Geometric Structures on Manifolds II: Complete affine structures

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Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's.
- Closely related to surfaces with hyperbolic structures and deformations which "stretch" or "shrink" the surface.

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Euclidean manifolds

- A *Euclidean manifold* is modeled on Euclidean space $\mathbb{R}^n$ with coordinate changes *affine transformations*

$$p \xrightarrow{\gamma} \mathsf{L}(\gamma)p + u(\gamma)$$

where the *linear part* $\mathsf{L}(\gamma)$ is an orthogonal linear map.

- If M is compact, it's geodesically complete and isometric to $\mathbb{R}^n/\Gamma$ where Γ finite extension of a subgroup of *translations* $\Lambda := \Gamma \cap \mathbb{R}^n \cong \mathbb{Z}^k$ (Bieberbach 1912);
- M finitely covered by flat torus $\mathbb{R}^n/\Lambda$ (where $\Lambda \subset \mathbb{R}^n$ is a lattice).
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Consequences of Bieberbach theorems

- For Euclidean manifolds:
  - Only finitely many topological types in each dimension.
  - Only one *commensurability* class.
  - $\pi_1(M)$ is finitely generated.
  - $\pi_1(M)$ is finitely presented.
  - $\chi(M) = 0$.
- None of these properties hold in general for complete affine manifolds!

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Proper affine actions

- Suppose $M = \mathbb{R}^n/\Gamma$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, Γ must act:
  - Discretely: ($\Gamma \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
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    Properly: (Go to ∞ in $\Gamma \Longrightarrow$ go to ∞ in every orbit Γx)

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    More precisely, the map

    $$\begin{aligned} \Gamma \times X &\longrightarrow X \times X \\ (\gamma, x) &\longmapsto (\gamma x, x) \end{aligned}$$

    is a proper map (preimages of compacta are compact).
    Unlike Riemannian isometries, discreteness does not imply properness.

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Geodesic completeness

- An affine structure is a *flat torsionfree affine connection.*
- Even if M is compact, it may be *incomplete.*
  - Example: *Hopf manifold*
    Quotient $V \setminus \{0\}/\langle A \rangle$, where $V \xrightarrow{A} V$ *linear expansion.*
  - Diffeomorphic to $S^{n-1} \times S^1$.
  - Geodesics aimed at the origin don't extend. . .
- *Geodesic completeness* $\Longleftrightarrow$ developing map bijective.
- Affine holonomy group $\Gamma \subset \mathrm{Aff(E)}$ acts properly, discretely, freely on E.

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Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - Γ is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathsf{L}(\Gamma) \subset \mathsf{O}(2,1)$ is isomorphic to Γ.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
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- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
    - No obvious metrics:
    - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available
    - Available tools: parallelism, geodesics.
    - Equivalently this structure is a geodesically complete torsionfree affine connection on M (a notion of parallelism).
    - Even Lorentzian structures are not metric spaces.

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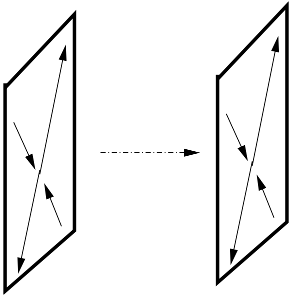

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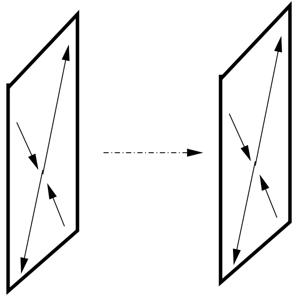

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Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $E^{2,1}/\Gamma$ is a *closed geodesic*. Like surfaces, most loops are freely homotopic to (unique) closed geodesics.

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- If $G = \pi_1(M)$ is virtually solvable (necessarily virtually polycyclic), then $G \hookrightarrow \mathrm{Aff}(\mathbb{R}^n)$ extends to $H \subset \mathrm{Aff}(\mathbb{R}^n)$, with $\pi_0(H)$ finite and H^0 acting simply transitively on $\mathbb{R}^n$.
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Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

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Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1x_2 + y_1y_2 - z_1z_2$$

  and Minkowski space $\mathrm{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathrm{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathrm{O}(2,1)$.
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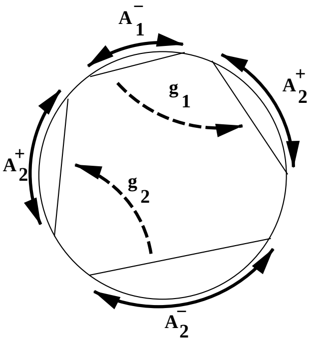
A Schottky group



- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
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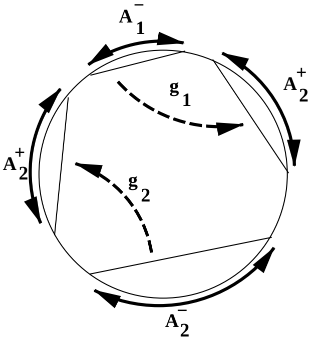

