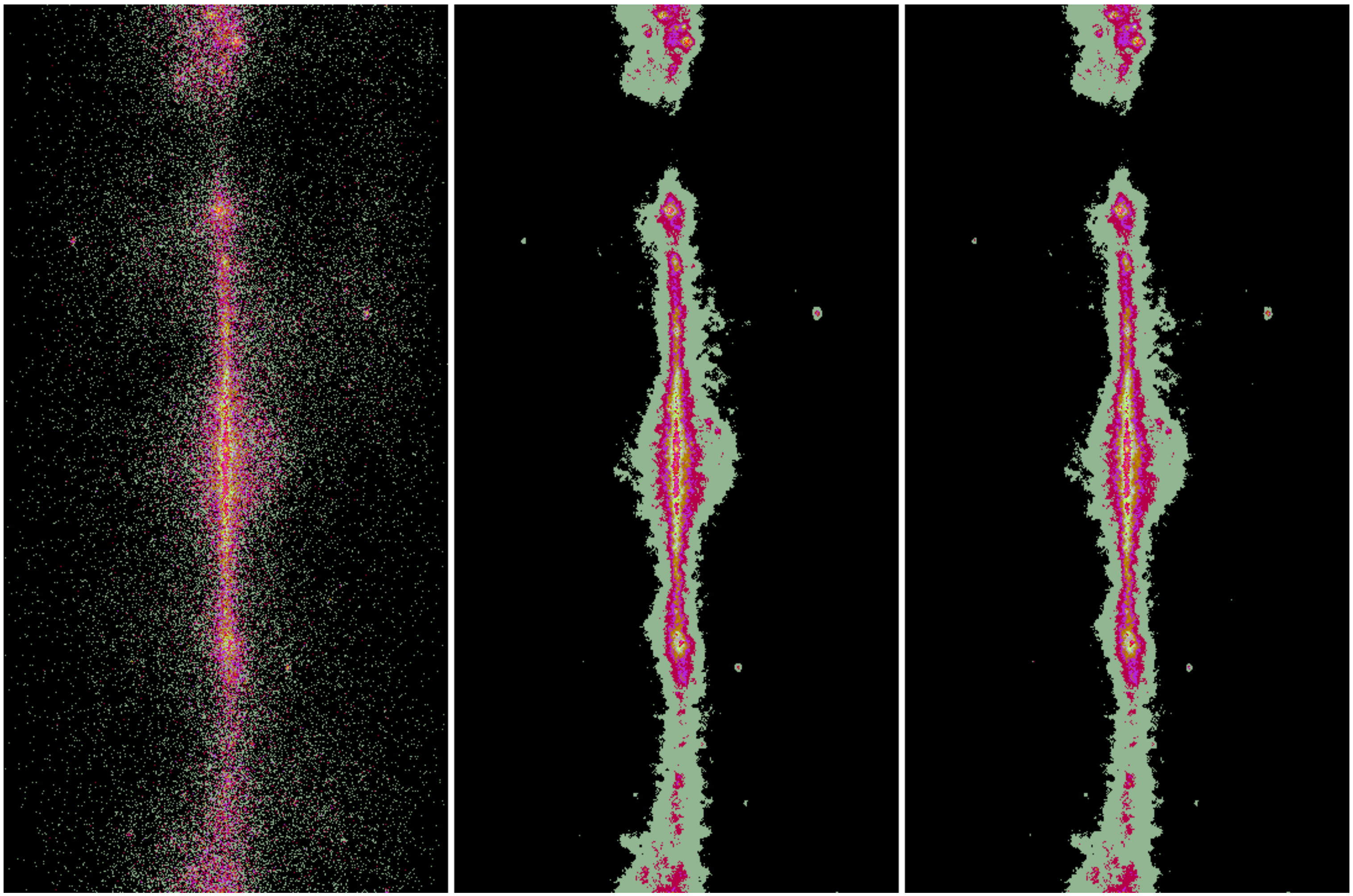
- **Markov random fields** arise naturally in **spatial context**.
- Spatial variables observed **indirectly**, via **treatments**, **covariates**, **blur**, **noise**, ...
- **Data** $\mathbf{y}$ = response to **linear predictor** $\boldsymbol{\eta}$

$$\begin{aligned} \boldsymbol{\eta} &= \mathbf{T}\boldsymbol{\tau} + \mathbf{F}\mathbf{x} + \mathbf{z} \\ \boldsymbol{\tau} &= \text{treatment / variety / covariate effects} \\ \mathbf{T} &= \text{design matrix (covariate information)} \\ \mathbf{x} &= \text{(secondary) } \textbf{spatial effects} \\ \mathbf{F} &= \text{linear filter (identity/incidence matrix, averaging operator, } \ldots) \\ \mathbf{z} &= \text{residual effects} \end{aligned}$$

- Usually, goal is to make **probabilistic inferences** about $\boldsymbol{\tau}$ (**MCMC** or ...). Unknown/unmeasured **covariates** might be identified via $\mathbf{x}$ and $\mathbf{z}$ (Rumsfeld).
- **Stochastic representation** of $\mathbf{x}$ via **MRF** : often "**prior ignorance**". E.g. **Ising/Potts model** or **Gaussian/non–Gaussian smoother**.

EGRET (energetic gamma–ray experiment telescope) astronomy

Raw photon counts L2 deblurring L1 deblurring

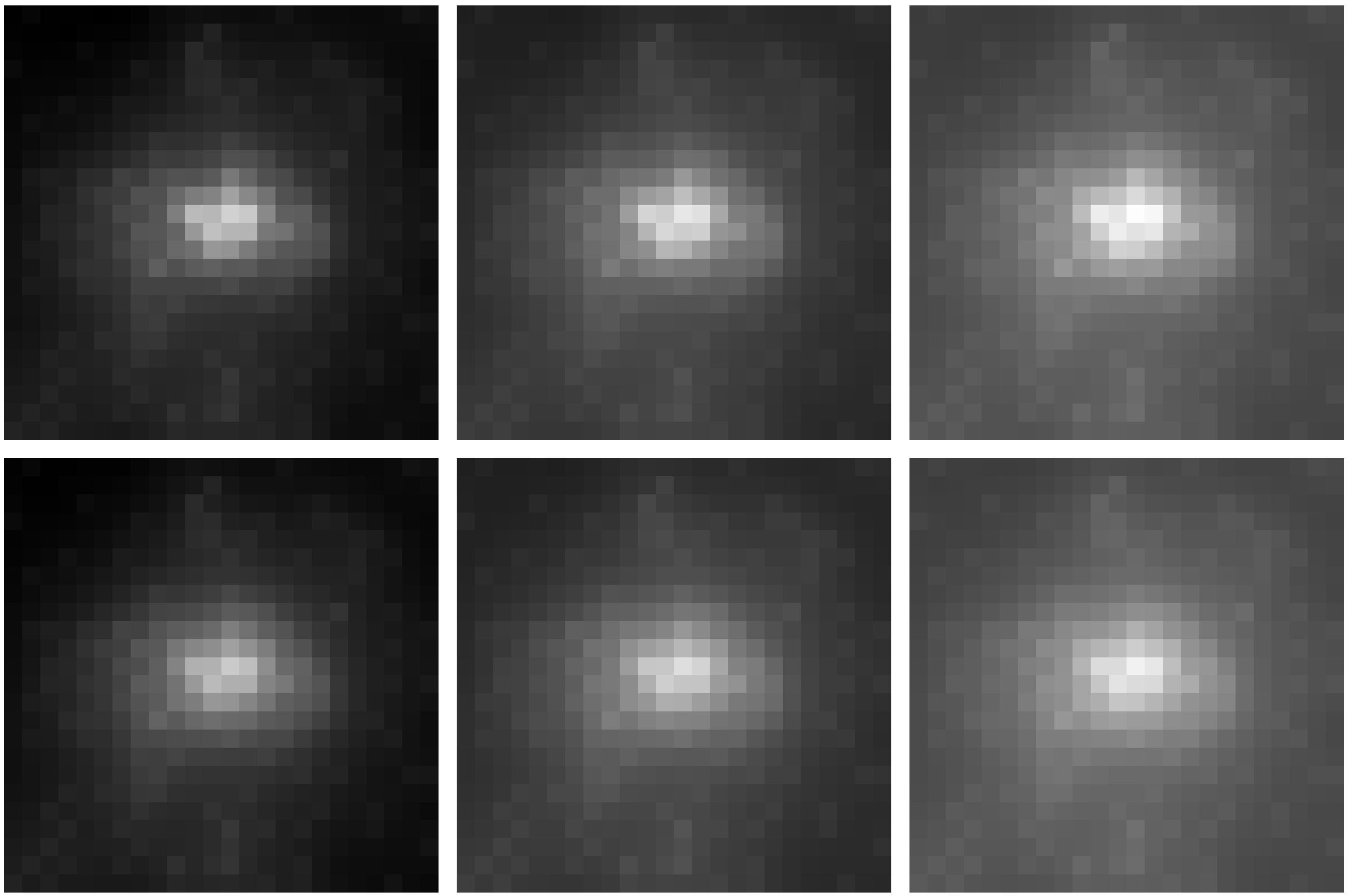


EGRET astronomy

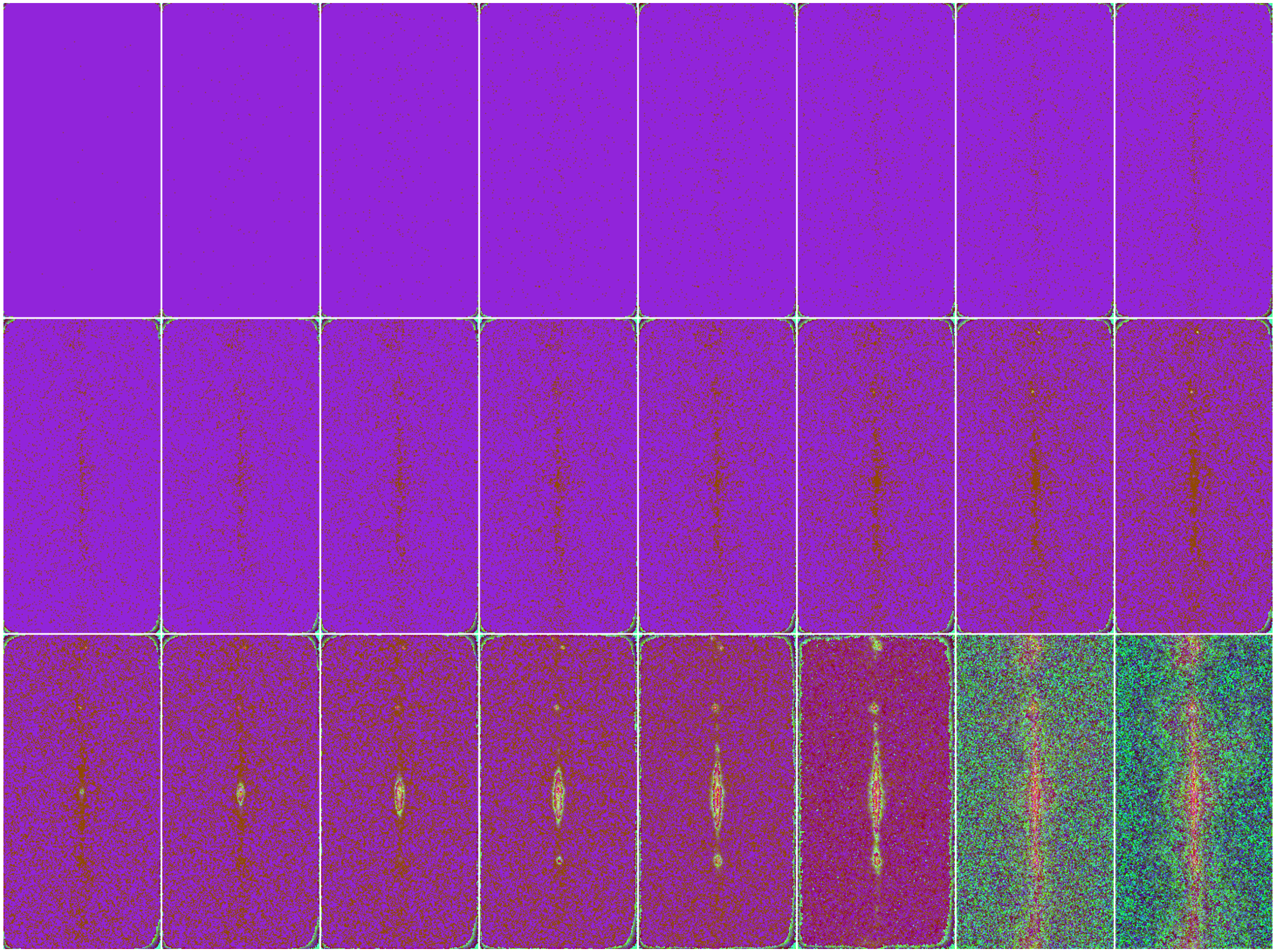
Lower 10% points | 50% points | Upper 10% points

L2

L1



Markov chain Monte Carlo every 2500 image updates



Geostatistical approach to spatial component

- Specify **continuum spatial process**, often chosen via family of Matérn variograms.
- Extract **covariance matrix** for observations.
- **Fit surface** and make **predictions**.
- **Rescaling** OK.
- Substantial **computational burden**.

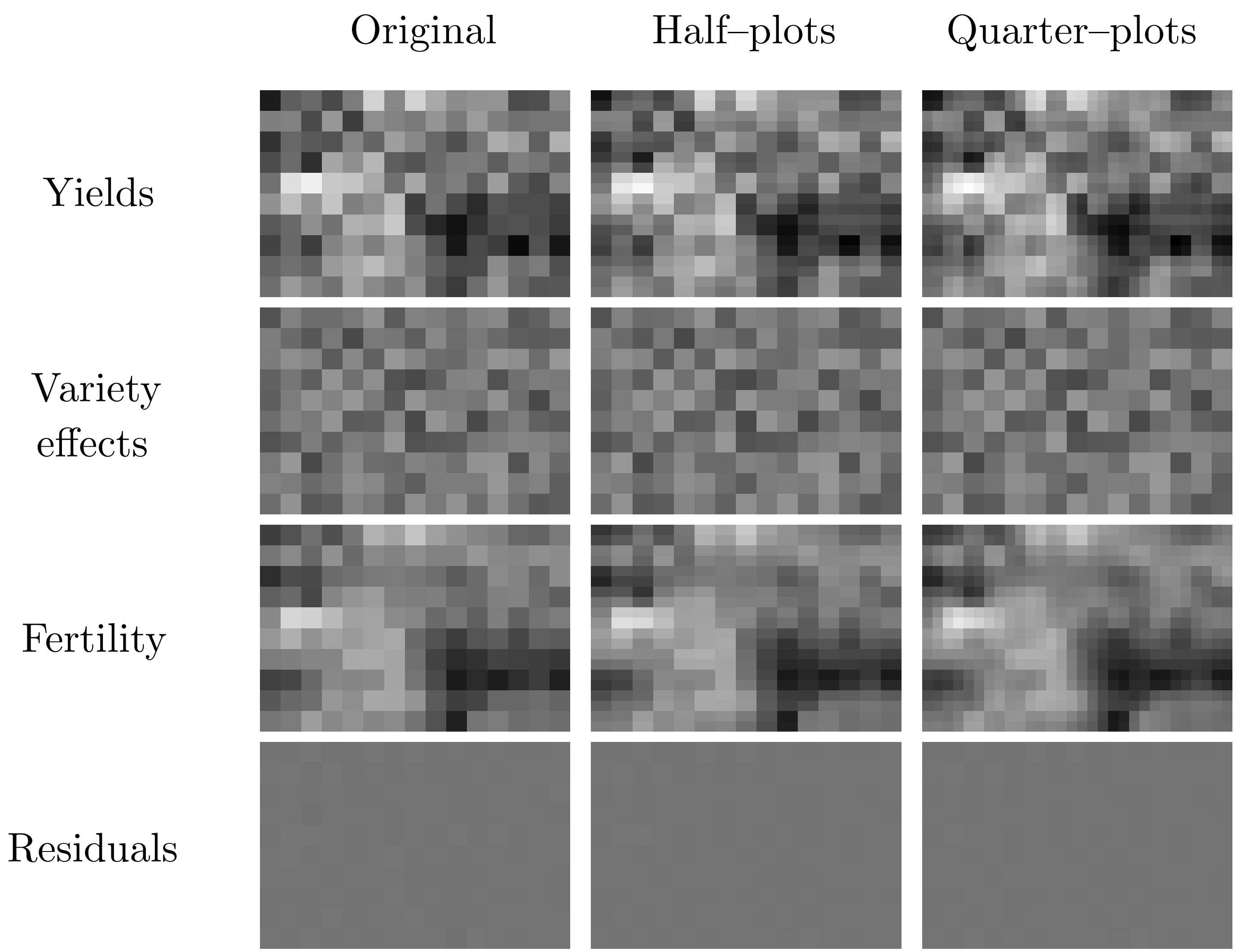
Gaussian Markov random field (MRF) approach

- Assume **discrete space** (?!) **Markov** property.
- If **Gaussian** $\Rightarrow$ locations of **nonzero** elements in **precision matrix**.
- **Estimate** parameters in overall scheme.
- **Sparse matrix computation** OK (e.g. cotton field with 500,000 pixels).
- **Scale** and **prediction problematic** at least aesthetically.

