Linear Algebra over a Ring

Ivo Herzog

New Directions in the Model Theory of Fields
Durham University
July 23, 2009

Linear Equations

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- The standard **axioms** for a left R-module are expressible in $\mathcal{L}(R)$. This collection of axioms, denoted $T(R)$, is usually infinite. For example, for every $r \in R$,
$$(\forall v, w)\ r(v + w) \doteq rv + rw$$
belongs to $T(R)$.

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- A finite conjunction of linear equations is a **system of linear equations:**

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  where A is an $m \times n$ matrix and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a *column* of n variables.
- A **positive-primitive formula** is an existentially quantified systems of linear equations:

$$\exists \mathbf{w}\ (A\mathbf{v} + B\mathbf{w} \doteq 0).$$

Examples

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- If $A = I_n$, the $n \times n$ identity matrix, then the pp-formula is equivalent, relative to $T(R)$, to the **divisibility condition**

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- Using this notation, we may express a general pp-formula in the free variables $\mathbf{v} = (v_1, \ldots, v_n)$ as

$$B|A\mathbf{v}.$$

Subgroups Defined by a Positive-primitive Formula

- If $\varphi(v_1, \ldots, v_n) = B|A\mathbf{v}$ is a positive-primitive formula in $\mathcal{L}(R)$, then

$$\varphi(M) := \{\mathbf{a} \in M^n : M \models \exists \mathbf{w}\, (B\mathbf{w} \doteq A\mathbf{a})\}$$

is a subgroup of M^n. A subgroup of the form $\varphi(M)$ is called an n-**ary pp-definable subgroup** of M.

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- If $\varphi(\mathbf{v})$ and $\psi(\mathbf{v})$ are positive-primitive formulae, then

$$(\varphi \wedge \psi)(\mathbf{v}) \text{ and } (\varphi + \psi)(\mathbf{v}) := \exists \mathbf{w}\ (\varphi(\mathbf{v} - \mathbf{w}) \wedge \psi(\mathbf{w}))$$

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are equivalent, relative to $T(R)$, to positive-primitive formulae.
- If ${}_R M$ is a left R-module, then
$$(\varphi \wedge \psi)(M) = \varphi(M) \cap \psi(M) \text{ and } (\varphi + \psi)(M) := \varphi(M) + \psi(M).$$

Lemma Presta

- **The Completeness Theorem:** $T(R) \vdash \psi(\mathbf{v}) \to \varphi(\mathbf{v})$ if and only if $\psi(M) \subseteq \varphi(M)$ for every left R-module ${}_R M$.

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- **Lemma Presta.** $T(R) \vdash B|A\mathbf{v} \rightarrow B'|A'\mathbf{v}$ iff there exist matrices U, V and G, of appropriate size, such that

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- **Proof of easy direction:**

$$\begin{aligned} T(R) \vdash B|A\mathbf{v} &\to UB|UA\mathbf{v} \\ &\leftrightarrow B'V|(A' + B'G)\mathbf{v} \\ &\to B'|(A' + B'G)\mathbf{v} \\ &\leftrightarrow B'|A'\mathbf{v}. \end{aligned}$$

Definition

- **Definition.** For $n \geq 0$, let $L'_n(R)$ be the set of pairs of matrices $(A \mid B)$ where A has n columns, and B has the same number of rows as A. The relation

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- **Proposition.** Let A be an $m \times n$ matrix; B an $m \times k$ matrix. The relation $\leq_n$ is the least partial order on $L'_n(R)$ satisfying:
  1. If U is a matrix with m columns, then $(A \mid B) \leq (UA \mid UB)$.
  2. If V is a matrix with k rows, then $(A \mid BV) \leq (A \mid B)$.
  3. If G is a $k \times n$ matrix, then $(A + BG \mid B) \leq (A \mid B)$.

Matrix Pairs

- Two pairs of matrices $(A \mid B)$ and $(A' \mid B')$ in $L'_n(R)$ are **equivalent** if

$$(A \mid B) \leq_n (A' \mid B') \text{ and } (A' \mid B') \leq_n (A \mid B).$$

An n**-ary matrix pair** $[A \mid B]$ is the equivalence class of $(A \mid B)$. Denote by $L_n(R)$ the partially ordered set of n-ary matrix pairs.

