Preconditioning saddle point problems arising from discretizations of partial differential equations

Part IV, Finite element exterior calculus

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based on joint work with: Douglas N. Arnold, Minnesota, Richard S. Falk, Rutgers

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Finite element exterior calculus (FEEC)

The development of FEEC leans heavily on earlier results taken from

- *Whitney, Bossavit, Raviart and Thomas, Nedelec, Hiptmair,...*

as well as on the theory of finite elements in general.
The presentation here is mostly based on

- D.N. Arnold, R.S. Falk, R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica 2006.

and later developments based on this paper.

The de Rham complex in three dimensions

We will utilize the de Rham complex in the form:

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0,$$

where $\Omega \subset \mathbb{R}^3$ and

$$\begin{aligned} H^1(\Omega) &= \{u \in L^2(\Omega) \mid \text{grad}\, u \in L^2(\Omega; \mathbb{R}^3)\}, \\ H(\text{curl}, \Omega) &= \{u \in L^2(\Omega; \mathbb{R}^3) \mid \text{curl}\, u \in L^2(\Omega; \mathbb{R}^3)\}, \\ H(\text{div}, \Omega) &= \{u \in L^2(\Omega; \mathbb{R}^3) \mid \text{div}\, u \in L^2(\Omega)\}. \end{aligned}$$

Discretizations and commuting diagrams

Stability of numerical methods utilizing the discrete spaces H_h^1, $H_h(\mathrm{curl})$, $H_h(\mathrm{div})$ and L_h^2 is frequently based on the existence of the following commuting diagram:

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\mathbb{R} \hookrightarrow H^1(\Omega) & \xrightarrow{\text{grad}} & H(\mathrm{curl},\Omega) & \xrightarrow{\text{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \longrightarrow & 0 \\
\downarrow \mathcal{I}_h^1 & & \downarrow \mathcal{I}_h^c & & \downarrow \mathcal{I}_h^d & & \downarrow \mathcal{I}_h^0 & & \\
\mathbb{R} \hookrightarrow \quad H_h^1 & \xrightarrow{\text{grad}} & H_h(\mathrm{curl}) & \xrightarrow{\text{curl}} & H_h(\mathrm{div}) & \xrightarrow{\text{div}} & L_h^2 & \longrightarrow & 0.
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A technical problem in most of the finite element literature: The canonical projections $\mathcal{I}_h$ are not defined on the entire space, but this problem can be fixed by using modified interpolation operators.

The de Rham complex and differential forms

By introducing differential forms the de Rham complex can be written as

$$\mathbb{R} \hookrightarrow \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0.$$

Here $\Lambda^k(\Omega) = C^\infty(\Omega; \mathrm{Alt}^k)$, where Alt^k is the vector space of alternating k–linear maps on $\mathbb{R}^n$.

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Here $\Lambda^k(\Omega) = C^\infty(\Omega; \mathrm{Alt}^k)$, where Alt^k is the vector space of alternating k–linear maps on $\mathbb{R}^n$.
The exterior derivative $d : \Lambda^k(\Omega) \rightarrow \Lambda^{k+1}(\Omega)$ is defined by

$$d\omega_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{v_j} \omega_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}),$$

for $\omega \in \Lambda^k(\Omega)$ and $v_1, \ldots, v_{k+1} \in \mathbb{R}^n$.

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for $\omega \in \Lambda^k(\Omega)$ and $v_1, \ldots, v_{k+1} \in \mathbb{R}^n$.
One easily checks that $d^2 = 0$.

Proxy fields

In the case of $n = 3$ the identification of $C^\infty(\Omega; \mathrm{Alt}^k)$ with the corresponding spaces of scalar/vector fields is based on

- $\mathrm{Alt}^0 \equiv \mathbb{R} \cong \mathbb{R}$
- $\mathrm{Alt}^1 \equiv (\mathbb{R}^3)^* \cong \mathbb{R}^3$ by $\mu \leftrightarrow u$ where $\mu(v) = u \cdot v$
- $\mathrm{Alt}^2 \cong \mathbb{R}^3$ by $\mu \leftrightarrow u$ where $\mu(v, w) = (u \times v) \cdot w$
- $\mathrm{Alt}^3 \cong \mathbb{R}$ by $\mu \leftrightarrow c$ where $\mu(u, v, w) = c\det(u, v, w)$