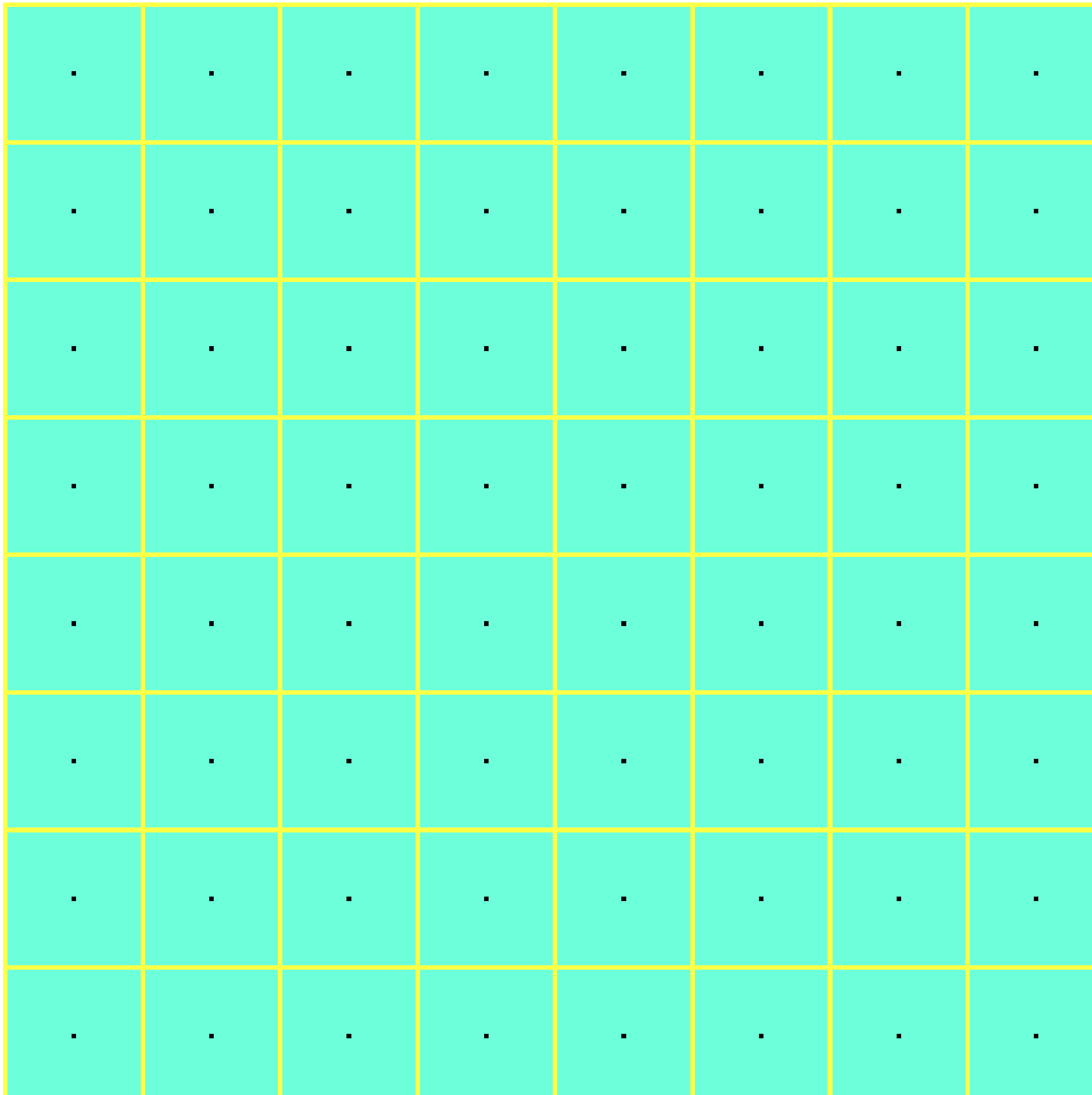
Variety trial for wheat at Plant Breeding Institute, UK

Besag and Higdon (JRSS B, 1999)

Bayesian spatial analysis: effect of scale



Markov random fields on pixel arrays



Gaussian Markov random fields on rectangular pixel arrays

- **Pixel centres** $i = (u, v) \in \mathcal{Z}^2$.

- Choose **neighbours** ∂i for each site i

  $\Rightarrow \quad \pi(x_i \,|\, \mathbf{x}_{-i}) \;\equiv\; \pi(x_i \,|\, \mathbf{x}_{\partial i})$.

  $\Rightarrow$ Undirected **conditional dependence graph** $\mathcal{G}$.

- Associated **Gaussian** random vector $\mathbf{X} = \{X_i : i \in \mathcal{Z}^2)$.

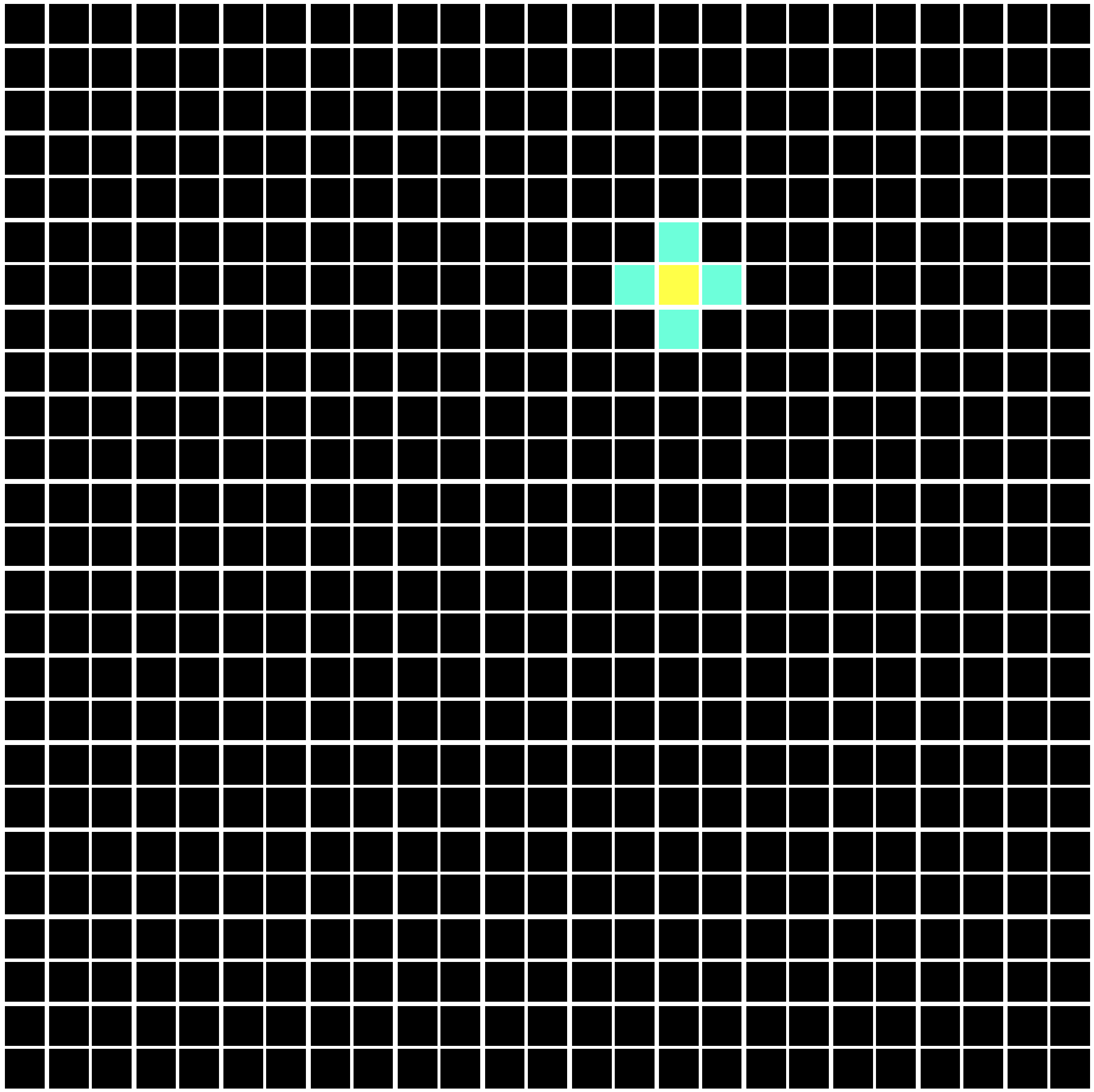
  Joint distribution $\{\pi(\mathbf{x})\}$, with **full conditionals** $\pi(x_i \,|\, \mathbf{x}_{-i})$,

  $\Rightarrow$ $\pi(\mathbf{x})$ honours the graph $\mathcal{G}$ and is a **Markov random field** w.r.t. $\mathcal{G}$.

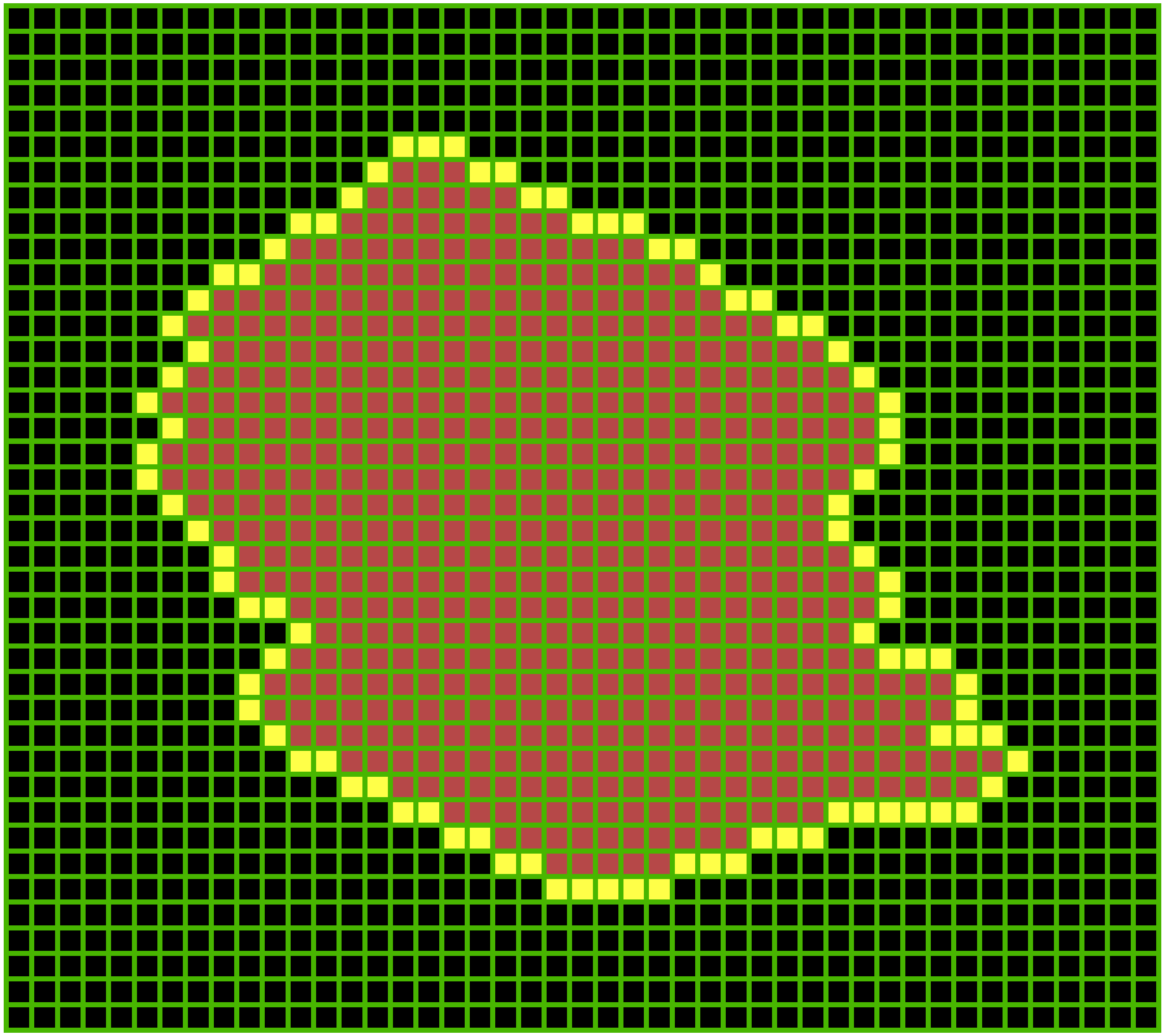
- **Cliquo**: any single site or set of mutual neighbours w.r.t. $\mathcal{G}$.

- **Clique**: maximal cliquo.

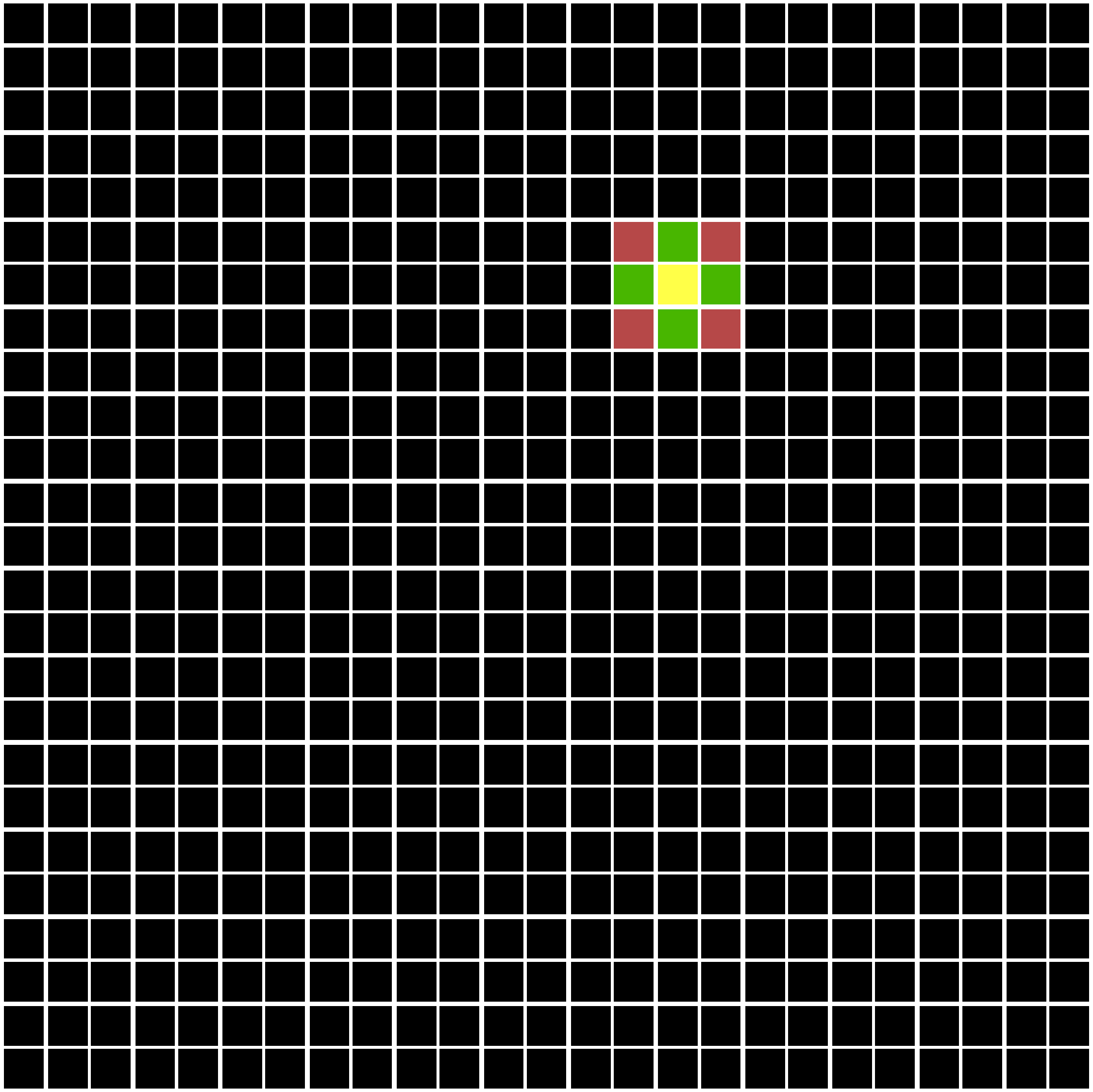
Neighbours for 1st–order Markov random field



