(1)

T a theory. QE, EI.

- ACF
- $ACVF_{0,0}$ $\quad V_i/k^n$
- $ACF_k + \{V_i\}_i + \{PV_i\}_i$

stable

fin. dim'l vector spaces.

$$T_P^0 = T + \text{Predicate } P \left(\begin{smallmatrix}\text{in each}\\ \text{sort}\end{smallmatrix}\right)$$
$$+ \text{"} dcl(P) = P \text{"}.$$

"RATIONALITY QUESTIONS".

$$T_P^{\hat{\mathbb{Z}}} = T_P + \text{"}Aut(P^{acl}/P) = \hat{\mathbb{Z}}\text{"}.$$

- $ACF_p^{\hat{\mathbb{Z}}} \subseteq PF$
- $ACVF_p = HF_0$

$$T_\sigma = T + \text{"}\sigma \text{ an automorphism"}.$$

$P = Fix(\sigma)$. $\quad T_\sigma \supseteq T_P^{\hat{\mathbb{Z}}}$.

Five examples.

1) QE for T_p $\longleftrightarrow$ isomorphism for K_T.

(Ax-Kochen).

2) Rational orbits $\longleftrightarrow$ T-Groupoids

Cluckers - Denef.

Existentially def. eq. rel's

3) $K(T_p^0)$ $\longleftrightarrow$ Linearization.

Denef-Loeser.

4) EI for T_σ, T_p^0

- Higher amalgamation
- Generalized imaginaries

  eq. rel's $\rightsquigarrow$ groupoids
- $(T^{gp})_\sigma$ has E.I.

5) $K_0(ACF_\sigma^0)$ and motives.

Maps from $K(ACVF_\sigma)$ — correspondence up to rational equivalence.

2)

Concrete groupoid = definable groupoid + functor $\to$ Def_T

:= A def. set Ob_G

For $x \in Ob_G$, a def. set A_x;

For $x, y \in Ob_G$, a definable family $M_{x,y}$ of def. bijections $A_x \to A_y$;

$Id_{A_x} = 1 \in M_{x,x} =: G_x$

$M_{x,y} \circ M_{u,x} = M_{u,y}$

$M_{x,y}^{-1} = M_{y,x}$

3)

Linearization $\mathbb{Q}\mathrm{Def}$.

- Same objects as Def.
- $\mathrm{Mor}(X, Y)$ = correspondences

  = definable maps $X \to \mathbb{Q}Y$.

- If $S \subseteq X \times Y$, S definable,

  $pr_X : S \to X$ with finite fibers,

  $$x \xmapsto[f_S]{} \sum \{y : (x,y) \in S\}$$

  The f_S span $\mathrm{Mor}(X, Y)$.

- NB: $\mathbb{Q}Y(P) \subset (\mathbb{Q}Y)(P)$.

Thm (D-L)

$$K(T_P^{\mathbb{Z}}) \longrightarrow K(\mathbb{Q}\mathrm{Def})$$

$$[V(P)] \longmapsto [V]$$

$V \to U$ finite

$$[u \mapsto \# V_u(P)] \longmapsto [V]$$

T^{eq} (imaginaries)

D a def. set, E a def. equiv. rel'n. New sort $S_{D,E}$.

$$E \rightrightarrows D \longrightarrow S_{D,E}$$

T^{grp} (generalized imaginaries)

G a definable groupoid.

New sorts Ob_G^*, $Mor^*_{a,b}$ ($a \in Ob\,G \cup Ob^*G$, $b \in Ob_G^*$) s.t.

the union forms a concrete grpd, with one object of Ob_G^* in each isomorphism class.

$\mathcal{C}_T = \{$ alg. closed substructures of models of $T\}$

$P(N) = \{$ subsets of $\{1, \dots, N\}\}$

$P(N)^- = \{$ proper " " " $\}$

An (independent) amalgamation problem for T = a function

$$a : P(N)^- \longrightarrow \mathcal{C}(T)$$

$$a(s) = \operatorname{acl} \bigotimes_{i \in s} a(i).$$

A solution = extension to $P(N)$.

Def. (acl)

$A \subseteq M \models T$

acl(A) IS THE UNION OF ALL FINITE A-DEFINABLE SETS.

Thm (T stable) T_G has EI

provided (independent) 4-amalgamation holds,

over any $A = acl(A)$.

Thm TFAE: (T stable)

EI

1) 4-amalgamation

2) Unique 3-amalgamation.

3) $\{(A, G) : A \in C_T, G \in \mathrm{Aut}\,A\}$ has 3-amalgamation.

4) Finite internal covers split. $/acl(\emptyset)$.

5) Every concrete groupoid w. one isom class, finite auto groups definably is equivalent to a finite group action.

Prop. $T = ACF_0 + \{V_i\}, \{PV_i\}$.

- $EI \leftrightarrow \{V_i\}$ closed under $\otimes$, duals.

(Tanaka, cf. Kamensky)

- Assume irreducibles are 1-dim'l.

T has $GEI \leftrightarrow$ interpret'l roots, i.e. $\forall V$, $\dim V = 1$ $\exists$ $U^{\otimes k} \cong V$.

- **Cor** $PF + \{V_i\}_{i \in \Gamma}$, $V_i \otimes V_j \cong V_{i+j}$ has E.I (Γ divisible.)

Def_T = category of

0-definable sets, maps.

$K(T)$ = Groth. semi-ring.

$= \pm \mathrm{Ob}\,\mathrm{Def}_T / \mathrm{iso}$, $\dot\cup$, $\times$.