Exterior product and pull backs

The wedge product maps $\mathrm{Alt}^j \times \mathrm{Alt}^k$ into Alt^{j+k}, and is defined by

$$\begin{aligned}\omega\wedge\mu(v_1, \dots v_{j+k}) \\ &= \sum_{\sigma}(\mathrm{sign}\sigma)\omega(v_{\sigma(1)}, \dots v_{\sigma(j)})\mu(v_{\sigma(j+1)}, \dots v_{\sigma(j+k)}).\end{aligned}$$

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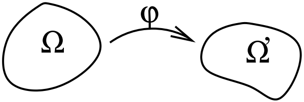

If $\phi : \Omega \to \Omega'$ then the pull back $\phi^* : \Lambda^k(\Omega') \to \Lambda^k(\Omega)$ is given by

$$(\phi^*\omega)_x(v_1, v_2, \ldots, v_k) = \omega_{\phi(x)}(D\phi_x(v_1), D\phi_x(v_2), \ldots, D\phi_x(v_k)),$$

where $D\phi_x$ is the derivative of ϕ at x mapping $T_x\Omega$ into $T_{\phi(x)}\Omega'$.

The pullback commutes with the exterior derivative, i.e.,

$$\phi^*(d\omega) = d(\phi^*\omega), \quad \omega \in \Lambda^k(\Omega'),$$

and distributes with respect to the wedge product:

$$\phi^*(\omega \wedge \eta) = \phi^*\omega \wedge \phi^*\eta.$$

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Stokes theorem:

$$\int_\Omega d\omega = \int_{\partial\Omega} \operatorname{Tr}\Omega, \quad \omega \in \Lambda^{n-1}$$

Variants of the de Rham complex

L^2 de Rham complex:

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

where $H\Lambda^k(\Omega) = \{\, \omega \in L^2\Lambda^k(\Omega) \,|\, d\omega \in L^2\Lambda^{k+1}(\Omega) \,\}$ and where the Hodge $\star$ operator is used to define the inner product in $L^2\Lambda^k(\Omega)$.

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The polynomial de Rham complex:

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0$$

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Here

$$\mathcal{P}_r\Lambda^k = \{\omega \in \Lambda^k \,|\, \omega(v_1, \dots v_k) \in \mathcal{P}_r,\ \forall v_1, \dots v_k \,\}$$

such that $\mathcal{P}_r\Lambda^k \cong \mathcal{P}_r \otimes \mathrm{Alt}^k$.

The Koszul complex

The Koszul differential κ of a k-form ω is the $(k-1)$-form given by

$$(\kappa\omega)_x(v_1, \ldots, v_{k-1}) = \omega_x\big(X(x), v_1, \ldots, v_{k-1}\big),$$

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For each r, κ maps $\mathcal{P}_{r-1}\Lambda^k$ to $\mathcal{P}_r\Lambda^{k-1}$, and the Koszul complex

$$0 \to \mathcal{P}_{r-n}\Lambda^n \xrightarrow{\kappa} \mathcal{P}_{r-n+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \xrightarrow{\kappa} \mathcal{P}_r\Lambda^0 \longrightarrow \mathbb{R} \to 0,$$

is exact.

The spaces $\mathcal{P}_r^- \Lambda^k$

The operators d and κ are related by the *homotopy relation*

$$(d\kappa + \kappa d)\omega = (r + k)\omega, \quad \omega \in \mathcal{H}_r \Lambda^k,$$

where $\mathcal{H}_r$ denotes the homogeneous polynomials of degree r.
As a consequence we obtain the identity

$$\mathcal{P}_r \Lambda^k = \mathcal{P}_{r-1} \Lambda^k + \kappa \mathcal{H}_{r-1} \Lambda^{k+1} + d\mathcal{H}_{r+1} \Lambda^{k-1}$$

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We note that $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$ and $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$. Furthermore,

$$0 \to \mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n \to 0$$

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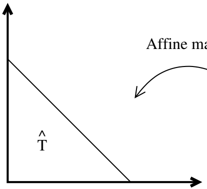
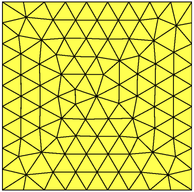
is an exact complex, and the space $\mathcal{P}_r^-\Lambda^k$ is *affine invariant.*