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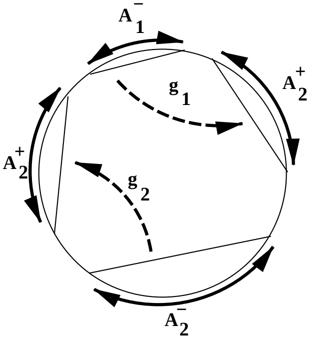

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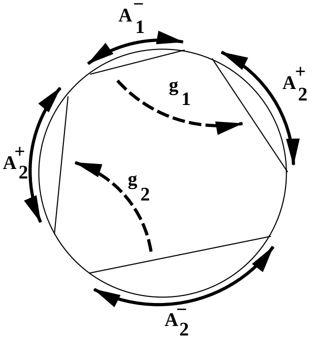

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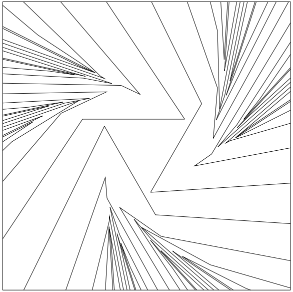

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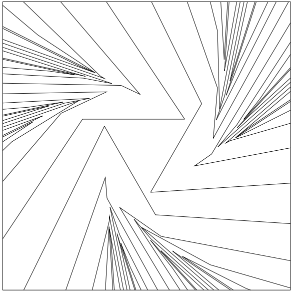

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{L} \mathrm{GL}(3,\mathbb{R})$ be the *linear part.*
  - $L(\Gamma)$ (conjugate to) a *discrete* subgroup of $O(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow S := \mathsf{H}^2/L(\Gamma)$$

  where S complete hyperbolic surface.
  - Mess (1990): Σ not compact .
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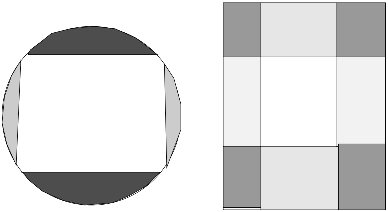

- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

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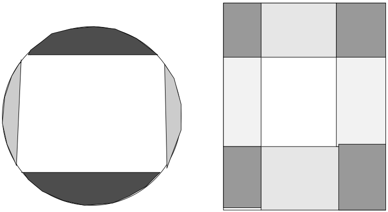

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- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
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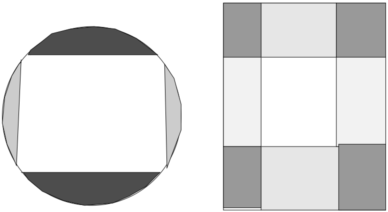
Slabs don't work!



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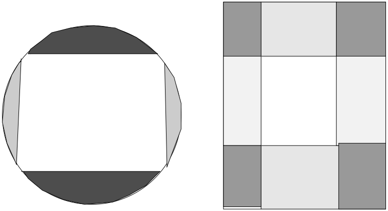

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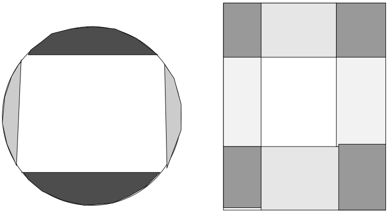

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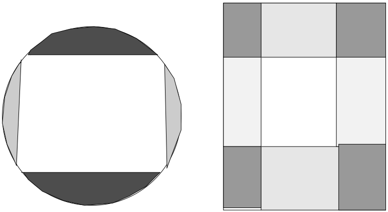

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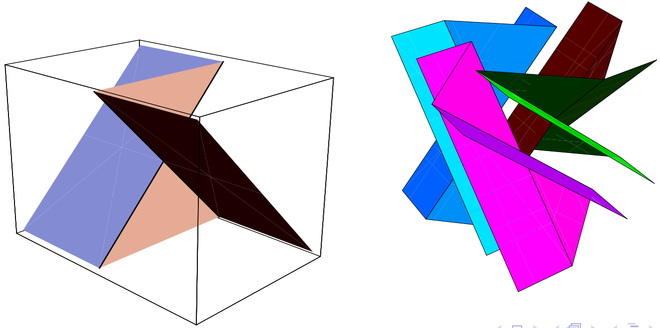
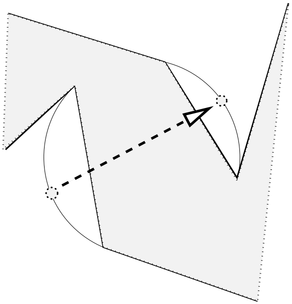
Drumm's Schottky groups

The classical construction of Schottky groups fails using affine half-spaces and slabs. Drumm's geometric construction uses *crooked planes*, PL hypersurfaces adapted to the Lorentz geometry which bound fundamental polyhedra for Schottky groups.

