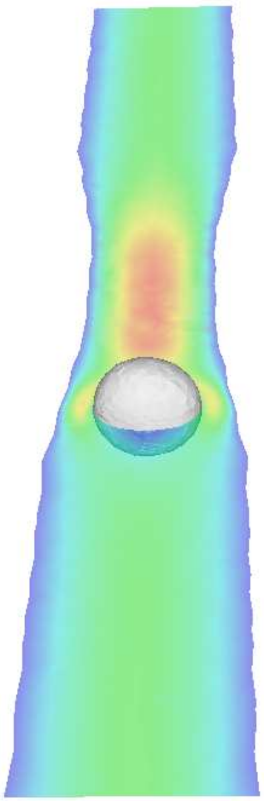
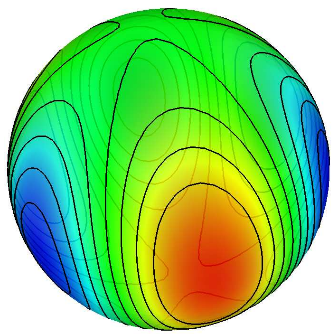
An Eulerian finite element method for elliptic equations on moving surfaces



levitated drop fluid/fluid



transport of surfactants

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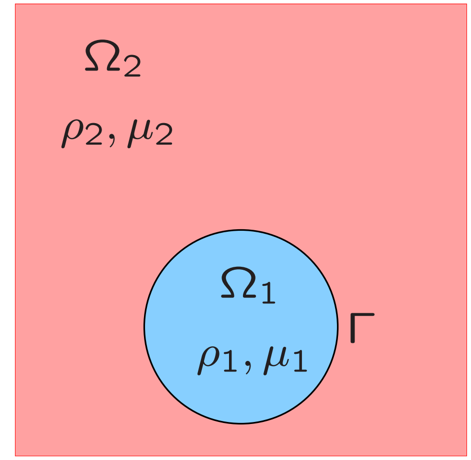
Motivation: Two-phase fluid dynamics + surfactants

- Standard model for two-phase flow.
- Level set for interface capturing.
- Treatment of surface tension.
- Special FE space for pressure.

An Eulerian FEM for elliptic equations on moving surfaces

- FEM for transport equation on the interface

Standard model for two-phase flows



Domains: $\Omega_1 = \Omega_1(t)$ and $\Omega_2 = \Omega_2(t)$

Interface: $\Gamma = \Gamma(t) = \partial\Omega_1 \cap \partial\Omega_2$

ρ_i: density in Ω_i
μ_i: viscosity in Ω_i

τ: surface tension coefficient

$$\begin{cases} \rho_i\big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) = \mathrm{div}\,(\sigma) + \rho_i\mathbf{g} \\ \qquad\qquad\qquad = -\nabla p + \mathrm{div}\,(\mu_i\mathbf{D}(\mathbf{u})) + \rho_i\mathbf{g} & \text{in } \Omega_i \quad \text{for } i = 1,2 \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega_i \end{cases}$$
$$[\sigma\mathbf{n}]_\Gamma = \tau\mathcal{K}\mathbf{n} - \nabla_\Gamma\,\tau, \qquad [\mathbf{u}]_\Gamma = 0\ .$$

$\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^T$: deformation tensor

$\sigma = -p\,\mathbf{I} + \mu\mathbf{D}(\mathbf{u})$: stress tensor

$\mathcal{K}$: curvature of Γ

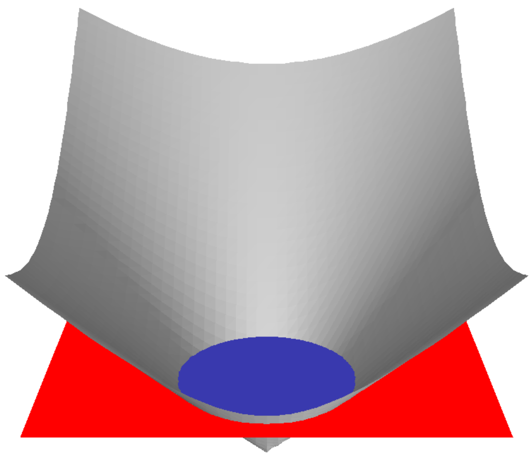
Assumption: τ constant

Appropriate model for a large class of two-phase flow problems.