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- **Proposition.** The following hold in $L_n(R)$:
  1. If P is an invertible matrix, then $[A \mid B] = [PA \mid PB]$.
  2. If Q is an invertible matrix, then $[A \mid BQ] = [A \mid B]$.
  3. If G is any matrix, then $[A + BG \mid B] = [A \mid B]$.

Maximum and Minimum Elements

- The **minimum** element of $L_n(R)$ is given by $0_n := [I_n \mid 0]$. For, if $[A \mid B]$ is an arbitrary n-ary matrix pair, then

$$0_n = [I_n \mid 0] \leq [A \cdot I_n \mid 0] = [A \mid B \cdot 0] \leq [A \mid B].$$

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- Similarly, the **maximum** element is given by $1_n := [I_n \mid I_n]$. If A is $m \times n$, let P be $n \times m$. Then since

$$[A \mid B] \leq [PA \mid I_n \cdot PB] \leq [PA \mid I_n] = [PA + I_n(I_n - PA) \mid I_n] = 1_n.$$

Also note that

$$1_n = [I_n \mid I_n] = [0 \mid I_n] \leq [0 \mid B].$$

Principal Ideal Domains

- If R is a **PID,** then there are invertible matrices P and Q such that $PBQ = D$ is a diagonal matrix. Thus

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- The **infimum** of two n-ary matrix pairs $[A \mid B]$ and $[A' \mid B']$ is given by

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- If $D = (d_{ij})$ is a diagonal matrix, then

$$[A \mid D] = \bigwedge_i [{}_iA \mid d_{ii}],$$

where ${}_iA$ denotes the i-th row of A.

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- **Proposition.** A matrix B is regular iff for every A, there is an A' such that
$$[A \mid B] = [A' \mid 0].$$
- **Proof:** If B is regular, then $[A \mid BC] \leq [A \mid B] \leq [A \mid BC]$, with BC **idempotent** $(BC)^2 = BCBC = BC$. If E is idempotent, then
$$[A \mid E] \leq [(I_m - E)A \mid 0] \leq [A - EA \mid E] = [A \mid E].$$

von Neumann Regular Rings

- For the converse, let I_n be the $n \times n$ identity matrix. If $[I_n \mid B] = [A' \mid 0]$, then
  1. there is a U such that $UB = 0$ and $U \cdot I_n = A'$, i.e., $A'B = 0$; and
  2. there are U and G such that $UA' = I_n + BG$.

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  Then $0 = UA'B = B + BGB$, so that $B = B(-G)B$.
- **Corollary.** A ring R is **von Neumann regular** iff for every matrix pair $[A \mid B]$ there is an A' such that $[A \mid B] = [A' \mid 0]$.

The Opposite Ring R^{op}

- Multiplication of matrices with entries in R^{op} is denoted $A * B$. It is related to multiplication of matrices over R by the equation
$$(A * B)^{\mathrm{tr}} = B^{\mathrm{tr}} * A^{\mathrm{tr}}.$$

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- **Theorem.** (Prest, Huisgen-Z./Zimmermann) If $(A \mid B) \leq (A' \mid B')$ in $L'_n(R)$, then in $L'_n(R^{\mathrm{op}})$,
$$\left(\begin{array}{c|c} I_n & (A')^{\mathrm{tr}} \\ 0 & (B')^{\mathrm{tr}} \end{array} \right) \leq \left(\begin{array}{c|c} I_n & A^{\mathrm{tr}} \\ 0 & B^{\mathrm{tr}} \end{array} \right).$$

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- **Proof:** We are given matrices U, V and G such that $UB = B'V$ and $UA = A' + B'G$, or
$$(A', B') \begin{pmatrix} I_n & 0 \\ G & V \end{pmatrix} = U(A, B).$$

- In $L_n(R^{\text{op}})$, this yields

$$\begin{pmatrix} I_n & G^{tr} \\ 0 & V^{tr} \end{pmatrix} * \begin{pmatrix} (A')^{\text{tr}} \\ (B')^{tr} \end{pmatrix} = \begin{pmatrix} A^{\text{tr}} \\ B^{tr} \end{pmatrix} * U^{\text{tr}}.$$

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- Let $U' = \begin{pmatrix} I_n & G^{tr} \\ 0 & V^{tr} \end{pmatrix}$, $V' = U^{\mathrm{tr}}$, and $G' = 0$.