Significance of affine invariant spaces





In fact, $\mathcal{P}_r^-\Lambda^k$ is nearly the only affine invariant polynomial space X satisfying

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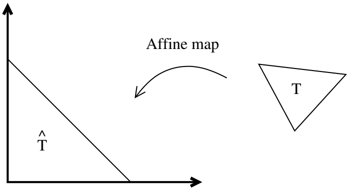

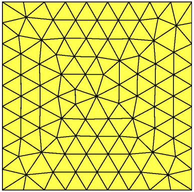

In fact, $\mathcal{P}_r^-\Lambda^k$ is nearly the only affine invariant polynomial space X satisfying

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More precisely, either $X = \mathcal{P}_{r-1}^-\Lambda^k$, or

$$X = \{\omega \in \mathcal{P}_r\Lambda^k \mid d\omega \in \mathcal{P}_{r-2}\Lambda^{k+1}\}.$$

The four exact sequences ending with $\mathcal{P}_r\Lambda^3(\mathcal{T})$ in 3D

$$0 \to \mathcal{P}_{r+1}\Lambda^0 \xrightarrow{d} \mathcal{P}^-_{r+1}\Lambda^1 \xrightarrow{d} \mathcal{P}^-_{r+1}\Lambda^2 \xrightarrow{d} \mathcal{P}_r\Lambda^3 \to 0$$

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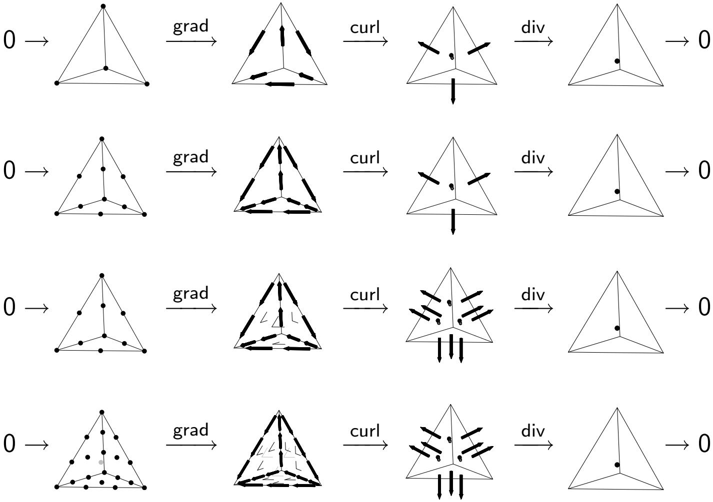
$$0 \to \mathcal{P}_{r+3}\Lambda^0 \xrightarrow{d} \mathcal{P}_{r+2}\Lambda^1 \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^2 \xrightarrow{d} \mathcal{P}_r\Lambda^3 \to 0$$

The four sequences ending with $\mathcal{P}_0\Lambda^3(\mathcal{T})$ in 3D



Piecewise smooth differential forms

It is a consequence of Stokes theorem that a piecewise smooth k–form ω, with respect to a simplicial mesh $\mathcal{T}_h$ of Ω, is in $H\Lambda^k(\Omega)$ if and only if the trace of ω, $\operatorname{Tr}\omega$, is continuous on the interfaces.

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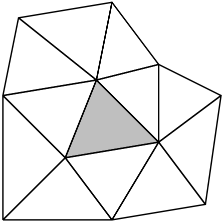

Here $\operatorname{Tr}\omega$ is defined by restricting the spatial variable x to the interface, and by applying ω only to tangent vectors of the interface.

Degrees of freedom

To obtain *finite element* differential forms—not just pw polynomials—we need *degrees of freedom*, i.e., a decomposition of the dual spaces $(\mathcal{P}_r\Lambda^k(T))^*$ and $(\mathcal{P}_r^-\Lambda^k(T))^*$ (with T a simplex), into subspaces associated to subsimplices f of T.
DOF for $\mathcal{P}_r\Lambda^k(T)$: to a subsimplex f of dimension d we associate

$$\omega \mapsto \int_f \operatorname{Tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d}^-\Lambda^{d-k}(f)$$

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Given a triangulation $\mathcal{T}$, we can then define $\mathcal{P}_r\Lambda^k(\mathcal{T})$, $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$. They are subspaces of $H\Lambda^k(\Omega)$.