Idea:(Sethian, Osher)

$$\Gamma(t) = \text{zero-level of a scalar function}$$

The level set function $\varphi(x,t)$



$$\varphi(x,t) = \begin{cases} < 0 & \text{for } x \text{ in phase } \Omega_1 \\ > 0 & \text{for } x \text{ in phase } \Omega_2 \\ = 0 & \text{at the interface} \end{cases}$$

should be an *"approximate signed distance function"*.

$$x(t) \in \Gamma(t) \quad \Rightarrow \quad \varphi(x(t),t) = 0.$$

Level set equation

$$\boxed{\varphi_t + \mathbf{u} \cdot \nabla\varphi \;=\; 0}$$

Navier-Stokes equations coupled with level set equation:

$$\begin{aligned} \rho(\varphi)\Big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) - \mathrm{div}\Big(\mu(\varphi)\,\mathbf{D}(\mathbf{u})\Big) + \nabla p &= \rho(\varphi)\,g - \tau\,\mathcal{K}(\varphi)\,\delta_\Gamma \mathbf{n}_\Gamma \\ \nabla\cdot\mathbf{u} &= 0 \\ \varphi_t + \mathbf{u}\cdot\nabla\varphi &= 0 \end{aligned}$$

where ρ, μ and $\mathcal{K}, \delta_\Gamma, \mathbf{n}_\Gamma$ depend on φ, e.g.:

$$\mathcal{K}(\varphi) = \nabla\cdot\left(\frac{\nabla\varphi}{||\nabla\varphi||}\right)$$ *second* derivatives.

Localized force term in weak formulation:

$$f_\Gamma(\mathbf{v}) = \tau \int_\Gamma \mathcal{K}\mathbf{n}_\Gamma\cdot\mathbf{v}\;ds$$

$$f_\Gamma \in H^{-1}(\Omega)$$

Discretization:

- Weak formulation + FE methods; velocity space: P_2.
- Discretization of localized force term f_Γ: Laplace-Beltrami.
- Finite element space for discontinuous pressure: XFEM $P_1 \rightsquigarrow Q_h^\Gamma$.
- Level set equation: P_2 FE + SDFEM stabilization.
- Time integration: θ-schema / fractional step.

Iterative solvers:

- Multigrid. Krylov subspace methods.
- Preconditioners: robustness w.r.t. μ, ρ, Δt, h.
- XFEM $\longrightarrow$ modified iterative solvers?

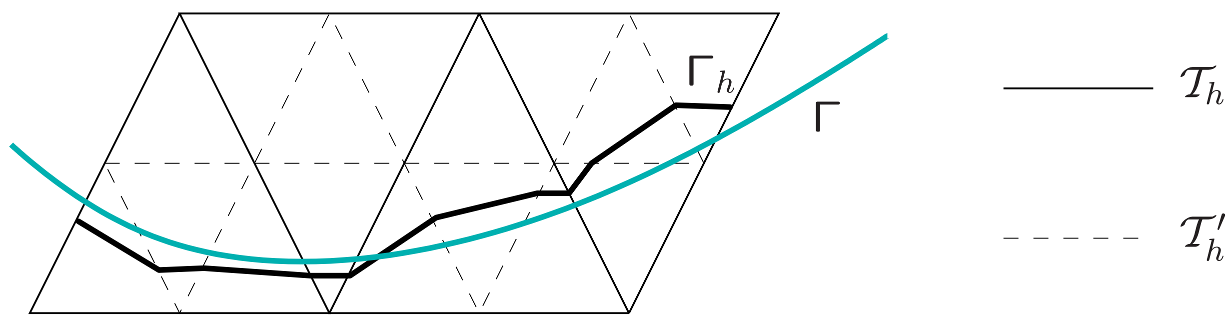
Γ = zero level of ϕ (= level set function = signed distance function)

ϕ_h = piecewise quadratic FE approximation of ϕ.

Our strategy:

$\phi \approx \phi_h$ (piecewise P_2) $\longrightarrow$ $I(\phi_h)$ (piecewise P_1 on refined mesh).