The Anti-isomorphism

- This induces an anti-morphism from $L_n(R)$ to $L_n(R^{\mathrm{op}})$, given by
$$[A \mid B] \mapsto [A \mid B]^* := \left[\begin{array}{c|c} I_n & A^{\mathrm{tr}} \\ 0 & B^{\mathrm{tr}} \end{array} \right].$$

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- To see that it is a anti-isomorphism, just note that

$$\left[\begin{array}{c|cc} I_n & I_n & 0 \\ 0 & A & B \end{array}\right] = \left[\begin{array}{c|cc} I_n & I_n & 0 \\ -A & 0 & B \end{array}\right] = [I_n \mid I_n] \wedge [-A \mid B] = [A \mid B].$$

Properties of Duality

- **Example.** The anti-isomorphism $[A \mid B] \mapsto \left[\begin{array}{c|c} I_n & A^{\mathrm{tr}} \\ 0 & B^{\mathrm{tr}} \end{array}\right]$ interchanges the respective families of annihilator and divisibility conditions: $[A \mid 0]^* = [I_n \mid A^{tr}]$.

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- **Theorem.** (IH) Let M be a right R-module, and N a left R-module. Given n-tuples $\mathbf{a} \in M^n$ and $\mathbf{b} \in N^n$, then $\mathbf{a} \otimes \mathbf{b} = 0$ in $M \otimes_R N$ iff there is a pp-formula $\varphi(\mathbf{v})$ such that $N \models \varphi(\mathbf{b})$ and $M \models \varphi^*(\mathbf{a})$.

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- **Proof of the easy direction:**

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{v}A \otimes \mathbf{b} = \mathbf{v} \otimes A\mathbf{b} = \mathbf{v} \otimes B\mathbf{w} = \mathbf{v}B \otimes \mathbf{w} = 0.$$

The Modular Lattice

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$$[A \mid B] + [A' \mid B'] = ([A \mid B]^* \wedge [A' \mid B']^*)^*.$$
- Explicitly,
$$[A \mid B] + [A' \mid B'] := \left[\begin{array}{c|cccc} I_n & I_n & 0 & I_n & 0 \\ 0 & A & B & 0 & 0 \\ 0 & 0 & 0 & A' & B' \end{array}\right].$$

Quantifiers

- If $\varphi(\mathbf{v}) \rightleftharpoons [A \mid B]$ is the pp-formula associated to the matrix pair, we also write $\varphi(\mathbf{u}, \mathbf{v}) \rightleftharpoons [A_1, A_2 \mid B]$.

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- There are two quantifiers, given by

$$\exists \mathbf{v}\ \varphi(\mathbf{u}, \mathbf{v}) \rightleftharpoons [A_1 \mid A_2, B] \text{ and } \varphi(\mathbf{u}, 0) \rightleftharpoons [A_1 \mid B].$$

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- These quantifiers are related by duality according to the following equations in $L_n(R^{\mathrm{op}})$:

$$[A_1, A_2 \mid B]^* = \left[\begin{array}{cc|c} I_{n_1} & 0 & A_1^{\mathrm{tr}} \\ 0 & I_{n_2} & A_2^{\mathrm{tr}} \\ 0 & 0 & B^{\mathrm{tr}} \end{array}\right];$$

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- Let $Y_k \leq X_k$ $(k = 0, 1)$ be the image of Γ under the natural projection onto X_k.
- Then
  1. $(Y_0 \cap \Gamma) \triangleleft Y_0$ and $(Y_0 \cap \Gamma) \triangleleft Y_0$; and
  2. there is an isomorphism $f_\Gamma : Y_0/(Y_0 \cap \Gamma) \cong Y_1/(Y_1 \cap \Gamma)$.

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- Then
  1. $(Y_0 \cap \Gamma) \lhd Y_0$ and $(Y_0 \cap \Gamma) \lhd Y_0$; and
  2. there is an isomorphism $f_\Gamma : Y_0/(Y_0 \cap \Gamma) \cong Y_1/(Y_1 \cap \Gamma)$.
- The graph of the isomorphism is the image of Γ in the quotient of the inclusion

$$[(Y_0 \cap \Gamma) \times (Y_1 \cap \Gamma)] \leq \Gamma \leq (Y_0 \times Y_1).$$

The Goursat Group

- The **Goursat group** $G(R)$ is the free group on the elements of $\cup_{n\geq 1} L_n(R)$, modulo the relations:
  1. for every three matrices A_1, A_2 and B with the same number of rows,
  $$[A_2|A_1, B] - [A_2|B] = [A_1|A_2, B] - [A_1|B];$$
  2. for $[A|B] \in L_m$ and $[A'|B'] \in L_n$,
  $$\left[\begin{array}{cc|cc} A & 0 & A' & 0 \\ 0 & B & 0 & B' \end{array}\right] = [A|B] + [A'|B']; \text{ and}$$
  3. for every $n \geq 1$, $0_n = 0$.

