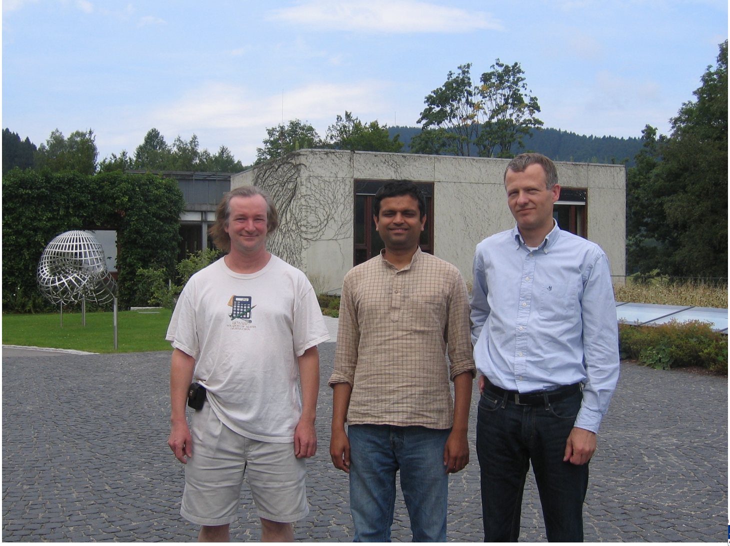
Localising subcategories of the stable module category of a finite group

Dave Benson
(Joint work with Srikanth Iyengar and Henning Krause)

Durham, July 2009

Benson, Iyengar and Krause



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DEFINITION

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- If $0 \to M_1 \to M_2 \to M_3 \to 0$ and two of M_1, M_2, M_3 are in $\mathcal{C}$ then so is the third
- $\mathcal{C}$ is closed under direct sums.

The Main Theorem

Theorem

There is a natural one to one correspondence between non-zero localising subcategories of $\mathsf{Mod}(kG)$ *and subsets of*

$$\operatorname{Proj} H^*(G,k) = \{\textit{non-maximal hgs prime ideals in } H^*(G,k)\}$$

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Remark

This is analogous to Neeman's classification of localising subcategories of $D(\mathrm{Mod}R)$ for a commutative ring R, but quite a bit harder.

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$$\underline{\mathrm{Hom}}_{kG}(M,N) = \frac{\mathrm{Hom}_{kG}(M,N)}{\mathrm{PHom}_{kG}(M,N)}$$

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Now, a localising subcategory is a full triangulated subcategory closed under direct sums.

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Tate resolutions: $\mathsf{StMod}(kG) \simeq \mathsf{K}_{\mathrm{ac}}\mathsf{Inj}(kG)$.

Recollement

$$\mathsf{StMod}(kG) \simeq \mathsf{K_{ac}Inj}(kG) \begin{array}{c} \xleftarrow{\mathrm{Hom}_k(tk,-)} \\ \longrightarrow \\ \xleftarrow[-\otimes_k tk]{} \end{array} \mathsf{KInj}(kG) \begin{array}{c} \xleftarrow{\mathrm{Hom}_k(pk,-)} \\ \longrightarrow \\ \xleftarrow[-\otimes_k pk]{} \end{array} D\,\mathsf{Mod}(kG).$$

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$$E = \langle g_1, \dots, g_r \mid g_i^p = 1, \quad g_i g_j = g_j g_i \rangle \cong (\mathbb{Z}/p)^r$$

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  $\deg X_i = 0$, $\deg Y_i = 1$.
  $Y_i^2 = 0$, $Y_i Y_j = -Y_j Y_i$, $d(Y_i) = X_i$, $d(X_i) = 0$.

Transfer of stratification

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give a one to one correspondence on localising subcategories.

Formality

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Still using dg Λ-modules, differential on Λ is zero.

BGG correspondence

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The final step in the proof is to classify localising subcategories of $D_{\mathsf{dg}}(S)$ using methods similar to Neeman's.

