

T a theory. QE, EI.

• ACF

stable ↗  
↘

• ACVF  $\frac{0,0}{k}$   $V_i/k^*$

• ACF +  $\{V_i\}_k = \{PV_i\}_k$

fin. dim'd vector spaces.

$T_P^0 = T + \text{predicate } P \text{ (in each sort)}$

+ "del(P) = P".

"RATIONALITY QUESTION".

$T_P^{\exists} = T_P + \text{"Aut}(P^{\text{rel}}/P) = \hat{\mathbb{Z}} \text{"}$

•  $ACF_P^{\exists} \subseteq PF$

•  $ACVF_P = HF_0$

$T_C = T + \text{"c an automorphism"}$

$P = \text{Fix}(c), \quad T_C \geq T_P^{\exists}$

Five examples.

- 1)  $QE$  for  $T_p \iff$  isomorphism for  $K_T$ .  
(Ax-Kocher-1).
- 2) Rational orbits  $\iff$  Groupoids  
Clusters - Denot.  
Entirely det. eq. rel's
- 3)  $K(T_p^0) \iff$  Linearization.  
Denot - Locia.
- 4) EI for  $T_c, T_p^0$ 
  - Higher amalgamation
  - Generalized imaginaries  
eq. rel's  $\rightarrow$  groupoids
  - $(T^{gr})_c$  has E.I.
- 5)  $K_c(ACF_c^0)$  and motives.

Maps from

$K(ACVF_c)$

correspondence  
up to rational  
equivalence.

2)

Concrete groupoid = definable  
groupoid + functor  $\rightarrow \text{Def}_T$

$\vdash =$  A def. set  $\text{Ob}_G$

For  $x \in \text{Ob}_G$ , a def. set  $A_x$ ;

For  $x, y \in \text{Ob}_G$ , a definable  
family  $\Pi_{x,y}$  of def.

bijections  $A_x \rightarrow A_y$ ;

$\text{Id}_{A_x} = 1 \in \Pi_{x,x} =: \Theta_x$

$\Pi_{x,y} \circ \Pi_{u,x} = \Pi_{u,y}$

$\Pi_{x,y}^{-1} = \Pi_{y,x}$

3)

# Linearization

$\mathbb{Q} \text{ Det}$ .

- Same objects as  $\text{Det}$ .
- $\text{Mor}(X, Y) = \text{Correspondences}$   
 $= \text{definable maps } X \rightarrow \mathbb{Q}Y$ .

- If  $S \subseteq X \times Y$ ,  $S$  definable,  
 $p_x: S \rightarrow X$  with finite fibers,  
 $X \xrightarrow{f_S} \Sigma \{y: (x, y) \in S\}$   
 The  $f_S$  spec  $\text{Mor}(X, Y)$ .

NB:  $\mathbb{Q}Y(P) \subseteq (\mathbb{Q}Y)(P)$ .

Thm (D-L)

$$\begin{array}{ccc}
 K(T_P^{\mathbb{Z}}) & \longrightarrow & K(\mathbb{Q} \text{ Det}) \\
 [V(P)] & \longmapsto & [V]
 \end{array}$$

$V \rightarrow U$  finite

$$[u \mapsto \#V_u] \longmapsto [V]$$

(P)

T<sub>09</sub> (imaginaries)

$D$  a def. set,  $E$  a def.

equiv. rel'n. New sort  $S_{D,E}$ .

$E \ni D \longrightarrow S_{D,E}$

T<sub>99p</sub> (generalized imaginaries)

$G$  a definable groupoid.

New sorts  $Ob_G^+$ ,  $Mor_{a,b}^+$

( $a \in Ob_G, b \in Ob_G^+$ ) s.t.  
 $\cup Ob_G^+$

The union forms a concrete sgd,  
with one object  $Ob_G^+$  in  
each isomorphism class.