The 0-Dimensional Goursat Group

- The 0-**dimensional Goursat group** $G_0(R)$ is the free group on the elements of $L_1(R)$, modulo the relations:
  1. $0_1 = 0$; and
  2. if A_1 and A_2 are column matrices, and all three matrices A_1, A_2, and B have the same number of rows, then

$$[A_2|A_1, B] - [A_2|B] = [A_1|A_2, B] - [A_1|B].$$

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  $$[A_2|A_1, B] - [A_2|B] = [A_1|A_2, B] - [A_1|B].$$
- There is an obvious morphism $\iota : G_0(R) \to G(R)$ induced by $[A \mid B] \mapsto [A \mid B]$.

Finitely Presented Modules

- A left R-module ${}_RM$ is **finitely presented** if there is an exact sequence, called a **free presentation,** of the form

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  We say that M is **presented** by the matrix A, and write $M = M_A$.
- Two matrices A and B are **equivalent,** denoted $A \sim B$, if they present isomorphic modules, $M_A \cong M_B$. The equivalence class of a matrix A is denoted by $\{A\}$.

The Grothendieck Group

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$$[M \oplus N] = [M] + [N].$$

It may also be defined as the free group on the equivalence classes $\{A\}$ of matrices, modulo the relations

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- **Adelman:** Let $\mathrm{Ab}(R)$ be the **free abelian category** over R. The subcategory of projective objects of $\mathrm{Ab}(R)$ is equivalent to R-mod. Thus $K_0(R\text{-mod}, \oplus)$ is isomorphic to the **Grothendieck group** $K_0(\mathrm{Ab}(R))$.

Some Homomorphisms

- **The Isomorphism Theorem.** (IH) There exist morphisms

$$\begin{tikzcd} G_0(R) \arrow[rr, "\iota"] & & G(R) \arrow[dl, "\gamma"] \\ & K_0(R\text{-mod}, \oplus) \arrow[ul, "\kappa"] & \end{tikzcd}$$

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- The morphism $\kappa : K_0(R\text{-mod}, \oplus) \to G_0(R)$ is induced by the function $\{A\} \mapsto \sum_{j=1}^{n} [A_j \mid A_{j+1}, \ldots, A_n]$, where A_j is the j-th column of A.

Equivalent Matrices

- **Theorem.** Two matrices with entries in the ring R are equivalent iff one can be obtained from the other by a sequence of operations of the following form:
  1. permutation of rows or columns;
  2. replacement of a matrix C by $\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$, or the reverse;
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- Reference: Theorem 6.1 in **Lickorish, W.B.R.,** *An Introduction to Knot Theory* GTM 175.

The Morphism $\gamma : G(R) \longrightarrow K_0(R\text{-mod}, \oplus)$

- **Lemma.** If $(A \mid B) \leq (A' \mid B')$, then

$$\begin{pmatrix} A & B & 0 \\ 0 & 0 & B' \end{pmatrix} \sim \begin{pmatrix} A & B & 0 \\ A' & 0 & B' \end{pmatrix}.$$

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- **Proof.** We are given U, V and G such that $UB = B'V$ and $UA = A' + B'G$. Thus
$$\begin{aligned}
\begin{pmatrix} A & B & 0 \\ 0 & 0 & B' \end{pmatrix} &\sim \begin{pmatrix} A & B & 0 \\ UA & UB & B' \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ UA & B'V & B' \end{pmatrix} \\
&\sim \begin{pmatrix} A & B & 0 \\ UA & 0 & B' \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ A' + B'G & 0 & B' \end{pmatrix} \\
&\sim \begin{pmatrix} A & B & 0 \\ A' & 0 & B' \end{pmatrix}.
\end{aligned}$$

γ Is Well-Defined

- If $[A \mid B] = [A' \mid B']$, then

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- So in $K_0(R\text{-mod}, \oplus)$, we have that

$$\{(A, B)\} + \{B'\} = \{(A', B')\} + \{B\}$$

and hence that $\{(A, B)\} - \{B\} = \{(A', B')\} - \{B'\}$.

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