Construction of bounded cochain projections

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If we apply the three operations:

- extend (E)
- regularize (R)
- canonical projection ($\mathcal{I}_h$)

we get a map $Q_h^k : H\Lambda^k(\Omega) \to \Lambda_h^k$ which is bounded and commutes with d. But it is not a projection.

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However the composition

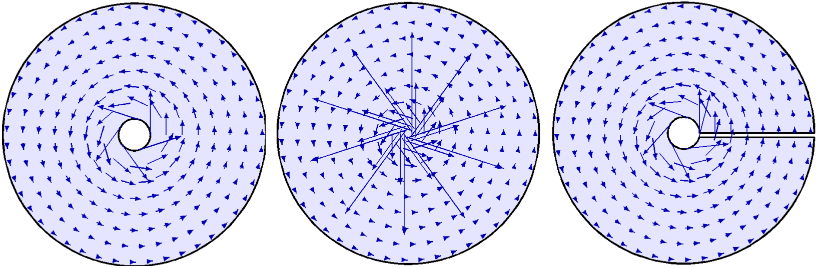
$$\pi_h^k = (Q_h^k|_{\Lambda_h^k})^{-1} \circ Q_h^k$$

can be shown to be a *bounded cochain projection*. Its operator norm depends on the shape regularity of the mesh.

De Rham cohomology

$$0 \longrightarrow \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \Lambda^2(\Omega) \longrightarrow 0$$

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^2) \xrightarrow{\text{curl}} C^\infty(\Omega) \longrightarrow 0$$



Cohomology

The de Rham complex

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$$\mathfrak{Z}^k := \ker d^k = \operatorname{range} d^{k-1} =: \mathfrak{B}^k.$$

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In general, $\mathfrak{B}^k \subset \mathfrak{Z}^k$ and we assume throughout that the kth cohomology group $\mathfrak{Z}^k/\mathfrak{B}^k$ is finite dimensional.
The space of harmonic k-forms, $\mathfrak{H}^k$, consists of all $q \in \mathfrak{Z}^k$ such that

$$\langle q, \mu \rangle = 0 \quad \mu \in \mathfrak{B}^k.$$

This leads to the Hodge decomposition
$H\Lambda^k(\Omega) = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k\perp} = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}$. Note that $\mathfrak{H}^k \cong \mathfrak{Z}^k/\mathfrak{B}^k$.

Hodge Laplace problem

$$H\Lambda^{k-1}(\Omega) \xrightarrow{d^{k-1}} H\Lambda^k(\Omega) \xrightarrow{d^k} H\Lambda^{k+1}(\Omega)$$

Formally: Given $f \in \Lambda^k$, find $u \in \Lambda^k$ such that

$$(d^{k-1}\delta^{k-1} + \delta^k d^k)u = f.$$

Here δ^k is a formal adjoint of d^k.

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The following mixed formulation is always well-posed: Given $f \in L^2\Lambda^k(\Omega)$, find $\sigma \in H\Lambda^{k-1}$, $u \in H\Lambda^k$ and $p \in \mathfrak{H}^k$ such that

$$\begin{aligned}
\langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0 && \forall \tau \in H\Lambda^{k-1} \\
\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle && \forall v \in H\Lambda^{k} \\
\langle u, q \rangle &= 0 && \forall q \in \mathfrak{H}^{k}
\end{aligned}$$

Hodge Laplacian

Well-posedness of the Hodge Laplace problem follows from the Hodge decomposition and Poincaré's inequality:

$$\|\omega\|_{L^2} \le c\,\|d\omega\|_{L^2}, \quad \omega \in (\mathfrak{Z}^k)^\perp.$$

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Special cases ($\dim \mathfrak{H}^k = 0$):

- $k = 0$: ordinary Laplacian
- $k = n$: mixed Laplacian
- $k = 1, n = 3$: $\sigma = -\operatorname{div} u, \quad \operatorname{grad}\sigma + \operatorname{curl}\operatorname{curl} u = f$
- $k = 2, n = 3$: $\sigma = \operatorname{curl} u, \quad \operatorname{curl}\sigma - \operatorname{grad}\operatorname{div} u = f,$

Abstract framework, Hilbert complex

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- $\ker d^k /$ range $d^{k-1} = \mathfrak{Z}^k/\mathfrak{B}^k$ has finite dimension

Discretization, Abstract setting

$$\cdots \to H\Lambda^{k-1} \xrightarrow{d^{k-1}} H\Lambda^k \to \cdots$$

Complex of Hilbert spaces with d^k bounded and closed range.