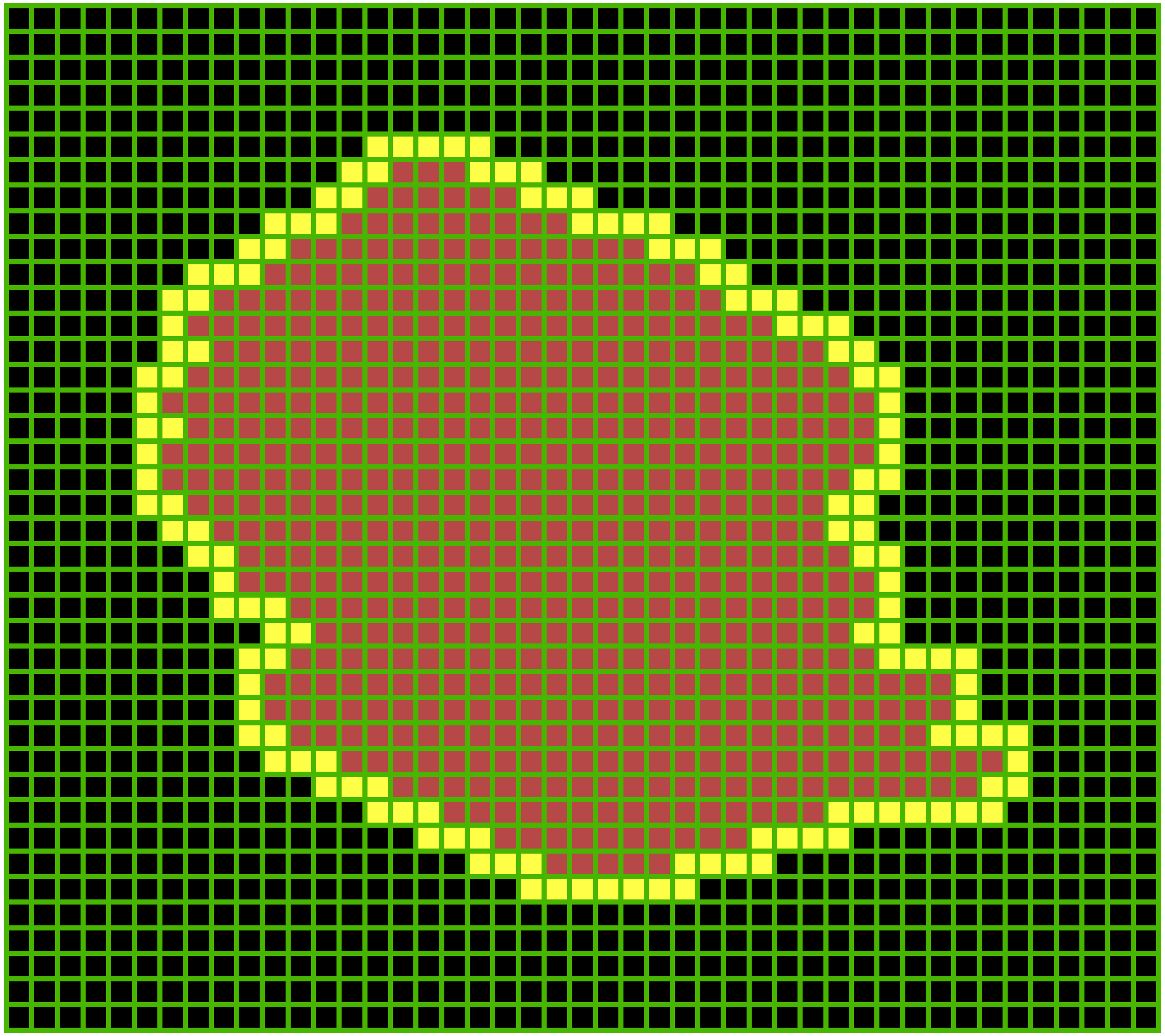
Global property for 1st–order Markov random field



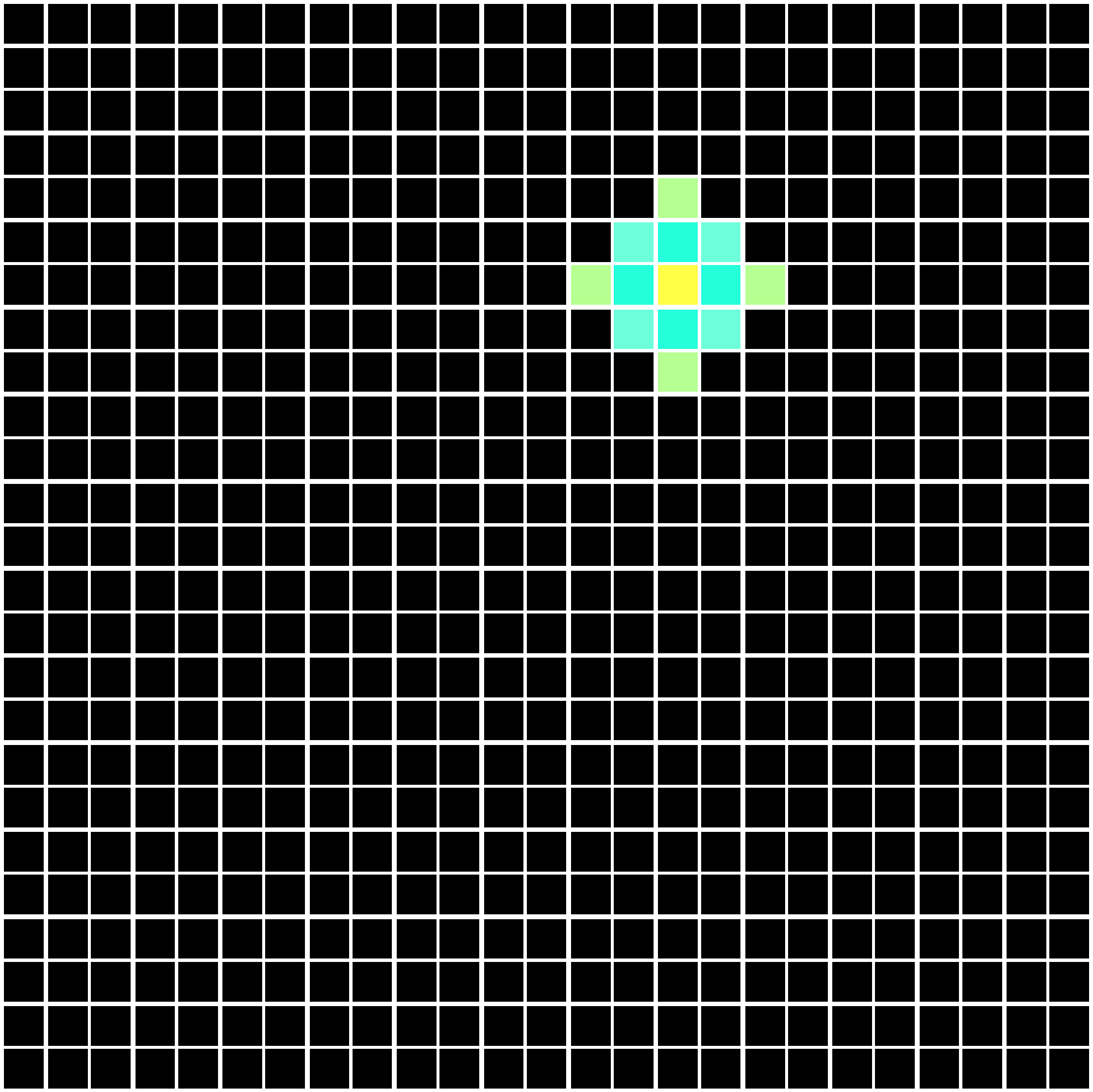
Neighbours for 2nd–order Markov random field



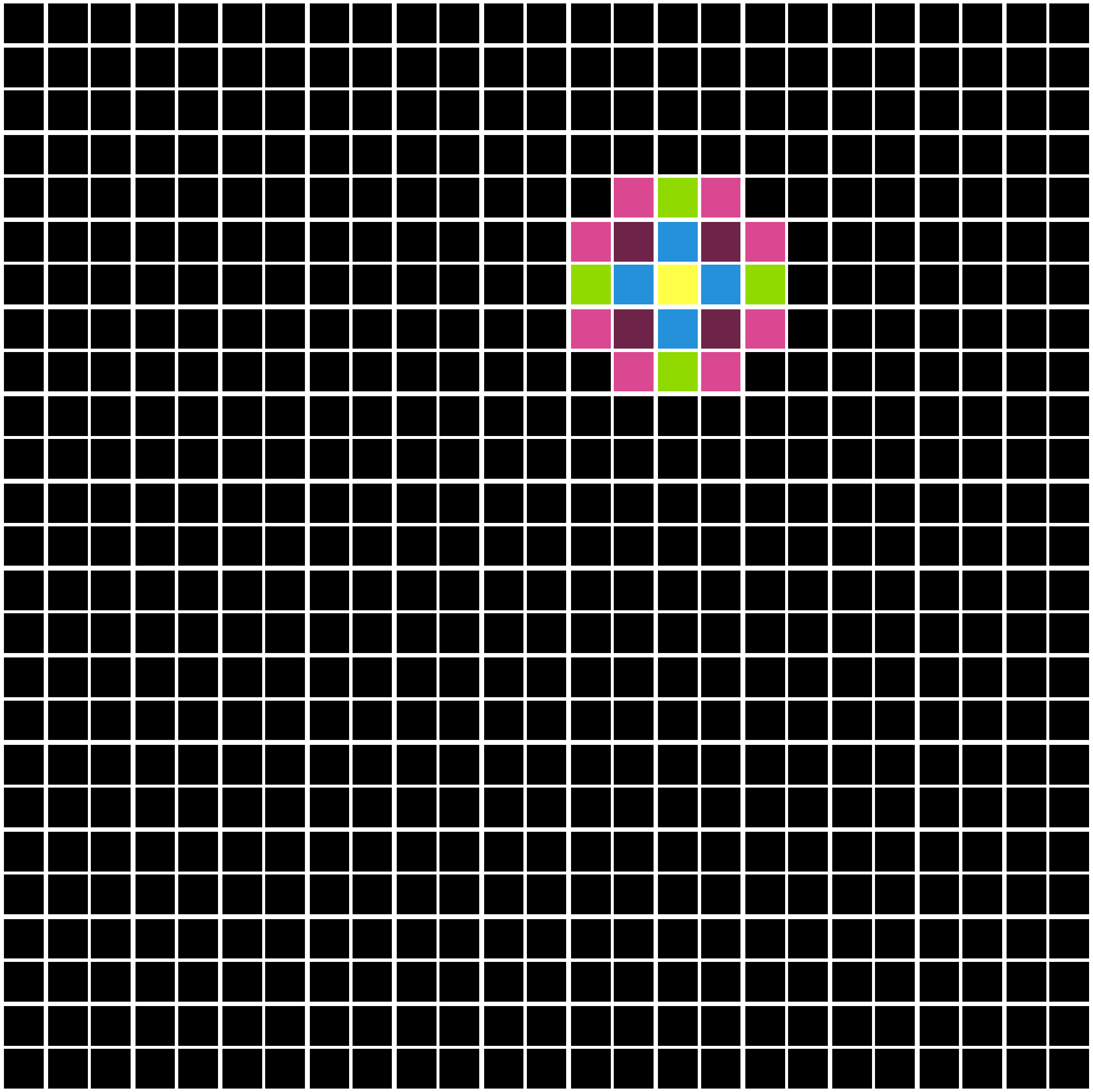
Global property for 2nd–order Markov random field



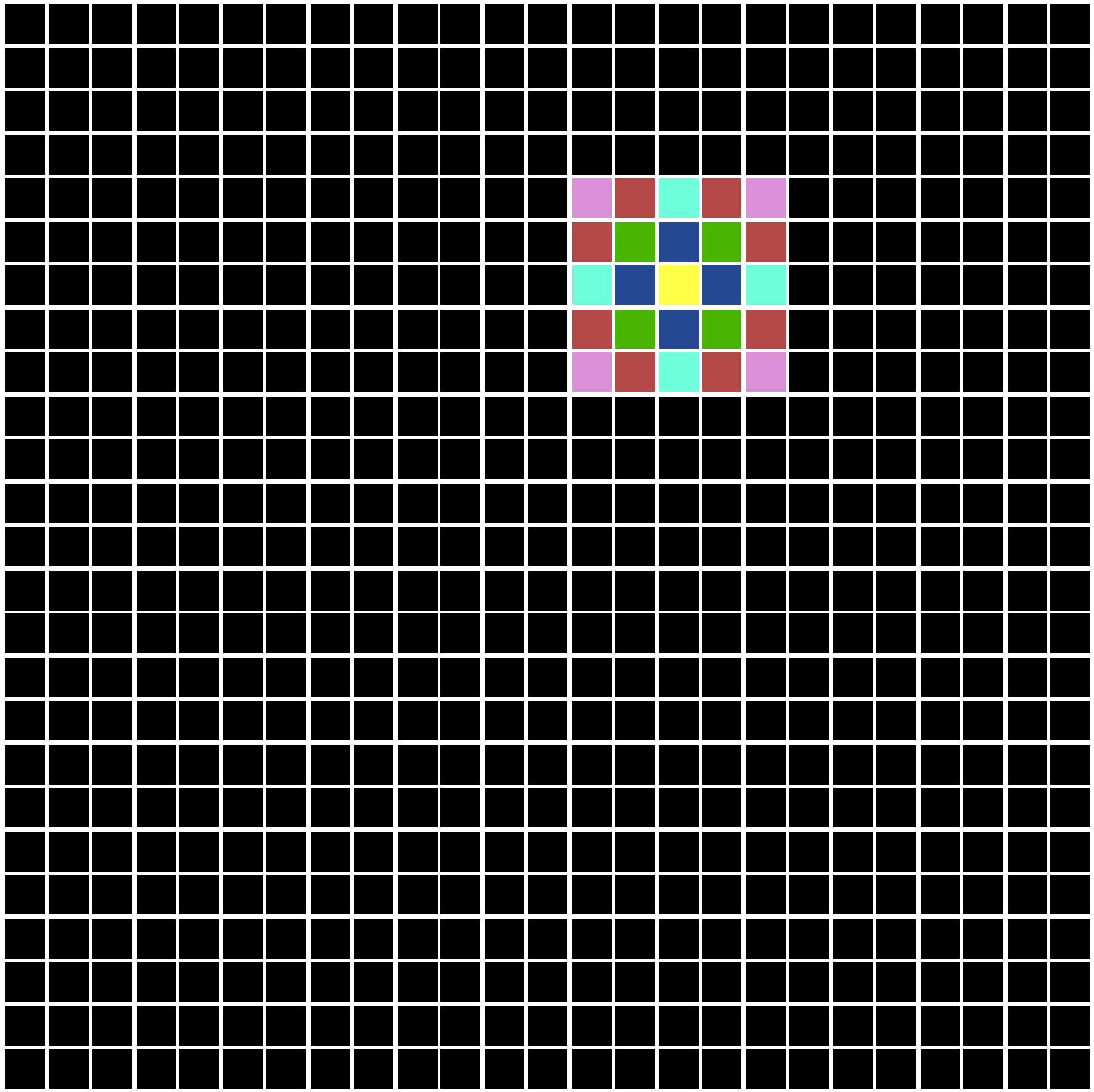
Neighbours for 3rd–order Markov random field