$\Gamma \approx \Gamma_h :=$ zero level of $I(\phi_h)$ (planar segments).



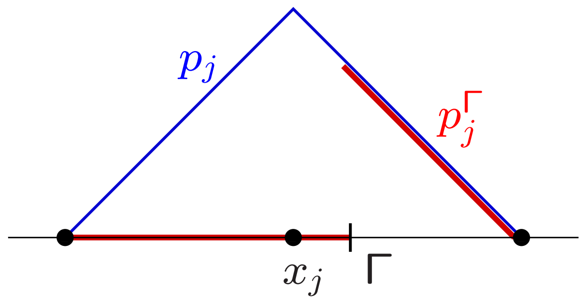
Under reasonable assumptions: dist$(\Gamma, \Gamma_h) \leq c\, h^2$.

Belytschko (1999)for elasticity problems.
Hansbo (2002) for interface problems.

Idea: Enrich FE space (e.g. P_1 FE) by additional discontinuous basis functions near Γ:

$$p_j^\Gamma(\mathbf{x}) := p_j(\mathbf{x}) H_\Gamma(\mathbf{x})$$

where $H_\Gamma(\mathbf{x}) = \begin{cases} 1 & x \in \Omega_2, \\ 0 & \text{else.} \end{cases}$



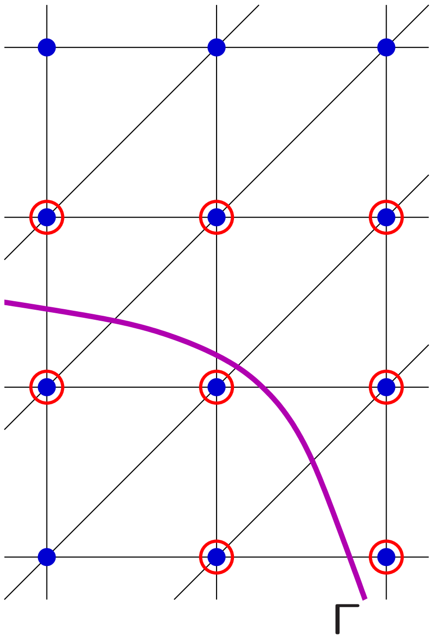
(Technical) difficulties:

- Integration over sub-elements $T \cap \Omega_2$:

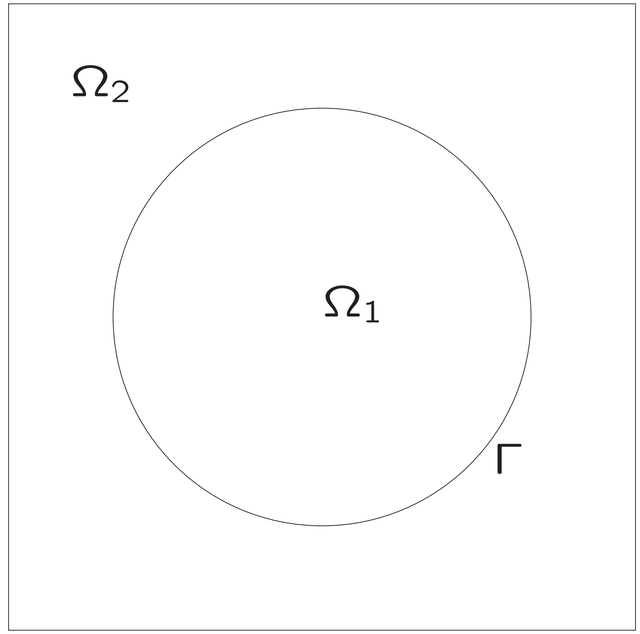
$$\int_T p_j^\Gamma(\mathbf{x}) \cdot f(\mathbf{x})\, d\mathbf{x} = \int_{T\cap\Omega_2} p_j(\mathbf{x}) \cdot f(\mathbf{x})\, d\mathbf{x}$$

- Q_h^Γ depends on Γ! (in practice: Γ_h)

- Reference: [Groß, R., JCP 07].