Discretization, Abstract setting

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When is it stable?

Bounded cochain projections

Key property: Suppose that there exists a *bounded cochain projection*.

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- π_h^k uniformly bounded
- $\|\pi_h^k\omega - \omega\| \to 0$.
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- *If* $\|v - \pi_h^k v\| < \|v\| \ \forall v \in \mathfrak{H}^k$*, then the induced map on cohomology is an isomorphism.*
- $\operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \leq \sup_{v\in\mathfrak{H}^k, \|v\|=1} \|v - \pi_h^k v\|$
- *The discrete Poincaré inequality holds uniformly in h.*
- *Galerkin's method is stable and convergent.*

Proof of discrete Poincaré inequality

Theorem: Assume that $\|\pi_h\|_{\mathcal{L}(\Lambda^k,\Lambda^k)} \leq c_\pi$. Then

$$\|\omega\| \leq c_p c_\pi \|d\omega\|, \quad \omega \in \mathfrak{Z}_h^{k\perp}.$$

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Proof: Given $\omega \in \mathfrak{Z}_h^{k\perp}$, define $\eta \in \mathfrak{Z}^{k\perp} \subset H\Lambda^k(\Omega)$ by $d\eta = d\omega$. By the *Poincaré inequality*, $\|\eta\| \le c_p\|d\omega\|$, so it is enough to show that $\|\omega\| \le c_\pi\|\eta\|$. Now, $\omega - \pi_h\eta \in \Lambda_h^k$ and $d(\omega - \pi_h\eta) = 0$, so $\omega - \pi_h\eta \in \mathfrak{Z}_h^k$. Therefore

$$\|\omega\|^2 = \langle\omega, \pi_h\eta\rangle + \langle\omega, \omega - \pi_h\eta\rangle = \langle\omega, \pi_h\eta\rangle \le \|\omega\|\|\pi_h\eta\|,$$

whence $\|\omega\| \le \|\pi_h\eta\|$. The result follows from *the uniform boundedness* of π_h.

Preconditioning the Hodge Laplace problem

Hodge Lapace problem (assume no harmonic forms):
Find $(\sigma, u) \in H\Lambda^{k-1} \times H\Lambda^k$ such that

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0 && \forall \tau \in H\Lambda^{k-1} \\ \langle d\sigma, v \rangle + \langle du, dv \rangle &= \langle f, v \rangle && \forall v \in H\Lambda^k \end{aligned}$$

with coefficient matrix

$$\mathcal{A} = \begin{pmatrix} I & d^{(k-1)*} \\ d^{k-1} & -d^{k*} d^k \end{pmatrix}$$

Here d^* is the formal adjoint of d.

Construction of a preconditioner

$$\mathcal{A} = \begin{pmatrix} I & d^{(k-1)*} \\ d^{k-1} & -d^{k*}d^k \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} (I + d^{(k-1)*}d^{k-1})^{-1} & 0 \\ 0 & (I + d^{k*}d^k)^{-1} \end{pmatrix}$$

where the operator $I + d^*d$ corresponds to the bilinear form

$$\langle \sigma, \tau \rangle + \langle d\sigma, d\tau \rangle$$

Special case, $n = 3$

The preconditioner $\mathcal{B}$ corresponds to:

$$k = 0 \qquad \mathcal{B} = (I - \operatorname{div}\operatorname{grad})^{-1} = (I - \Delta)^{-1}$$

$$k = 1 \qquad \mathcal{B} = \begin{pmatrix} (I - \Delta)^{-1} & 0 \\ 0 & (I + \operatorname{curl}\operatorname{curl})^{-1} \end{pmatrix}$$

$$k = 2 \qquad \mathcal{B} = \begin{pmatrix} (I + \operatorname{curl}\operatorname{curl})^{-1} & 0 \\ 0 & (I - \operatorname{grad}\operatorname{div})^{-1} \end{pmatrix}$$

$$k = 3 \qquad \mathcal{B} = \begin{pmatrix} (I - \operatorname{grad}\operatorname{div})^{-1} & 0 \\ 0 & I \end{pmatrix}$$