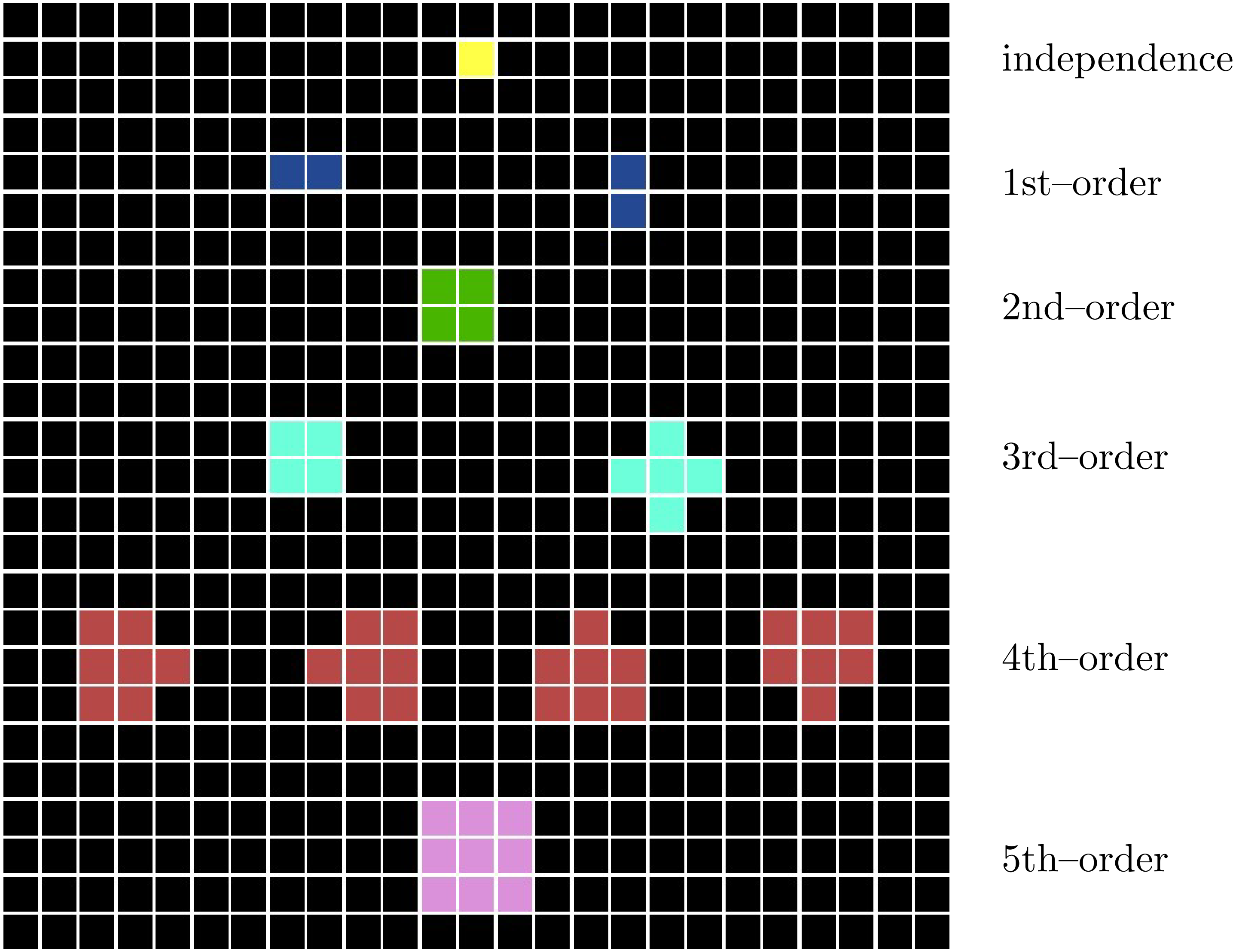
Neighbours for 4th–order Markov random field



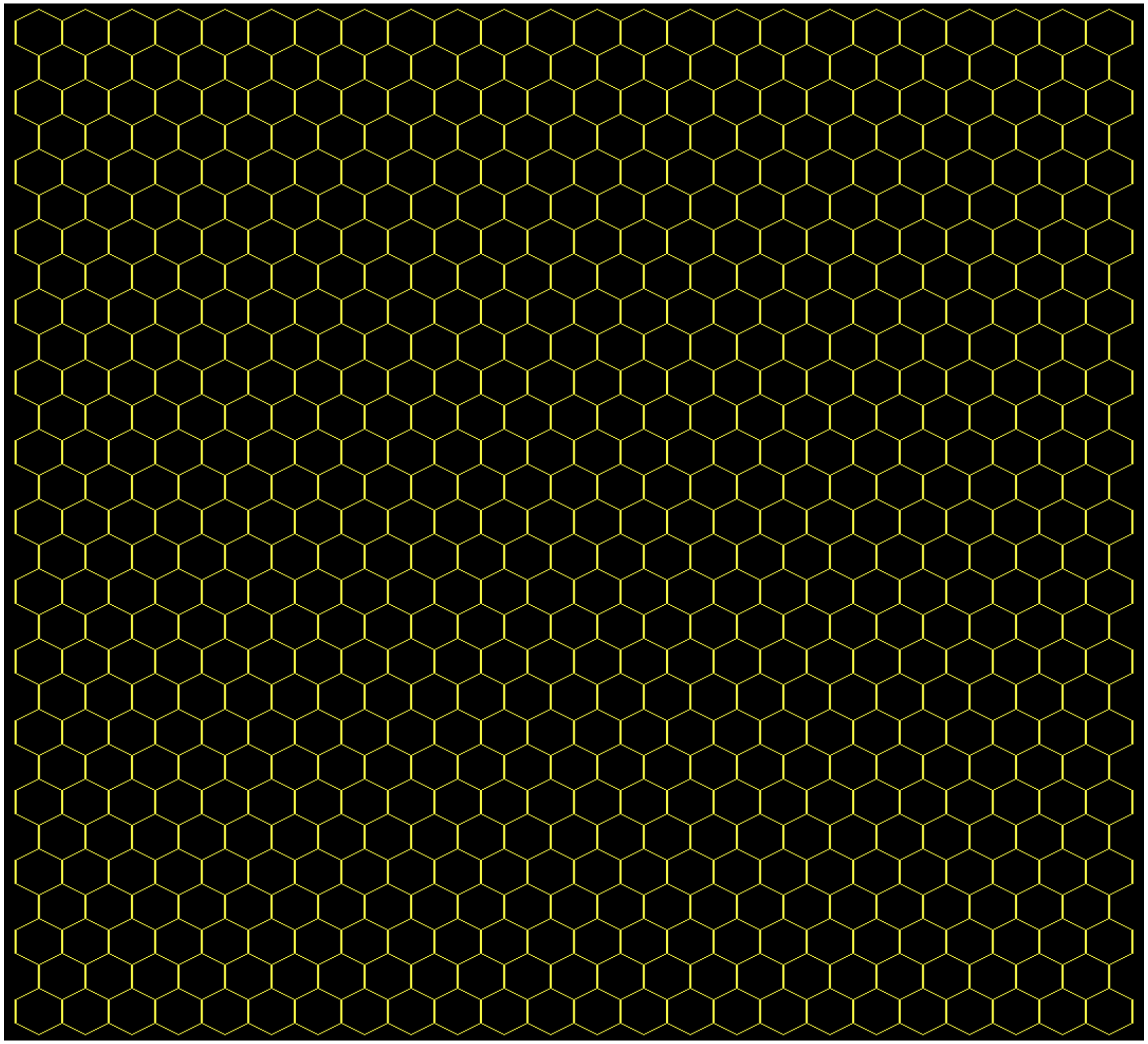
Neighbours for 5th–order Markov random field



Cliques for MRF's on rectangular arrays



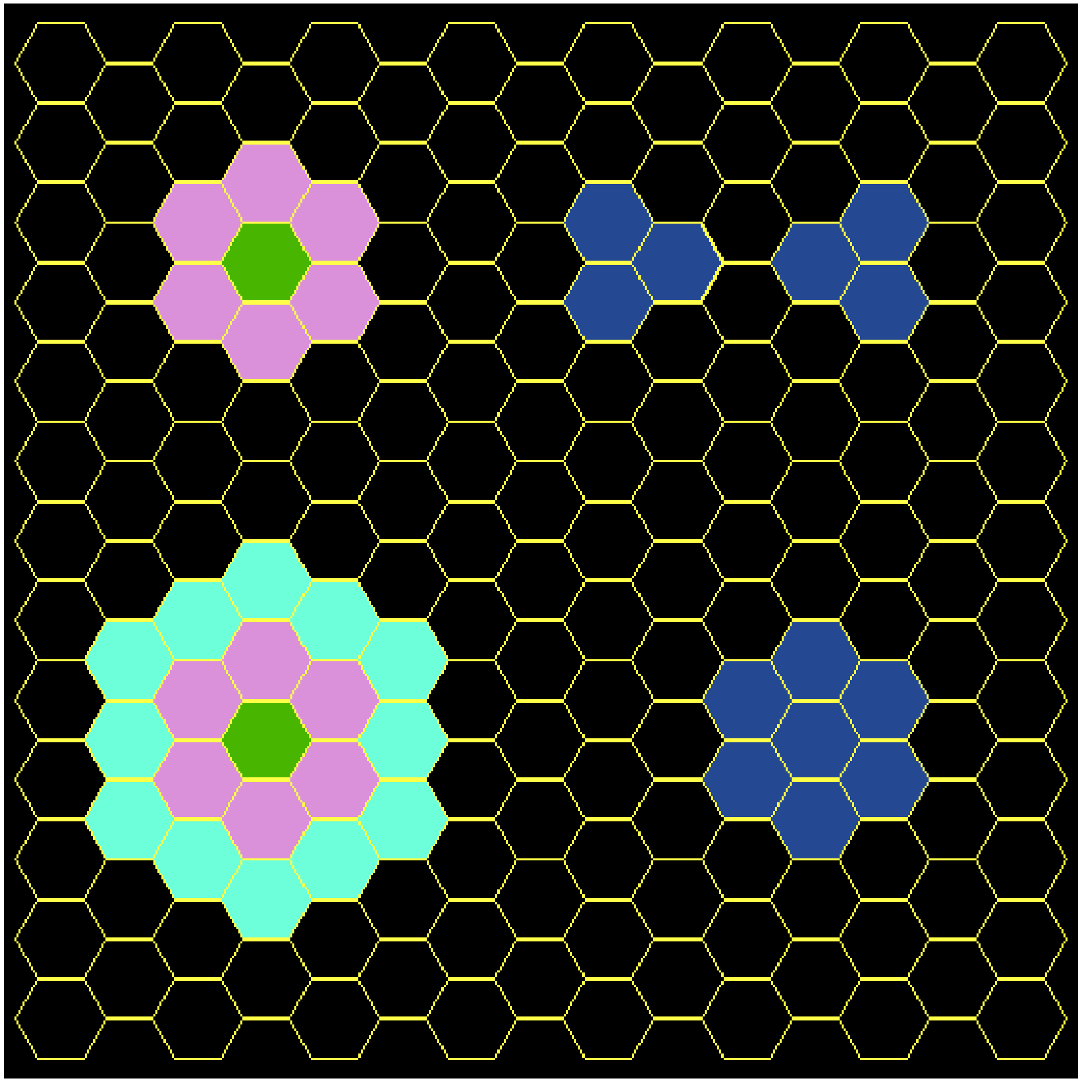
Markov random fields on hexagonal arrays



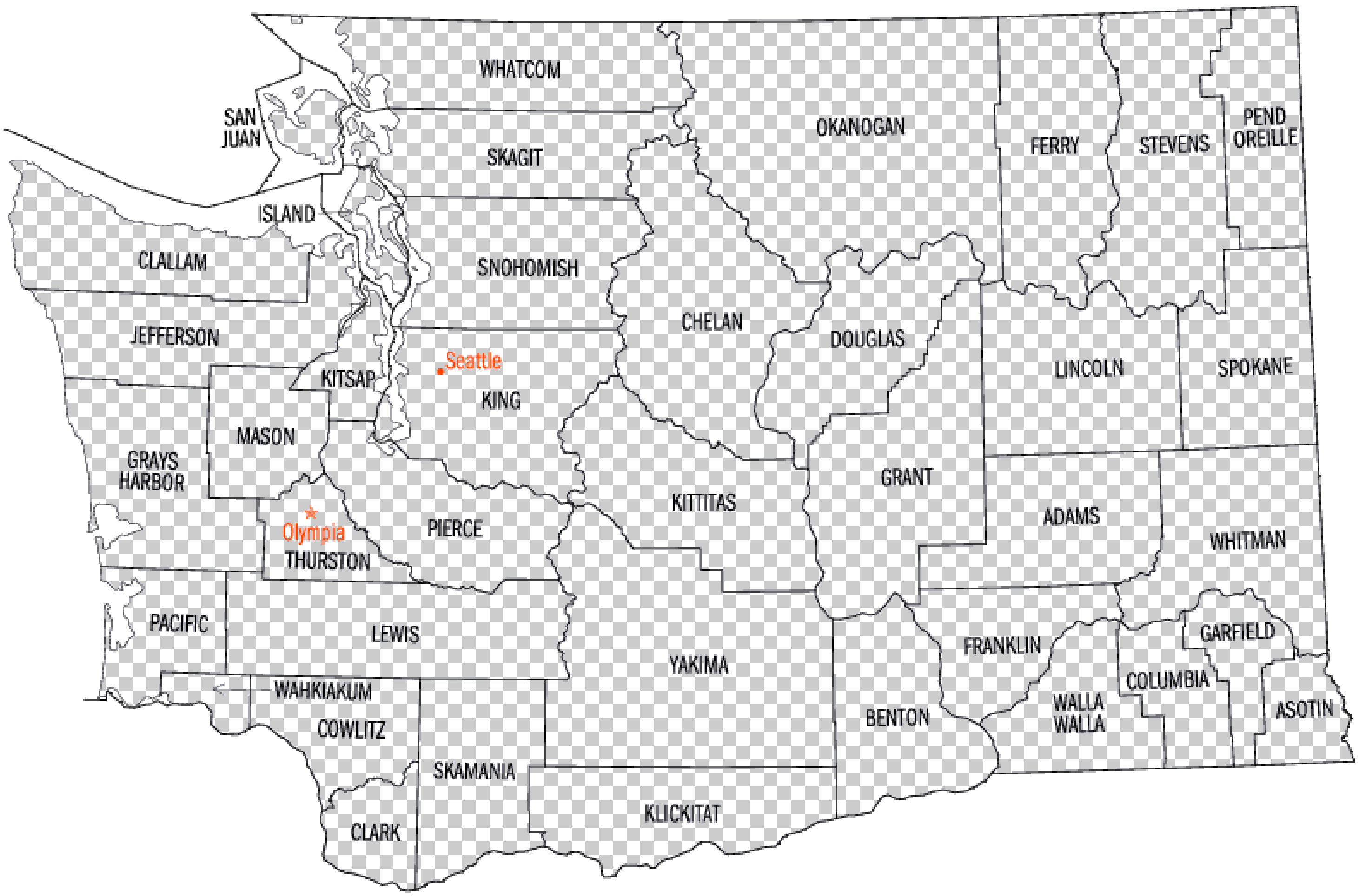
Neighbours and cliques for MRF's on hexagonal arrays

1st–order

2nd–order



Example of irregular regions: Washington State



Besag, Green, Higdon & Mengersen (1995)

Pairwise difference distributions

- **Sites** (e.g. pixels) i, j, ..., with associated **random variables** X_i, X_j, ...
- Joint **generalized probability density function** of X_i's:

$$\pi(\mathbf{x}) \;\propto\; \exp\{-\textstyle\sum_{i\heartsuit j} \lambda_{ij}\, g(|x_i - x_j|)\}, \qquad x_i \in \mathcal{R},$$

where $i \heartsuit j$ indicates that i and j are **neighbours**.

- At best $\pi(.)$ is **informative** about some or all **contrasts** among X_i's.

NB. $\sum_i c_i X_i$ is a contrast if the constants c_i satisfy $\sum_i c_i = 0$.

Gaussian pairwise difference distributions

$$\pi(\mathbf{x}) \;\propto\; \exp\{-\textstyle\sum_{i\heartsuit j} \lambda_{ij}\,(x_i - x_j)^2\}$$

- $\lambda_{ij} > 0$ for all $i \heartsuit j$ $\Rightarrow$ $\sum_{i\heartsuit j} \lambda_{ij}\,(x_i - x_j)^2$ is **positive semidefinite**

$\Rightarrow$ simple **differences** have well–defined distributions

$\Rightarrow$ **variogram** $\nu_{ij} := \frac{1}{2}\,\mathrm{var}\,(X_i - X_j)$ is well defined.

Künsch (1987), Besag & Kooperberg (1995)

First–order Gaussian intrinsic autoregressions on $\mathcal{Z}^2$

- Let $\{X_{u,v} : (u,v) \in \mathcal{Z}^2\}$ be **Gaussian** with **conditional** means and variances

$$\begin{aligned} \mathrm{E}\,(X_{u,v} \mid \ldots) &= \beta\,(x_{u-1,v} + x_{u+1,v}) + \gamma\,(x_{u,v-1} + x_{u,v+1}), \\ \mathrm{var}\,(X_{u,v} \mid \ldots) &= \kappa > 0, \end{aligned}$$

  where $\beta, \gamma > 0$ and $\boldsymbol{\beta} + \boldsymbol{\gamma} = \mathbf{\frac{1}{2}}$. Symmetric special case : $\beta = \gamma = \frac{1}{4}$.