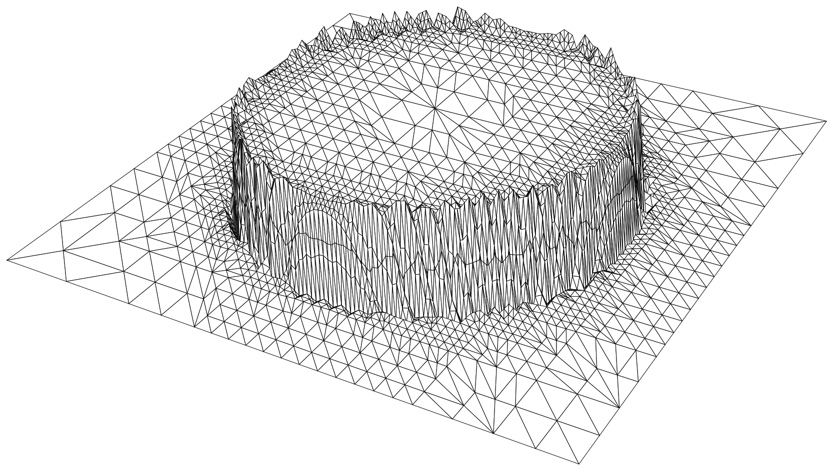


$\Omega_1 = \{\, x \in \mathbb{R}^3 \mid \|x\| \leq \frac{2}{3} \,\}$.

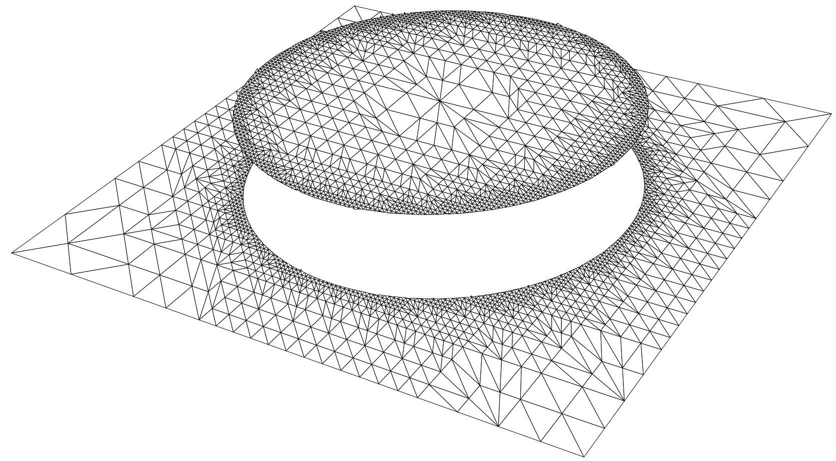


$f_\Gamma(\mathbf{v}) = \tau \int_\Gamma \mathcal{K} \mathbf{n}_\Gamma \cdot \mathbf{v} \, ds$ with $\tau = 1$. Note $\mathcal{K} = 2/r = 3$.

Solution: $\quad u^* = 0, \quad p^* = \begin{cases} C & \text{in } \Omega_2, \\ C + \tau\mathcal{K} & \text{in } \Omega_1. \end{cases}$



Q_h^1 FE (standard P_1)



Q_h^Γ FE (XFEM space)

ref.	$p_h \in Q_h^1$		$p_h \in Q_h^{\Gamma_h}$	
	$\|e_p\|_{L^2}$	order	$\|e_p\|_{L^2}$	order
0	1.60E+00	–	1.64E-01	–
1	1.07E+00	0.57	4.97E-02	**1.73**
2	8.23E-01	0.38	1.66E-02	**1.58**
3	5.80E-01	0.51	7.16E-03	**1.22**
4	4.13E-01	0.49	2.83E-03	**1.34**

Pressure errors for the $P_2 - Q_h^1$ and $P_2 - Q_h^{\Gamma}$ pair.