$\mathcal{C}_T = \{ \text{alg. closed} \\ \text{substructures of models of } T \}$

$P(N) = \{ \text{subsets of } \{1, \dots, N\} \}$

$P(N)^- = \{ \text{proper " " " } \}$

An (independent) amalgamation  
problem for  $T$  is a functor

$$a: P(N)^- \rightarrow \mathcal{C}(T)$$

$$a(s) = \text{acl} \bigoplus_{i \in s} a(i).$$

A solution = extension to  $P(N)$ .

Def. (acl)

$$A \subseteq M \models T$$

acl(A) IS THE UNION OF ALL  
FINITE A-DEFINABLE SETS.

Thm ( $T$  stable)  $T_G$  has EI  
 provided 4-<sup>(independent)</sup>amalgamation holds,  
 over any  $A = \text{acl}(A)$ .

Thm TFAE: ( $T$  stable)  
 EI

1) 4-amalgamation

2) Unique 3-amalgamation.

3)  $\{(A, \sigma) : A \in \mathcal{C}_T, \sigma \in \text{Aut}(A)\}$   
 has 3-amalgamation.

4) Finite internal covers split. (coll.)

5) Every concrete groupoid  
 w. one iso'm class, finite and  
 is equivalent to a finite group action.  
 (this is a group action)

Prop.  $T = ACF_0 + \{V_i\}, \{P V_i\}$ .

- $EI \iff \{V_i\}$  closed under  $\otimes$ , duals.

(Tannaka, cf. Kemerby)

- Assume irreducibles are 1-dim.

$T$  has  $EI \iff$  interpret.  $\mathcal{U}$

cases, i.e.  $\forall V, \dim V = 1$

$$\exists \mathcal{U}^{\otimes k} \cong V.$$

- Cor  $PF + \{V_i\}_{i \in P}$ ,

$$V_i \otimes V_j \cong V_{i+j}$$

has  $EI$  ( $P$  divisible.)



$\text{Def}_T = \text{category of}$

0. definable sets, maps.

$K(T) = \text{Groth. ring.}$  <sup>semi.</sup>

$= \pm \text{Cb } \text{Def}_T / \text{iso}, \cup, \times.$

Given unit, definable  $X_a = \{x: (x_a) \in X\}$ ,

write:  $[X] = \sum_a [X_a]$

$[X_a] \in K(T_a).$

$K_C =$  obtained from  $K$  by

imposing:

$[X] \mapsto [X]$

$K(T_b) \longrightarrow K(T_{ab})$

injective,  $X \in \text{Def}_{T_b}$

(can replace to tensor, central group.)

LEM. 2 The incarnation  
 $K \xrightarrow{\alpha} R$

Gives:  $K_c(\text{Th}(K)) \xrightarrow{K_c(\alpha)} K_c(\text{Th}(R))$

$\alpha_*$  IS A RING ISOMORPHISM.

(ANALOG FOR  $K_0$  IS WRONG.)

A basic

THM Let  $U$  be a stably embedded  
union of sorts of  $V$ ;

Assume  $V$  is  $U$ -analyzable

( $V = U_n \supseteq U_{n-1} \supseteq \dots \supseteq U_1 = U$ ,  $U_{i+1}$   
internal to  $U_i$ )

$U$   
STABLE  
EMBEDDED  
in  
 $V$

(WEAK)

+ CHAIN CONDITION ON INTERSECTIONS  
OF CONJUGATES OF STABILIZERS  
OF LIAISON GROUPS OF  $U_{i+1}/U_i$ .  
(e.g. SOLVABLE, LINEAR, ...)

Then the natural homomorphism

$K_c(\text{Th}(U)) \longrightarrow K_c(\text{Th}(V))$

is an isomorphism.

Proof uses liaison groupoids.