- **Pairwise difference distribution** with

$$\pi(\mathbf{x}) \propto \exp\{-\lambda\beta \textstyle\sum_u \sum_v (x_{u,v} - x_{u+1,v})^2 - \lambda\gamma \sum_u \sum_v (x_{u,v} - x_{u,v+1})^2\},$$

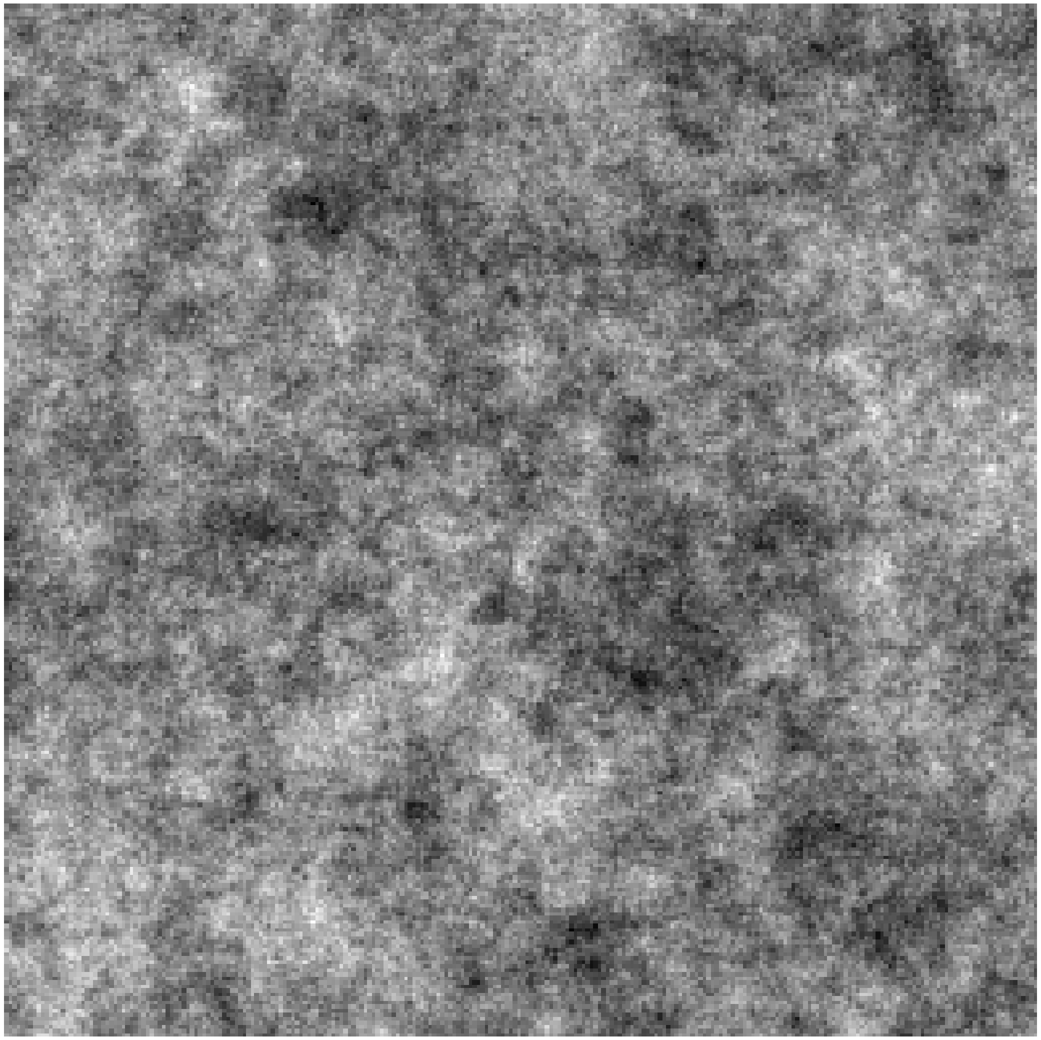
  where $\lambda = 1/(2\kappa)$. All $\{X_{u,v} - X_{u+s,v+t}\}$ have well–defined distributions.

- **Variogram** $\{\nu_{s,t} : s, t \in \mathcal{Z}\}$ is well defined and translation invariant :

$$\nu_{s,t} := \tfrac{1}{2}\,\mathrm{var}\,(X_{u,v} - X_{u+s,v+t}) = ???$$

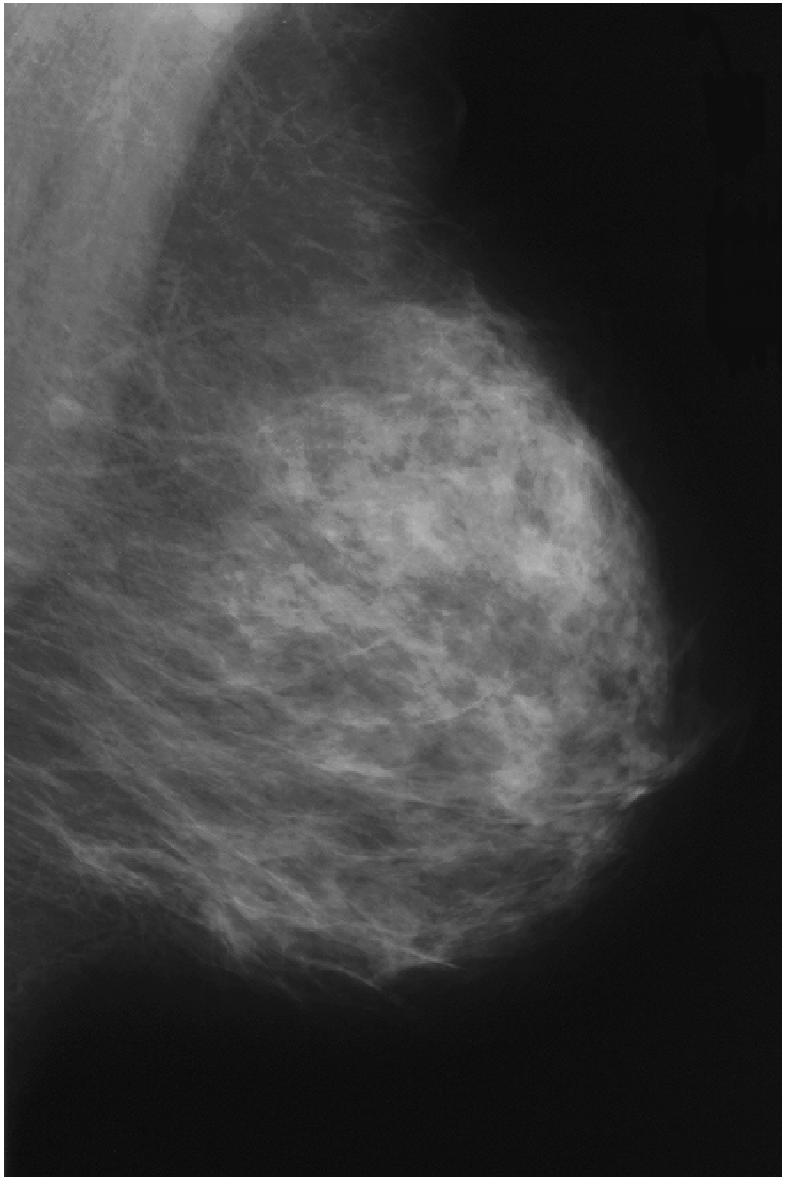
- **Computational advantage** : sparse precision matrix.
- **Disadvantage** : defined w.r.t. regular grid; what are effects of **rescaling**?

Symmetric first–order intrinsic autoregression



256×256 array

X–ray mammography (film)



Analysis: Larissa Stanberry

Data: Ruth Warren

Stephen Duffy

Spectral density diagram for simple Gaussian time series

Discrete time stationary AR(1) $(1 - \rho \cos \omega)^{-1}$	————————————> $\rho \to 1$	Discrete time random walk $(1 - \cos \omega)^{-1}$
| | | | | V		| | | | | V
Continuous time Ornstein–Uhlenbeck $(\alpha + \omega^2)^{-1}$	————————————> $\alpha \to 0$	Continuous time Brownian motion ω^{-2}

First–order Gaussian intrinsic autoregressions on $\mathcal{Z}^2$

- Let $\{X_{u,v} : (u,v) \in \mathcal{Z}^2\}$ be **Gaussian** with **conditional** means and variances

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  where $\beta,\, \gamma > 0$ and $\beta + \gamma = \frac{1}{2}$.

- $\{X_{u,v}\}$ has **generalized spectral density function**

$$f(\omega, \eta) = \kappa \,/\, (1 - 2\beta \cos\omega - 2\gamma \cos\eta)$$

  and finite **variogram** $\{\nu_{s,t} : s, t \in \mathcal{Z}\}$

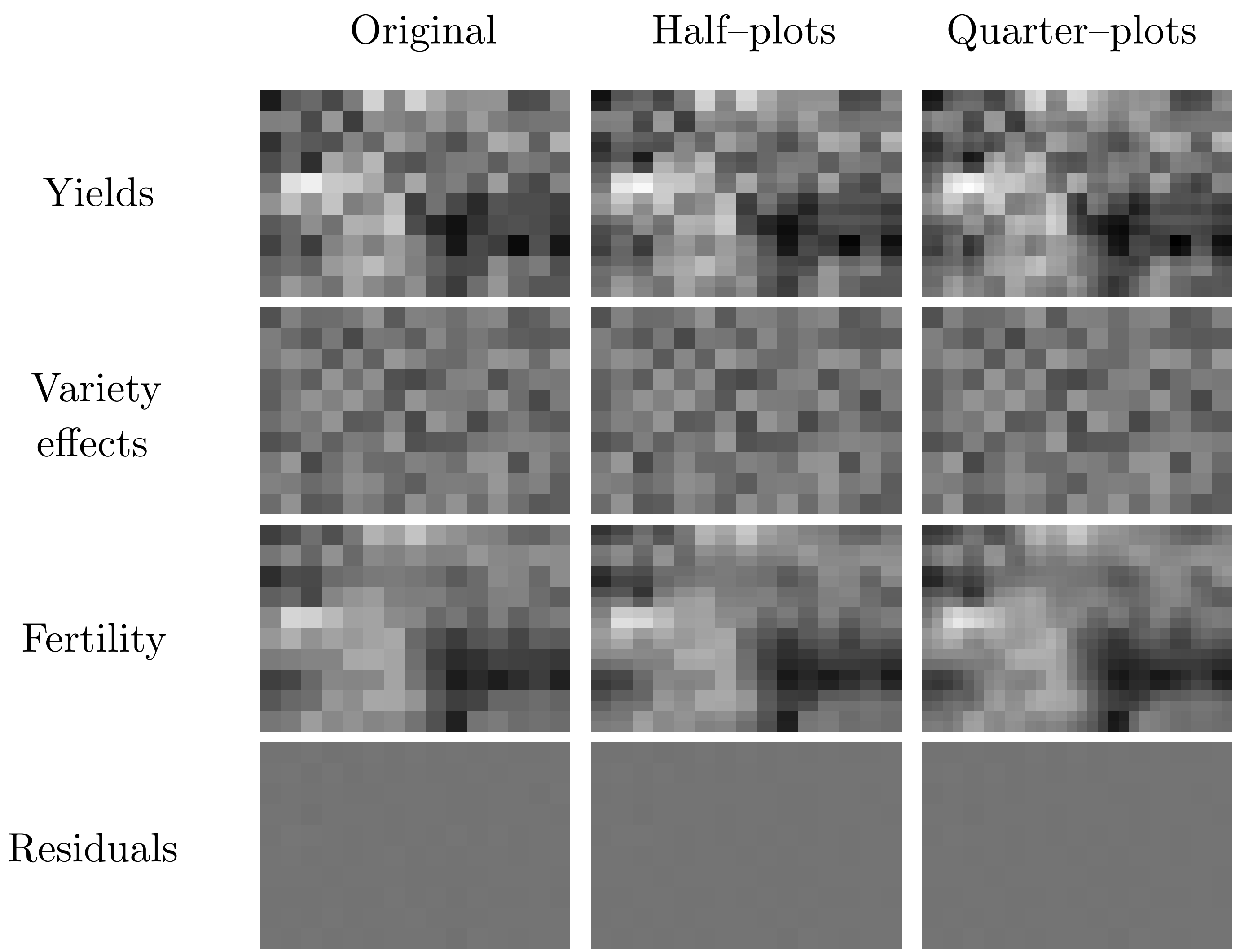
$$\nu_{s,t} := \tfrac{1}{2}\,\mathrm{var}\,(X_{u,v} - X_{u+s,v+t}) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{1 - \cos s\omega \, \cos t\eta}{1 - 2\beta\cos\omega - 2\gamma\cos\eta}\, d\omega\, d\eta.$$

- **Computational advantage** : sparse precision matrix.
- **Disadvantage** : defined w.r.t. regular grid; what are effects of **rescaling**?

Variety trial for wheat at Plant Breeding Institute, UK

Besag and Higdon (JRSS B, 1999)

Bayesian spatial analysis: effect of scale



Calculating the exact variogram $\{\nu_{s,t}\}$

$$\nu_{s,t} \;=\; \tfrac{1}{2}\,\mathrm{var}\,(X_{u,v} - X_{u+s,v+t}) \;=\; \frac{1}{\pi^2}\int_0^{\pi}\!\!\int_0^{\pi} \frac{1-\cos s\omega\,\cos t\eta}{1-2\beta\cos\omega-2\gamma\cos\eta}\,d\omega\,d\eta$$

... **but** extremely awkward in general, both analytically and numerically.

- **Symmetric case** $\beta = \gamma = \frac{1}{4}$ (McCrea & Whipple, 1940; Spitzer, 1964)

- **General case** $\beta \neq \gamma$ (Besag & Mondal, 2005)

  Obtain **delicate** finite summations for $\nu_{s,0}$ and $\nu_{0,t}$. Then

$$\left.\begin{aligned} \pi\,(\beta\gamma)^{\frac{1}{2}}\,\nu_{s,s} \;&=\; 1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots + \tfrac{1}{2s-1} \\ \nu_{s,t} \;&=\; -\delta_{s,t} + \beta\,(\nu_{s-1,t}+\nu_{s+1,t}) + \gamma\,(\nu_{s,t-1}+\nu_{s,t+1}) \end{aligned}\right\} \quad\Rightarrow\quad \nu_{s,t}$$

Asymptotic expansion of the variogram

- Exact results for $\nu_{s,0}$ and $\nu_{0,t}$ are numerically **unstable** for large s and t; but

$$\pi\,(\beta\gamma)^{\frac{1}{2}}\,\nu_{s,s} \;=\; 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2s-1}$$

$$\overset{?}{\Rightarrow} \qquad \nu_{s,t} \;\approx\; \textbf{logarithm} \;+\; \textbf{constant} \;+\; \ldots$$

cf. **de Wijs process** (logarithm) + **white noise** (constant).

- **Symmetric case** $\beta = \gamma = \frac{1}{4}$ (Duffin & Shaffer, 1960)

$$\pi\,\nu_{s,t} \;=\; 2\ln r \,+\, 3\ln 2 \,+\, 2\rho \,-\, \tfrac{1}{6}r^{-2}\cos 4\phi \,+\, O(r^{-4}),$$

where $r^2 = s^2 + t^2$, $\rho = 0.5772\ldots$ is Euler's constant and $\tan\phi = s/t$.

- **General case** $\beta + \gamma = \frac{1}{2}$, $\beta \neq \gamma$ (Besag & Mondal, 2005)

$$4\pi\,(\beta\gamma)^{\frac{1}{2}}\,\nu_{s,t} \;=\; 2\ln r \,+\, 3\ln 2 \,+\, 2\rho \,-\, \tfrac{1}{6}r^{-2}\{\cos 4\phi \,-\, 4(\beta-\gamma)\cos 2\phi\} + O(r^{-4}),$$

where $r^2 \;=\; 4\beta s^2 + 4\gamma t^2$ and $\tan\phi \;=\; \gamma^{\frac{1}{2}}s/(\beta^{\frac{1}{2}}t)$.

De Wijs process $\{Y(\mathbf{r})\}$ on $\mathcal{R}^2$

- $\{Y(\mathbf{r})\}$ is **Gaussian** and **Markov** with **spectral density function**

$$g(\omega,\eta) \;=\; \kappa \,/\, (\omega^2+\eta^2)$$

Realizations defined w.r.t. differences between **regional averages**.

Generalized functions : **Schwarz space**.