Leitfaden

$$D_{\mathsf{dg}}(S) \rightsquigarrow \mathsf{KInj}_{\mathsf{dg}}(\Lambda) \rightsquigarrow \mathsf{KInj}_{\mathsf{dg}}(A) \rightsquigarrow$$
$$\mathsf{KInj}(kE) \rightsquigarrow \mathsf{KInj}(kG) \rightsquigarrow \mathsf{StMod}(kG) \rightsquigarrow \mathsf{Mod}(kG).$$

Leitfaden

Let $\mathcal{T}$ be a triangulated category with direct sums and with a compact generator C.

Details: Stratifying Triangulated Categories

Let $\mathcal{T}$ be a triangulated category with direct sums and with a compact generator C.
$Z^n(\mathcal{T})$: natural transformations $x\colon \mathrm{Id} \to \tau^n$ satisfying $x\tau = (-1)^n \tau x$.

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For each X in $\mathcal{T}$, regard $H^*_C(X) = \mathrm{Hom}_{\mathcal{T}}(C, X)$ as a graded R-module via $R \to \mathrm{End}_{\mathcal{T}}(C)$.

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Definition

A subset V of $\mathrm{Spec}^*(R)$ is specialisation closed if $\mathfrak{p} \in V$, $\mathfrak{q} \supseteq \mathfrak{p}$ implies $\mathfrak{q} \in V$.

Support

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If V is specialisation closed, set

$$\mathcal{T}_V = \{X \in \mathcal{T} \mid \mathsf{supp}_R H^*_C(X) \subseteq V\}$$

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By Brown representability: There is a localisation functor $L_V \colon \mathcal{T} \to \mathcal{T}$ such that $L_V X = 0 \iff X \in \mathcal{T}_V$.

Support, contd.

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Then the "local cohomology functor" $\Gamma_\mathfrak{p} = \Gamma_V L_W$ is independent of these choices.

There is a functorial Rickard triangle

$$\Gamma_V X \to X \to L_V X$$

If $\mathfrak{p} \in \operatorname{Spec}^* R$, choose $V, W \subseteq \operatorname{Spec}^* R$ specialisation closed such that $\mathfrak{p} \notin W$, $V = W \cup \{\mathfrak{p}\}$.

Then the "local cohomology functor" $\Gamma_{\mathfrak{p}} = \Gamma_V L_W$ is independent of these choices.

Definition

The support of an object X is defined to be

$$\operatorname{supp} X = \{\mathfrak{p} \mid \Gamma_{\mathfrak{p}} X \neq 0\}$$

Stratification

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We say $\mathcal{T}$ is stratified by R if $\forall\, \mathfrak{p} \in \mathrm{Spec}^* R$, $\Gamma_{\mathfrak{p}}\mathcal{T}$ is either zero or a minimal non-zero localising subcategory.

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Theorem

$H^*(G, k)$ stratifies $\mathrm{KInj}(kG)$ and hence also $\mathrm{StMod}(kG)$.

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$$\operatorname{supp}(X \otimes_k Y) = \operatorname{supp} X \cap \operatorname{supp} Y$$

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  4. the complement of the set of primes corresponding to $\mathcal{C}$ is specialisation closed.

Costratification and Cosupport

Recall that for $X \in \mathsf{KInj}(kG)$,

$$\operatorname{supp} X = \{\mathfrak{p} \in \operatorname{Spec}^* H^*(G,k) \mid \Gamma_{\mathfrak{p}} X = \Gamma_{\mathfrak{p}} k \otimes_k X \neq 0\}$$

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A colocalising subcategory of a triangulated category is a triangulated subcategory closed under products.

Classification of Colocalising Subcategories

Theorem

Cosupport defines a one to one correspondence between colocalising subcategories of $\mathsf{KInj}(kG)$ *and subsets of* $\mathsf{Spec}^* H^*(G,k)$*, and also between colocalising subcategories of* $\mathsf{StMod}(kG)$ *and subsets of* $\mathsf{Proj}\, H^*(G,k)$*.*

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Theorem

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This tensor product corresponds to ordinary tensor product over k with diagonal G-action for objects in $\mathsf{KInj}(kG)$.