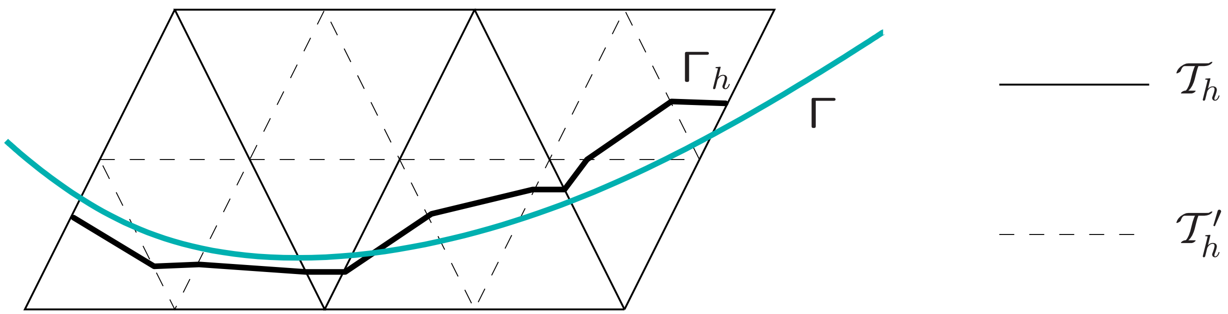
An Eulerian FEM for elliptic equations on moving surfaces

Convection-diffusion equation for $c(x,t)$, $x \in \Gamma(t)$:

$$\partial_{t,n} c - D_\Gamma \Delta_\Gamma c + \nabla_\Gamma \cdot (c\mathbf{u}_\Gamma) - \mathcal{K} u_\perp c = 0,$$

How can we discretize this equation on Γ ?.

Obvious idea: use a FE space induced by the "outer" triangulation $\mathcal{T}_h$.



Define

$$\omega_h := \cup_{T \in \mathcal{F}_h} S_T \ : \quad \text{tetrahedra in } \mathcal{T}_h' \text{ intersected by } \Gamma_h$$

$$V_h := \{\, v_h \in C(\omega_h) \mid v_{|S_T} \in P_1 \ \text{ for all } \ T \in \mathcal{F}_h \} : \quad \text{outer space}$$

$$V_h^\Gamma := \{\psi_h \in H^1(\Gamma_h) \,|\, \exists\ v_h \in V_h : \ \psi_h = v_h|_{\Gamma_h} \} : \quad \text{interface space}$$

Laplace-Beltrami equation

$$-\Delta_\Gamma u + u = f \quad \text{on } \Gamma,$$

with $\Gamma = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\|_2 = 1\}$ and $\Omega = (-2, 2)^3$.

Solution:

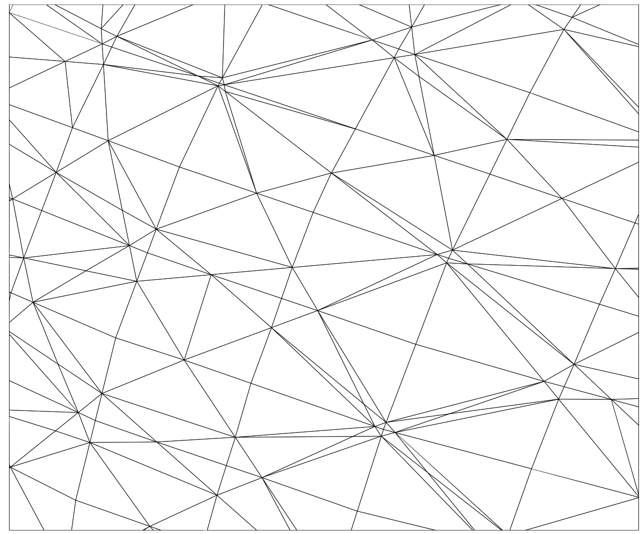
$$u(\mathbf{x}) = a\frac{\|\mathbf{x}\|^2}{12 + \|\mathbf{x}\|^2}\left(3x_1^2 x_2 - x_2^3\right).$$

Tetrahedral triangulations: $\{\mathcal{T}_l\}_{l\geq 0}$ constructed by local refinement close to Γ. Mesh size $h_l \sim \sqrt{3}\ 2^{-l}$.