- **Integrated de Wijs process** $\{Y(A)\}$

$$Y(A) \;=\; \frac{1}{|A|}\int_A dY(\mathbf{x}), \qquad\qquad A \subset \mathcal{R}^2.$$

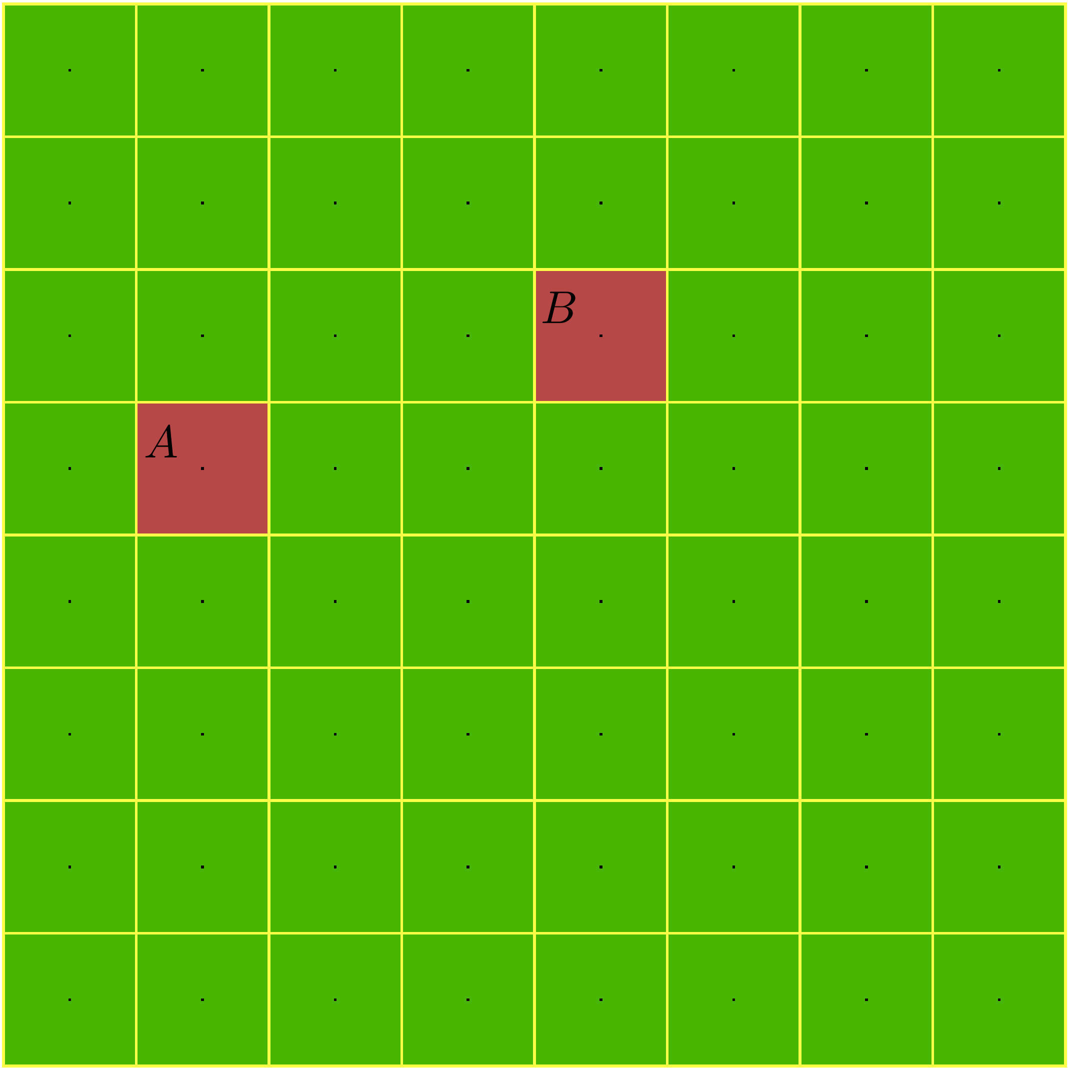
- **Variogram intensity** is **logarithmic**: process is **conformally invariant**.

Let A, $B \subset \mathcal{R}^2$ with $|A| = |B| = 1$ and $\phi(\mathbf{x}) := 1_A(\mathbf{x}) - 1_B(\mathbf{x})$, $\mathbf{x} \in \mathcal{R}^2$.

$$\Rightarrow \quad \nu(A,B) \;:=\; \mathrm{var}\,\{Y(A)-Y(B)\} \;=\; -\int_{\mathcal{R}^2}\int_{\mathcal{R}^2}\phi(\mathbf{x})\,\phi(\mathbf{y})\,\log\|\mathbf{x}-\mathbf{y}\|\,d\mathbf{x}\,d\mathbf{y}.$$

- Can incorporate **asymmetry** and more general **anisotropy**.

Original lattice $\mathcal{L}_1$ with array $\mathcal{D}_1$ and cells A and B



Integrated de Wijs process on $\mathcal{D}_1$

- Recall that De Wijs process on $\mathcal{R}^2$ has **spectral density function**

$$g(\omega,\eta) \;=\; \kappa \,/\, (\omega^2+\eta^2).$$

$$\Rightarrow \qquad \nu(A,B) \;=\; \frac{4\kappa}{\pi^2}\int_0^\infty\!\!\int_0^\infty \frac{\sin^2\!\omega\,\sin^2\!\eta\,\sin^2(s\omega+t\eta)}{\omega^2\eta^2\,(\omega^2+\eta^2)}\,d\omega\,d\eta,$$

where (s,t) denotes the $\mathcal{L}_1$–separation of A and B.

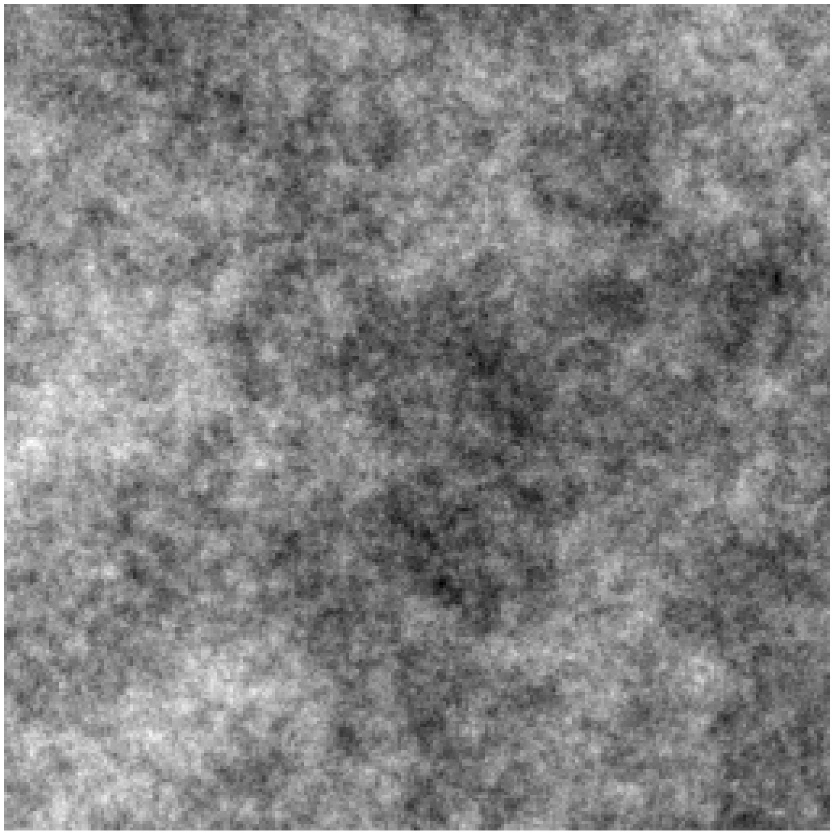
- NB. If $\phi(\mathbf{x})$ and $\varphi(\mathbf{x})$ are **test functions**, i.e. integrate to zero, then

$$-\int_{\mathcal{R}^2}\!\int_{\mathcal{R}^2}\phi(\mathbf{x})\,\varphi(\mathbf{y})\,\log\|\mathbf{x}-\mathbf{y}\|\,d\mathbf{x}\,d\mathbf{y} \;\equiv\; \int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty}\frac{\tilde\phi(\omega,\eta)\,\tilde\varphi(-\omega,-\eta)}{\omega^2+\eta^2}\,d\omega\,d\eta,$$

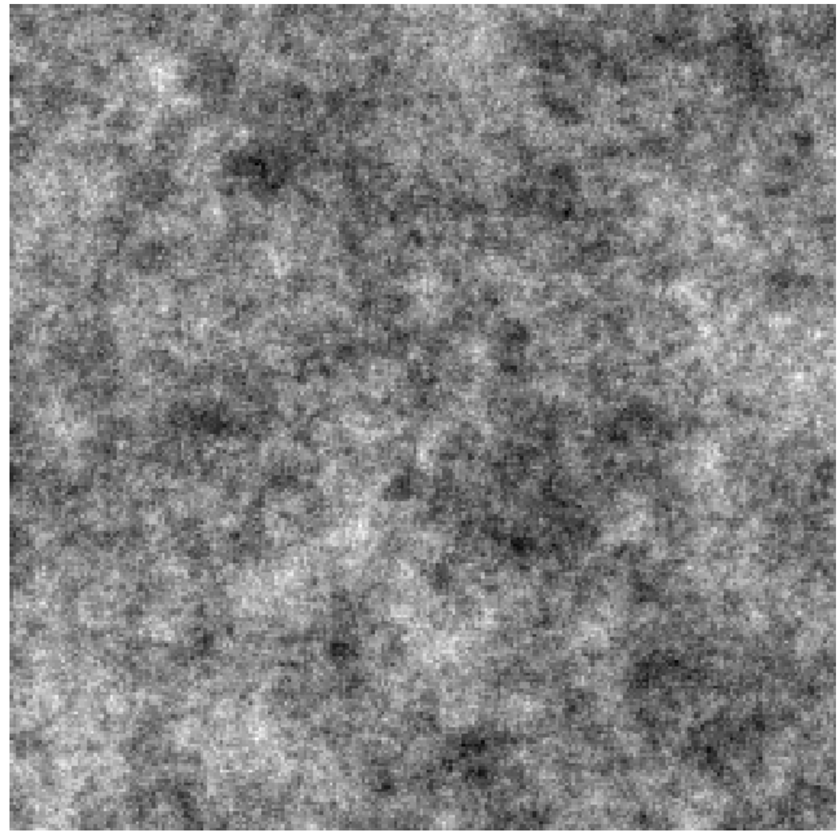
where $\tilde\phi$ and $\tilde\varphi$ are **Fourier transforms** of ϕ and φ. Here

$$\phi(\mathbf{x}) \;=\; \varphi(\mathbf{x}) \;=\; 1_A(\mathbf{x}) - 1_B(\mathbf{x}), \qquad \mathbf{x}\in\mathcal{R}^2.$$

Integrated de Wijs process

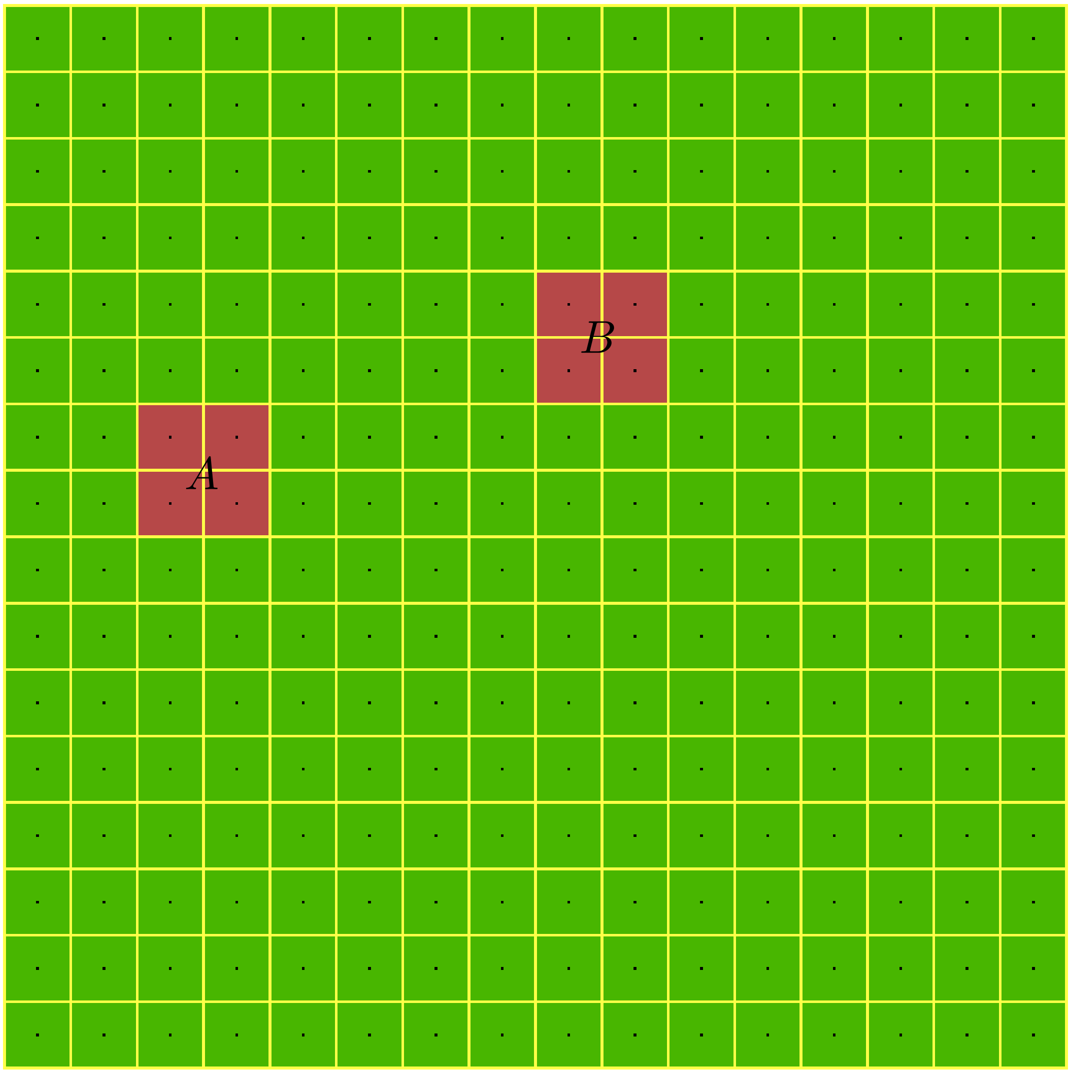


Intrinsic autoregression



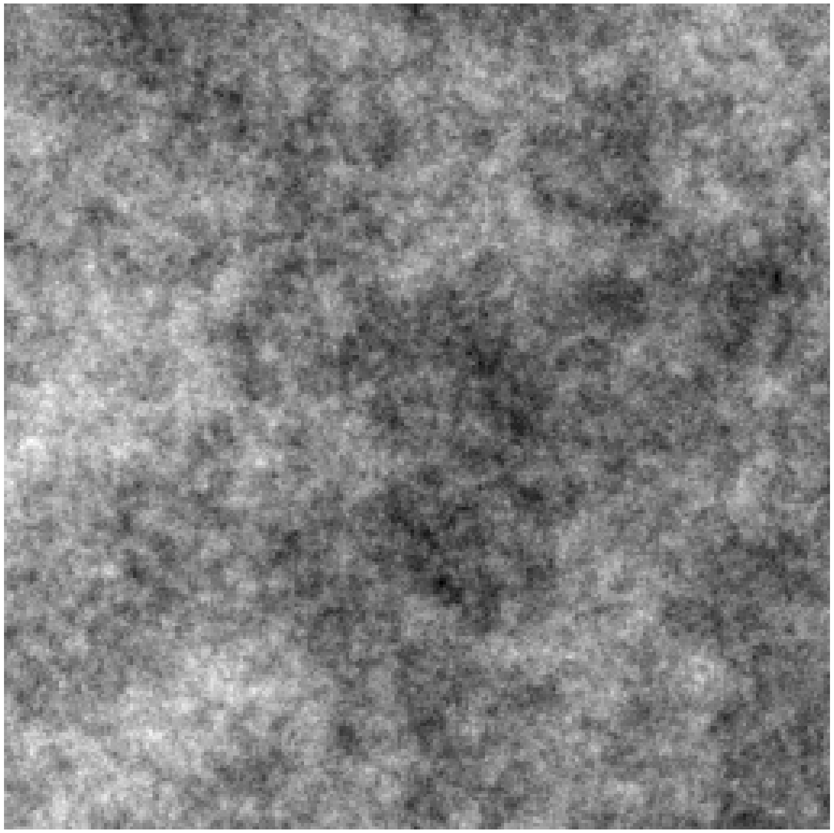
256×256 arrays

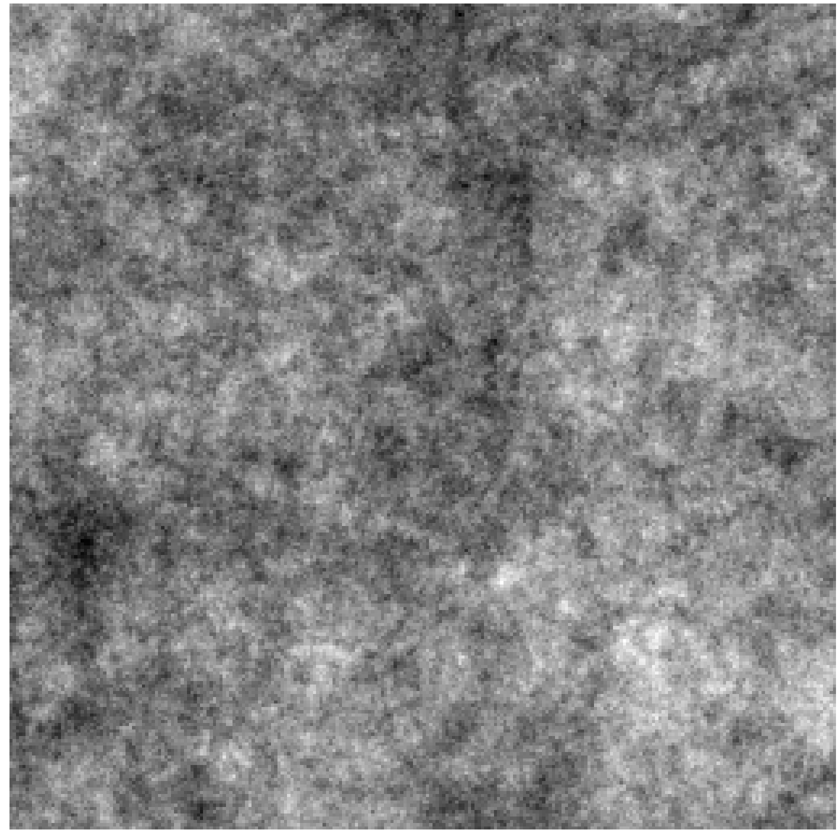
Sublattice $\mathcal{L}_2$ with subarray $\mathcal{D}_2$ and cells A and B