Given unif. definable $X_a = \{x : (x,a) \in X\}$

write: $[X] = \sum_a [X_a]$

$[X_a] \in K(T_a)$.

K_c = obtained from K by

imposing:

· $[X] \longmapsto [X]$

$K(T_b) \longrightarrow K(T_{ab})$

injective, $X \in \mathrm{Def}_{T_b}$

(Can restrict to tucas of certain groups.)

LEM. 2 The interpretation

$$k \xrightarrow[\alpha]{\text{in}} R$$

Gives:

$$K_c(Th(k)) \xrightarrow[K_c(\alpha)]{} K_c(Th(R))$$

α_* IS A RING ISOMORPHISM.

(ANALOG FOR K_0 IS WRONG.)

A basic
THM Let $\mathbb{U}$ be a stably embedded union of sorts of $\mathbb{V}$;

Assume $\mathbb{V}$ is $\mathbb{U}$-analysable

($\mathbb{V} = \mathbb{U}_n \supseteq \mathbb{U}_{n-1} \supseteq \dots \supseteq \mathbb{U}_1 = \mathbb{U}$, $\mathbb{U}_{i+1}$ internal to $\mathbb{U}_i$)

$\mathbb{U}$ Stab[illegible]

Embedded in $\mathbb{V}$

\+ (WEAK) CHAIN CONDITION ON INTERSECTIONS OF CONJUGATES OF STABILIZERS OF LIAISON GROUPS OF $\mathbb{U}_{i+1}/\mathbb{U}_i$
(e.g. SOLVABLE, LINEAR, ...)

Then the natural homomorphism

$$K_c(Th(\mathbb{U})) \longrightarrow K_c(Th(\mathbb{V}))$$

is an isomorphism.

Proof uses liaison groupoids.

Sketch of talk: (ACFA = FA)

$$K_0(ACF) \ni V \quad \text{VARIETY}$$

$$\downarrow$$

$$K_c(PF) \quad \text{(CAVALIERI)} \quad V(F)$$

$$\downarrow$$

$$K_c(\underline{FA^0}) \quad V(Fix(\sigma)) \quad \bar{x} \in V,\ \sigma x_i = x_i$$

residue field

DIFF. VARIETY, FINITE TOTAL DIM'N

= 0 - TRANSF. DIM'N.

$$\begin{tikzcd}
 & K_c(FA^0) \arrow[dl, "\not\cong"'] \arrow[dr, "\cong"] & \\
K_c(\underline{VFA^0_t}) & & K_c(VFA^0_t)
\end{tikzcd}$$

$VAL(t) > 0$ $\qquad$ $VAL(t) \leq 0$ $\qquad$ $[V(Fix(\sigma))]$

$$\begin{tikzcd}
K_c(\underline{VFA^0_t}) & & K_c(VFA^0_t) \\
 & K_c(FA^0_t) \arrow[ul, "\beta_0"] \arrow[ur, "\beta_\infty"'] & \\
K_c(VF^0_t) \arrow[uu, leftarrow, "\sigma = \mathrm{FROB}_p"] & & K_c(VF^0_t) \arrow[uu, leftarrow, "\sigma = \mathrm{FROB}_p"'] \\
 & KC(FA^0_t) \arrow[ul] \arrow[ur] \arrow[d] & \\
 & \mathbb{Z} \arrow[ul, leftarrow, bend left] \arrow[ur, leftarrow, bend right] &
\end{tikzcd}$$

infinitesimally

t = a transcendental constant

0 - finite

Poncelet

(numbers) $\qquad \mathbb{Z} \qquad |V(\mathbb{F}_p)|$

Definition

$$K_w = K_c(FA^0) \otimes K_c(FA^0)$$

$$K_c(FA^0) \xrightarrow{\beta_0} K_c(FA^0), \quad K_c(FA^0) \xrightarrow{\beta_\infty} K_c(FA^0)$$

COMPARISON WITH MOTIVES

QUESTION U, V SMOOTH PROJECTIVE VARIETIES.

IF U, V HAVE ISOMORPHIC MOTIVES,

SHOW: $[U(\mathrm{Fix}(\sigma))] = [V(\mathrm{Fix}(\sigma))] \in K_W$.

Pf $\quad S \subseteq U \times V$

$\quad T \subseteq V \times U$

$$T \circ S \underset{\mathrm{RAT}}{\sim} n \cdot \Delta_U, \quad S \circ T \underset{\mathrm{RAT}}{\sim} n \cdot \Delta_V$$

After replacing S, T by sufficiently "general position" rep's,

$$X(T \circ S) := \{x \in U : (x, \sigma(x)) \in T \circ S\}$$

$$[X(T \circ S)] = n \cdot [U(\mathrm{Fix}(\sigma))] = n\,[X(\Delta_U)]$$

$$[X(S \circ T)] = n \cdot [V(\mathrm{Fix}(\sigma))]$$

Let $Y = \{(x, y) \in S : (\sigma(x), y) \in T\}$

$$\begin{tikzcd} & Y \arrow[dl, "x"'] \arrow[dr, "\sigma^{-1}(y)"] & \\ X(T \circ S) & & X(S \circ T) \end{tikzcd}$$

Problems

1) Categoricity $K(RV)/_{I} \simeq K(VF)$.

2) Existentially definable equiv. relations.

$x E y \iff \exists u \ \varphi(x,y,u)$

$Mon'(x,y) = \{u : \ \varphi(x,y,u)\}$

4) Describe $T^{\text{a-gph}}$, higher generalized imaginaries with n-amalgamation.

5)

$K_0(VF_0^d) \rightrightarrows K_0(ACF_0^0) \longleftarrow$ K Motivel