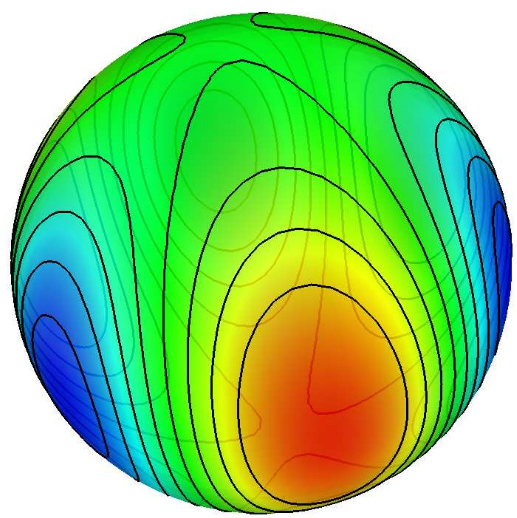
Level set function $\phi(\mathbf{x}) = \|\mathbf{x}\|^2 - 1$; $\phi_h := I(\phi)$ piecewise linear on $\mathcal{T}_h$.

$$\Gamma_h := \{\, \mathbf{x} \in \Omega \mid I(\phi_h)(\mathbf{x}) = 0 \,\}$$

Note: the interface triangulation Γ_h is not shape-regular:



Solution u:



Determine $u_h \in V_h^\Gamma$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \psi_h + u_h \psi_h \, \mathrm{ds}_h = \int_{\Gamma_h} f_h \psi_h \, \mathrm{ds}_h \qquad \text{for all} \quad \psi_h \in V_h^\Gamma,$$

with f_h an extension of f.

Results:

level l	$\|u - u_h\|_{L^2(\Gamma_h)}$	factor
1	0.1124	–
2	0.03244	3.47
3	0.008843	3.67
4	0.002186	4.05
5	0.0005483	3.99
6	0.0001365	4.02
7	0.0000341	4.00

Error analysis [Olshanskii, AR., submitted]:

Theorem. For each $u \in H^2(\Gamma)$ the following holds

$$\inf_{v_h \in V_h^\Gamma} \|u^e - v_h\|_{L^2(\Gamma_h)} \leq \|u^e - (I_h u^e)_{|\Gamma_h}\|_{L^2(\Gamma_h)} \leq C\, h^2 \|u\|_{H^2(\Gamma)},$$

$$\inf_{v_h \in V_h^\Gamma} \|u^e - v_h\|_{H^1(\Gamma_h)} \leq \|u^e - (I_h u^e)_{|\Gamma_h}\|_{H^1(\Gamma_h)} \leq C\, h \|u\|_{H^2(\Gamma)}.$$

Implementation very easy:

$$\int_{\Gamma_h} \nabla_{\Gamma_h}\phi_i \cdot \nabla_{\Gamma_h}\phi_j + \phi_i\phi_j \mathrm{ds}_h = \sum_{T \in \mathcal{F}_h} \int_T \ldots \mathrm{ds}_h$$

Furthermore:

- No data structure for triangulation of Γ_h needed.
- No shape regularity of triangulation of Γ_h required.

Let $(\phi_i)_{1\leq i\leq m}$ be all nodal basis functions in V_h with support intersected by Γ_h. Then

$$\text{span}\{\,(\phi_i)_{|\Gamma_h} \mid 1\leq i\leq m\,\} = V_h^\Gamma,$$

but $(\phi_i)_{|\Gamma_h}$ are not necessarily independent. Mass matrix:

$$M_{i,j} = \int_{\Gamma_h} \phi_i\phi_j \,\mathrm{d}s_h, \quad 1\leq i,j\leq m, \qquad \tilde{M} := D_M^{-\frac{1}{2}} M D_M^{-\frac{1}{2}}.$$

Spectrum of $\tilde{M}$:

level l	m	λ_1	λ_2	λ_m	λ_m/λ_2
1	112	3.8 e-17	0.0261	2.86	109
2	472	4.0 e-17	0.0058	2.83	488
3	1922	1.0 e-17	0.0012	2.83	2358
4	7646	3.6 e-17	0.00029	2.83	9660

Remarks:

- Scaling with D_M is essential.
- Analysis: in progress.
- Similar results for stiffness matrix.

More information:

www.igpm.rwth-aachen.de/DROPS/