Consider first–order intrinsic autoregression on $\mathcal{L}_2$ averaged to $\mathcal{D}_1$

Integrated de Wijs process **Intrinsic autoregression**

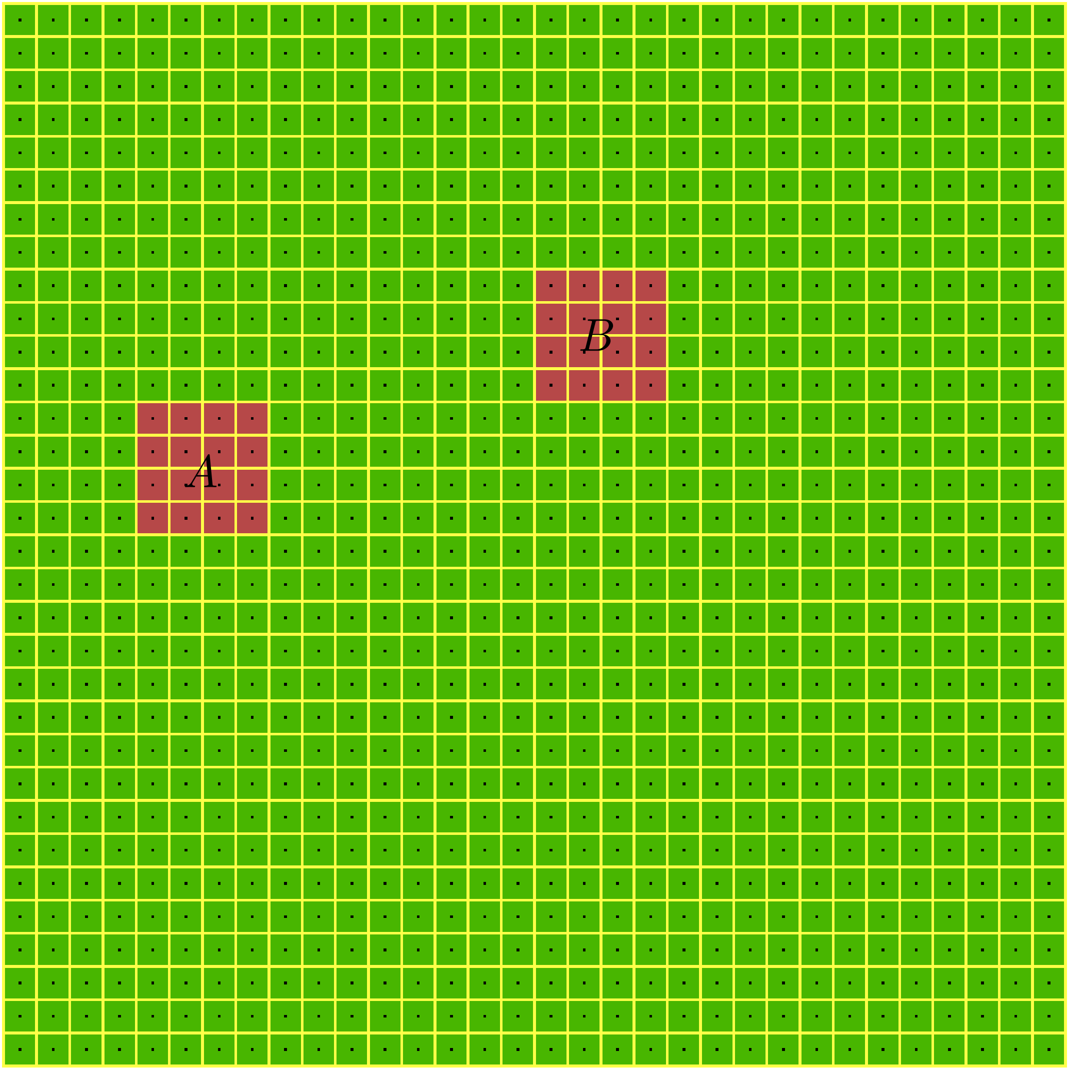




averaged over 2×2 blocks

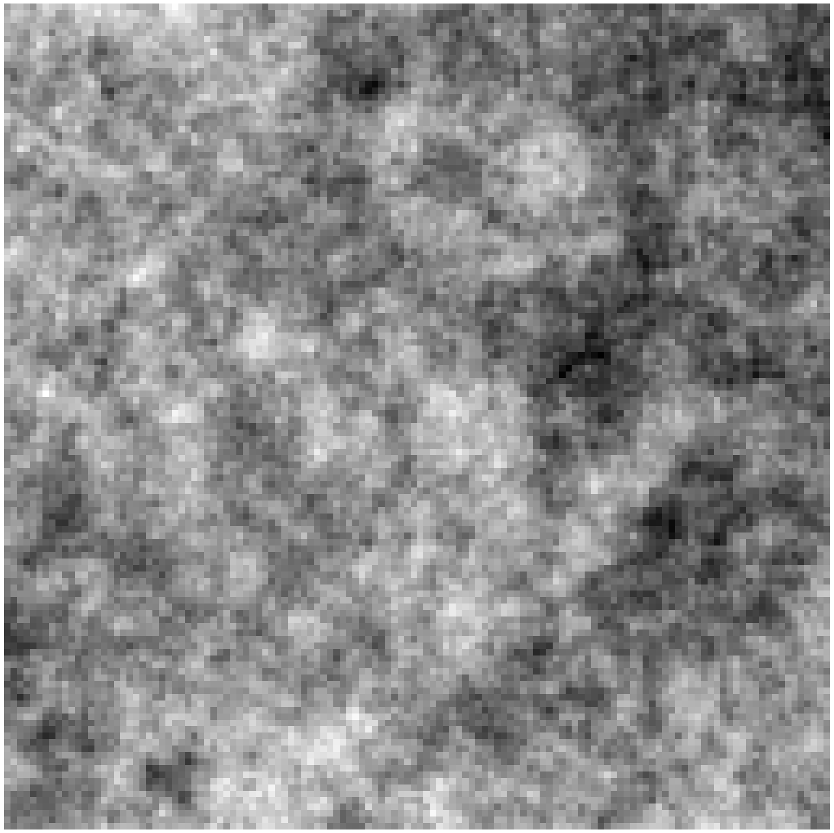
256×256 arrays

Sublattice $\mathcal{L}_4$ with subarray $\mathcal{D}_4$ and cells A and B

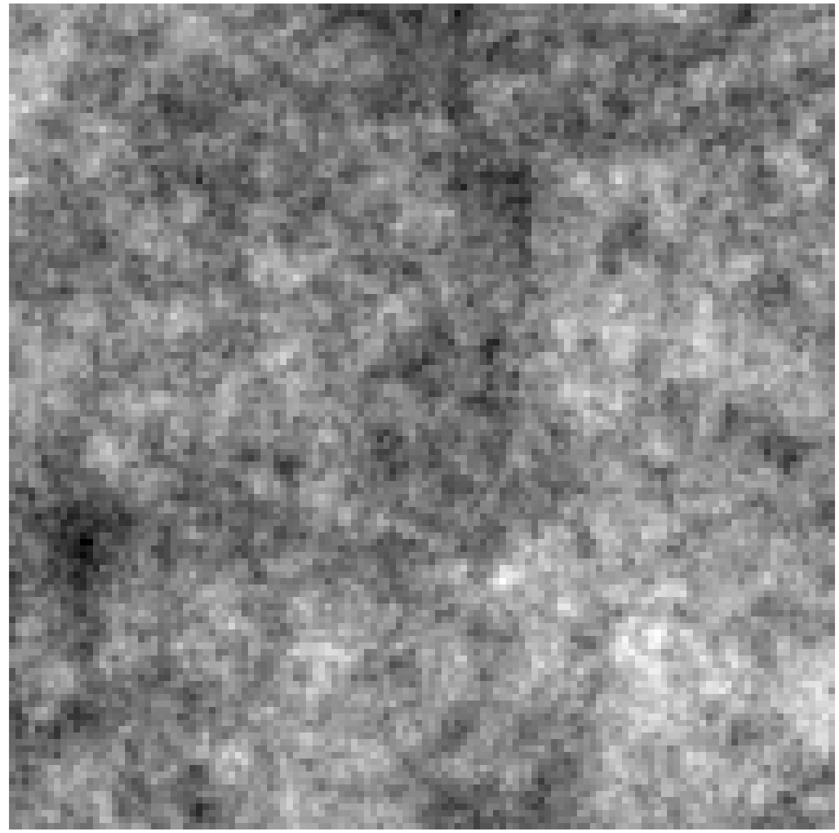


Consider first–order intrinsic autoregression on $\mathcal{L}_4$ averaged to $\mathcal{D}_1$

Integrated de Wijs process



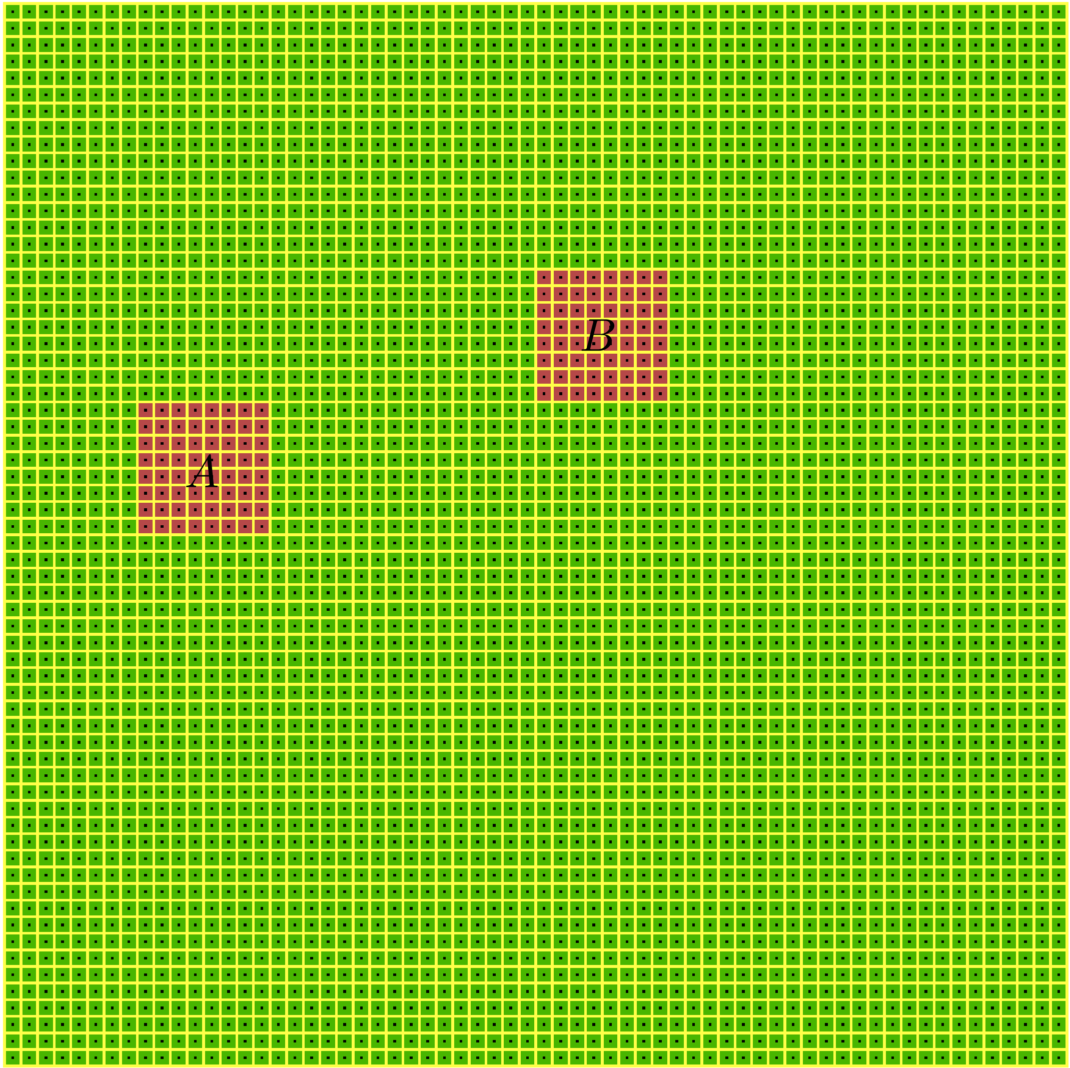
Intrinsic autoregression



averaged over 4×4 blocks

128×128 arrays

Sublattice $\mathcal{L}_8$ with subarray $\mathcal{D}_8$ and cells A and B



Consider first–order intrinsic autoregression on $\mathcal{L}_8$ averaged to $\mathcal{D}_1$

First–order intrinsic autoregressions on $\mathcal{L}_m$ averaged to $\mathcal{D}_1$

- $\mathcal{L}_1$ denotes original **lattice** at unit spacing.

  $\mathcal{L}_m$ denotes corresponding **sublattice** at spacing $1/m$: $m = 2, 3, \ldots$.

  $\mathcal{L}_m$ partitions $\mathcal{R}^2$ into **subarray** $\mathcal{D}_m$ of cells, each of area $1/m^2$.

- $\{X_{u,v}^{(m)}\}$ denotes symmetric first–order intrinsic autoregression on $\mathcal{L}_m$.

- Define sequence of **averaging processes** $\{Y_m(A)\}$ on cells $A \in \mathcal{D}_1$ by

$$Y_m(A) \;=\; \frac{1}{m^2} \sum_{(u,v)\in A} X_{u,v}^{(m)}.$$

  All contrasts have well–defined distributions with zero mean and finite variance.

- **What happens to** $\{Y_m(A)\}$ **as** $m \to \infty$ **?**

Limiting behaviour of $\{Y_m(A)\}$ as $m \to \infty$ (Besag & Mondal, 2005)

$$Y_m(A) \;=\; \frac{1}{m^2} \sum_{(u,v)\in A} X_{u,v}^{(m)}, \qquad\qquad A \in \mathcal{D}_1,$$

with **variogram** for $A, B \in \mathcal{D}_1$ separated by (s,t)

$$\begin{aligned}
\nu_m(A,B) &:= \tfrac{1}{2}\,\mathrm{var}\,\{Y_m(A) - Y_m(B)\} \\
&= \frac{8\kappa}{m^6} \int_0^{\frac{1}{2}m\pi}\!\!\int_0^{\frac{1}{2}m\pi} \frac{\sin^2\!\omega\,\sin^2\!\eta\,\sin^2(s\omega + t\eta)\,d\omega\,d\eta}{\sin^2(\omega/m)\,\sin^2(\eta/m)\,\{\sin^2(\omega/m) + \sin^2(\eta/m)\}} \\
&\to 8\kappa \int_0^{\infty}\!\!\int_0^{\infty} \frac{\sin^2\!\omega\,\sin^2\!\eta\,\sin^2(s\omega + t\eta)}{\omega^2\eta^2\,(\omega^2 + \eta^2)}\,d\omega\,d\eta, \qquad \text{as } m \to \infty,
\end{aligned}$$

the variogram of an **integrated de Wijs process** $\{Y(A) : A \in \mathcal{D}_1\}$.