Sketch of talk:

$(ACFA = FA)$

$K_0(ACF) \Rightarrow V$  VARIETY



$K_c(PF)$  (COVALIERI)  $V(F)$



$K_c(\underline{FA^0})$

$V(\text{Fix}(c))$

$\bar{x} \in V$   
 $\sigma x_i = x_i$

DIFF. VARIETY, FINITE TOTAL DIM

= 0 - TRANSF DIM'N.

residue field



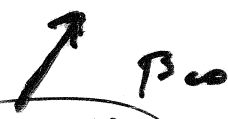
$K_c(\underline{VFA^0_t})$

$K_c(\underline{VFA^0_t})$

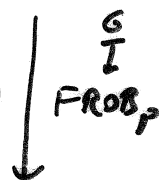
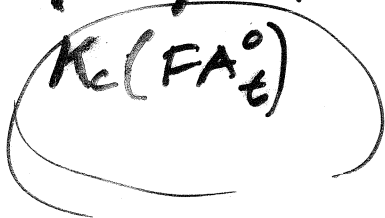
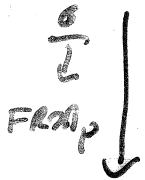
$[V(\text{Fix}(c))]$

$\text{VAL}(c) > 0$

$\text{VAL}(c) < 0$



uniformly  
 $t = c$  transcendental  
constant



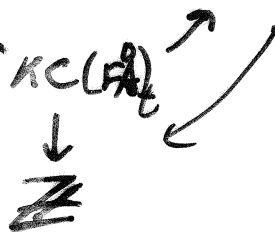
$K_c(\underline{VF^0_t})$

$K_c(\underline{VF^0_t})$

0 - linear

← Pencil

(numbers)



$|V(\underline{GF_p})|$

De Lincien

$K_w$

=

$K_c(\underline{FA^0}) \otimes K_c(\underline{FA^0})$

$\beta_0 \nwarrow K_c(\underline{FA^0}) \nearrow \beta_\infty$

COMPARISON WITH MOTIVES

QUESTION  $U, V$  <sup>SMOOTH PROJECTIVE</sup> VARIETIES.

IF  $U, V$  HAVE ISOMORPHIC MOTIVES,

SHOW:  $[U(\text{Fix}(c))] = [V(\text{Fix}(c))] \in K.W.$

Pf  $S \subseteq U \times V$   
 $T \subseteq V \times U$

$$T \circ S \underset{\text{RAT}}{\sim} n \cdot \Delta_U, \quad S \circ T \underset{\text{RAT}}{\sim} n \cdot \Delta_V$$

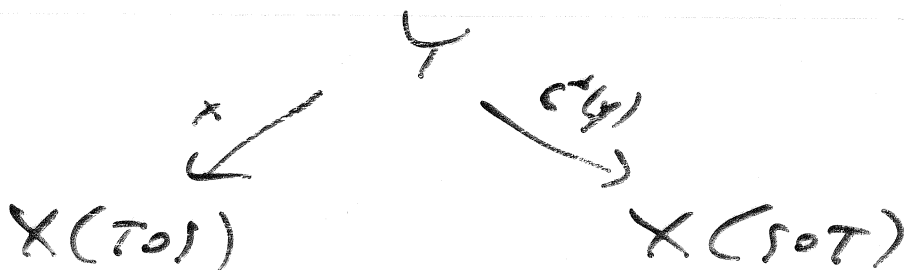
After replacing  $S, T$  by sufficiently "general position" reps,

$$X(T \circ S) := \{x \in U : (x, c(x)) \in T\}$$

$$[X(T \circ S)] = n \cdot [U(\text{Fix}(c))] = n [X(\Delta_U)]$$

$$[X(S \circ T)] = n \cdot [V(\text{Fix}(c))]$$

Let  $Y = \{(x, y) \in S : (c(x), y) \in T\}$



# Problems

- 1) Categoricity  $K(RV)_{\mathbb{R}} \cong K(VF)$
- 2) Extremally detachable equiv. relations.

$$x \in y \iff \exists u \varphi(x, y, u)$$

$$\text{Mod}(x, y) = \{u : \varphi(x, y, u)\}$$

- 4) Describe  $T^{\text{nspt}}$ , <sup>higher</sup> generalised imaginaries with  $\text{nspt}$ -analyticity.

5)

$$K_c(VF_c^0) \rightarrow K_c(ACF_c^0) \leftarrow K \text{ Motives}$$