Result generalizes rigorously to any non–empty $A, B \subset \mathcal{R}^2$.

In practice, $\mathbf{m = 2}$ or **4 adequate** because of **rapid convergence**.

Spectral density diagram for simple Gaussian time series

Discrete time stationary AR(1) $(1-\rho\cos\omega)^{-1}$	$\longrightarrow$ $\rho \to 1$	Discrete time random walk $(1-\cos\omega)^{-1}$
$\downarrow$		$\downarrow$
Continuous time Ornstein–Uhlenbeck $(\alpha+\omega^2)^{-1}$	$\longrightarrow$ $\alpha \to 0$	Continuous time Brownian motion ω^{-2}

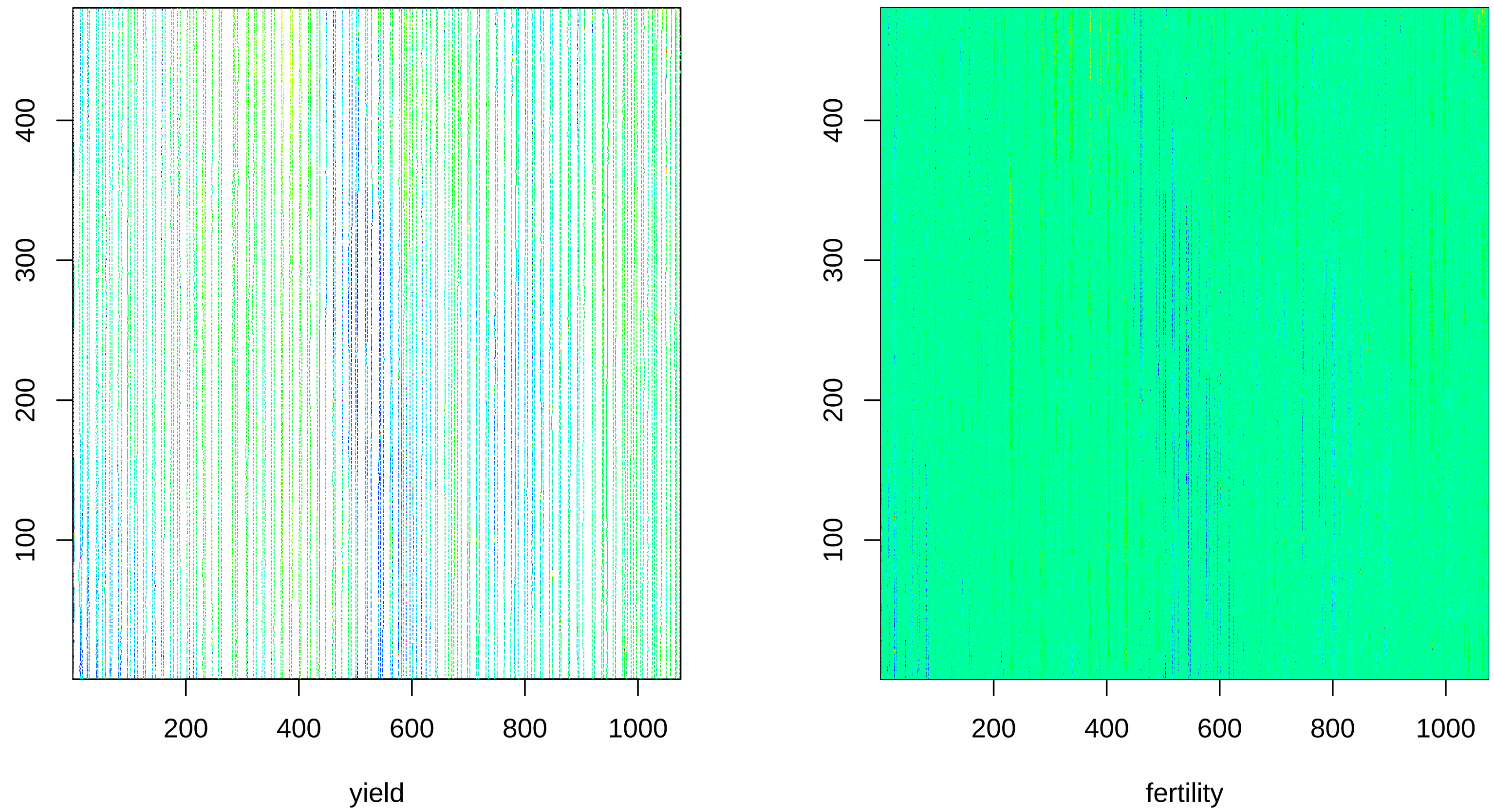
Spectral density diagram for 2-d Gaussian intrinsic processes

Discrete space stationary autoregression $(1 - 2\beta\cos\omega - 2\beta\cos\eta)^{-1}$	———————> $\beta \to \frac{1}{4}$	Discrete space intrinsic autoregression $(1 - \frac{1}{2}\cos\omega - \frac{1}{2}\cos\eta)^{-1}$
¦ ∨		¦ ∨
Continuous space K_0 Bessel process $(\alpha + \omega^2 + \eta^2)^{-1}$	———————> $\alpha \to 0$	Continuous space de Wijs process $(\omega^2 + \eta^2)^{-1}$

Extends to asymmetric case and some higher–order autoregressions.

Cotton picking time in NSW, Australia

(Virtually) de Wijs analysis of 500,000 cotton plots



Debashis Mondal, 2005

Higher–order intrinsic autoregressions

- Let $\{X_{u,v} : (u,v) \in \mathcal{Z}^2\}$ be **Gaussian** with **conditional** means and variances

$$\mathrm{E}\,(X_{u,v} \mid \ldots) \;=\; \sum_{k,l} \beta_{k,l}\, x_{u-k,v-l}, \qquad \mathrm{var}\,(X_{u,v} \mid \ldots) \;=\; \kappa \;>\; 0,$$

where (i) $\beta_{0,0} = 0$ (ii) $\beta_{k,l} \equiv \beta_{-k,-l}$ (iii) $\sum_{k,l} \beta_{k,l} = 1$ (iv) ...

- $\{X_{u,v}\}$ has **generalized spectral density function**

$$f(\omega,\eta) \;=\; \kappa \;/\; \{1 - \sum_{k,l} \beta_{k,l} \cos(\omega k + \eta l)\}.$$

- **Autoregression** is **simple** if variogram $\{\nu_{s,t} : s,t \in \mathcal{Z}\}$ exists $\Rightarrow$

$$\nu_{s,t} \;=\; \tfrac{1}{2}\,\mathrm{var}\,(X_{u,v} - X_{u+s,v+t}) \;=\; \frac{1}{\pi^2} \int_0^\pi \!\! \int_0^\pi \frac{1 - \cos s\omega\, \cos t\eta}{1 - \sum_{k,l} \beta_{k,l} \cos(\omega k + \eta l)}\, d\omega\, d\eta.$$

Generalizations of limiting behaviour (Besag & Mondal, 2005)

Second–order intrinsic autoregressions

$$\begin{aligned} \mathrm{E}\,(X_{u,v} \mid \ldots) &= \beta_{10}\,(x_{u-1,v} + x_{u+1,v}) + \beta_{01}\,(x_{u,v-1} + x_{u,v+1}) \\ &+ \beta_{11}\,(x_{u-1,v-1} + x_{u+1,v+1}) + \beta_{-11}\,(x_{u-1,v+1} + x_{u+1,v-1}) \end{aligned}$$

with $\beta_{10} + \beta_{01} + \beta_{11} + \beta_{-11} = \frac{1}{2}$ etc.

- **Diagonally symmetric** : $\beta_{10} = \beta,\ \beta_{01} = \gamma,\ \beta_{11} = \frac{1}{2}\delta = \beta_{-11}$

  $\nu_m(A, B) \rightarrow$ variogram of **asymmetric integrated de Wijs process**.

  i.e. limiting **spectral density** $\propto\ 1 / \{(\beta + \delta)\,\omega^2 + (\gamma + \delta)\,\eta^2)\}$.

  NB. includes **first–order** case with $\delta = 0$ but $\beta \neq \gamma$.

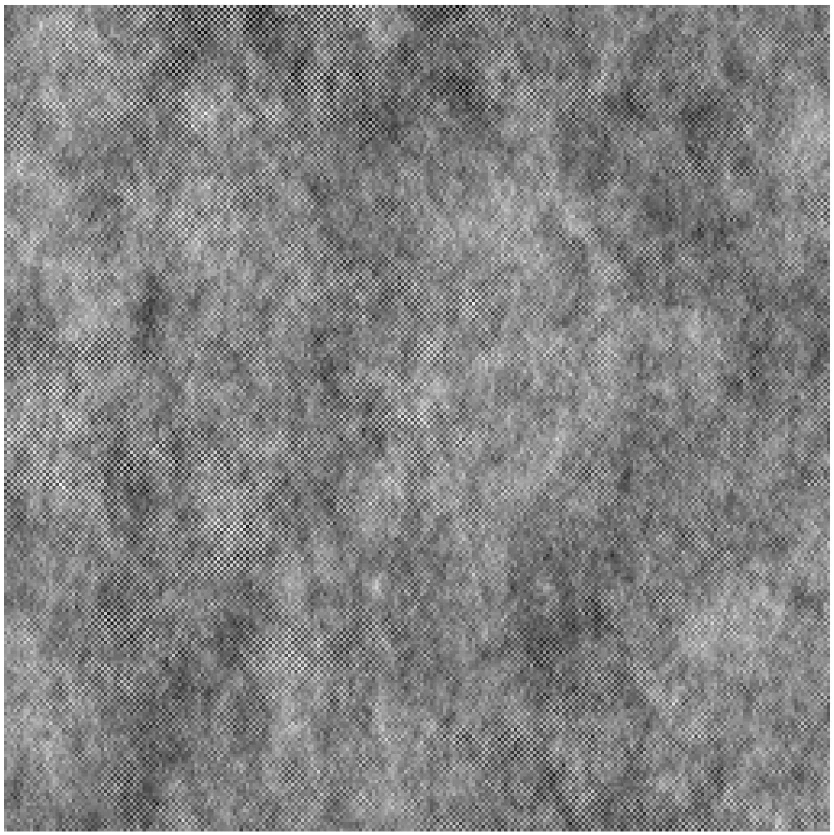
- **Diagonally antisymmetric** : $\beta_{10} = \beta,\ \beta_{01} = \gamma,\ \beta_{11} = \frac{1}{2}\delta = -\beta_{-11}$

  $\nu_m(A, B) \rightarrow$ variogram of **anisotropic integrated de Wijs process**.

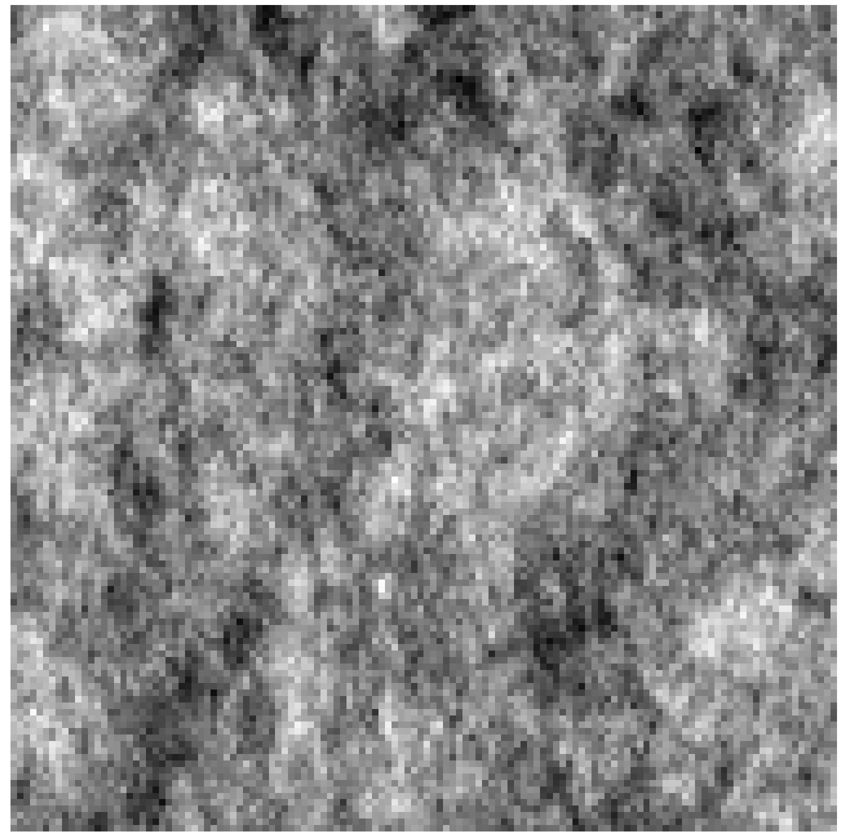
  i.e. limiting **spectral density** $\propto\ 1 / (\beta\omega^2 + 2\delta\omega\eta + \gamma\eta^2)$.

Extreme special case

$$\begin{aligned} \mathrm{E}\left(X_{u,v} \mid \ldots\right) &= \tfrac{1}{4}\left(x_{u-1,v} + x_{u+1,v}\right) - \tfrac{1}{4}\left(x_{u,v-1} + x_{u,v+1}\right) \\ &+ \tfrac{1}{4}\left(x_{u-1,v-1} + x_{u+1,v+1}\right) + \tfrac{1}{4}\left(x_{u-1,v+1} + x_{u+1,v-1}\right) \end{aligned}$$



256×256



128×128 averaged over 2×2 blocks

Generalizations of limiting behaviour

Third–order intrinsic autoregressions

- **Symmetric simultaneous intrinsic autoregression** (cf. Whittle, 1954)

$$X_{u,v} = \tfrac{1}{4}\left(X_{u-1,v} + X_{u+1,v} + X_{u,v-1} + X_{u,v+1}\right) + Z_{u,v}$$

where $\{Z_{u,v}\}$ is **Gaussian white noise** $\Rightarrow$

$$\begin{aligned}
\mathrm{E}\left(X_{u,v} \mid \ldots\right) &= \tfrac{2}{5}\left(x_{u-1,v} + x_{u+1,v} + x_{u,v-1} + x_{u,v+1}\right) \\
&\quad - \tfrac{1}{10}\left(x_{u-1,v-1} + x_{u+1,v+1} + x_{u-1,v+1} + x_{u+1,v-1}\right) \\
&\quad - \tfrac{1}{20}\left(x_{u-2,v} + x_{u+2,v} + x_{u,v-2} + x_{u,v+2}\right)
\end{aligned}$$

Requires **higher–order** differences or contrasts for well–defined distributions.

Limiting process corresponds to **thin–plate smoothing spline**

i.e. limiting **spectral density** $\propto 1 / (\omega^2 + \eta^2)^2$.

Generalizations of limiting behaviour

Third–order intrinsic autoregressions

- **Locally quadratic intrinsic autoregression** (Besag and Kooperberg, 1995)

$$\begin{aligned}
\mathrm{E}\,(X_{u,v} \mid \ldots) &= \tfrac{1}{4}\big(x_{u-1,v} + x_{u+1,v} + x_{u,v-1} + x_{u,v+1}\big) \\
&+ \tfrac{1}{8}\big(x_{u-1,v-1} + x_{u+1,v+1} + x_{u-1,v+1} + x_{u+1,v-1}\big) \\
&- \tfrac{1}{8}\big(x_{u-2,v} + x_{u+2,v} + x_{u,v-2} + x_{u,v+2}\big)
\end{aligned}$$

Requires genuine **two–dimensional differences** for well–defined distributions.

Limiting **spectral density** $\propto\; 1\,/\,(\omega^4 \,-\, \omega^2\eta^2 \,+\, \eta^4)$.

Wrap up

- **Gaussian Markov random fields** are alive and well!!
- **Precision matrix** of **Gaussian** MRF's **sparse** $\Rightarrow$ **efficient** computation.
- **Regional averages** of Gaussian MRF's $\xrightarrow{\text{rapid}}$ continuum **de Wijs** process.
- **Reconciliation** between Gaussian MRF and original geostatistical formulation.
- **Empirical evidence** for **de Wijs** process in **agriculture** :

  P. McCullagh & D. Clifford (2006), "Evidence of conformal invariance for crop yields", *Proc. R. Soc. A*, **462**, 2119–2143.

  Consistently selects **de Wijs** within **Matérn** class of **variograms** (25 crops!).
- **de Wijs** process also alive and well and can be fitted via **Gaussian MRF's**.