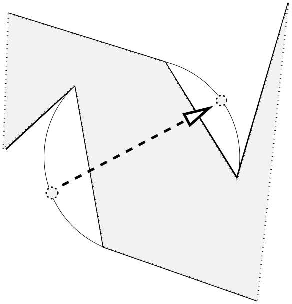


Crooked polyhedron for a boost



- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $E^{2,1}$;
- Extend outside light cone in $E^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



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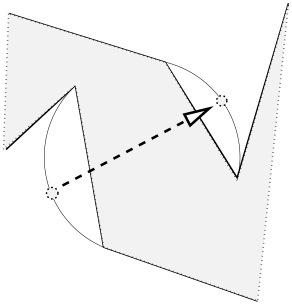

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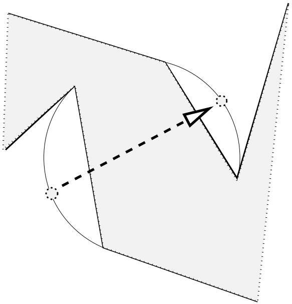

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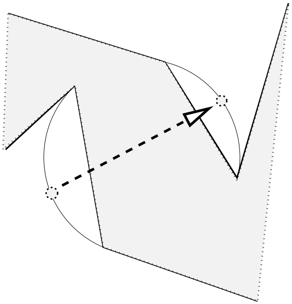

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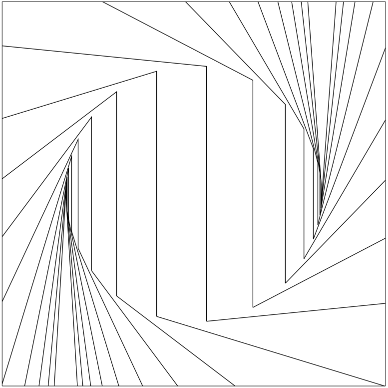
Images of crooked planes under a linear cyclic group



The resulting tessellation for a linear boost.

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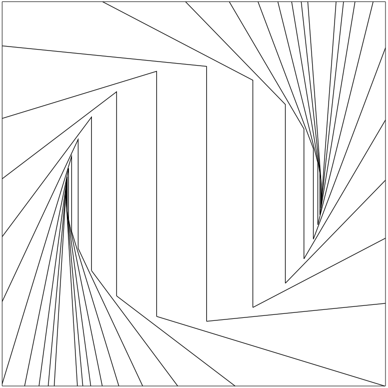

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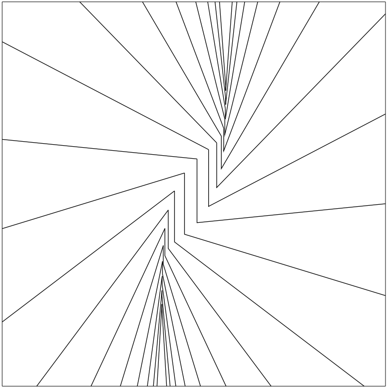
Images of crooked planes under an affine deformation



- Adding translations frees up the action
- — which is now proper on *all* of $E^{2,1}$.

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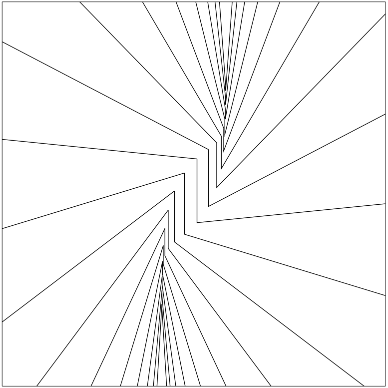

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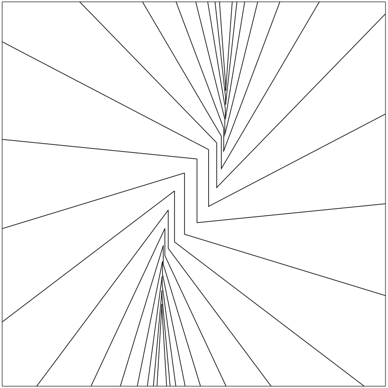

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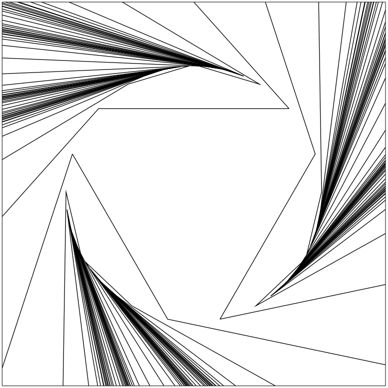
Linear action of Schottky group



Crooked polyhedra tile H^2 for subgroup of $O(2,1)$.

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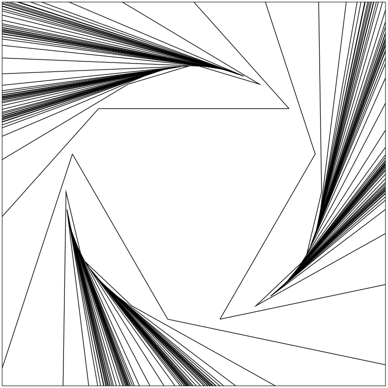

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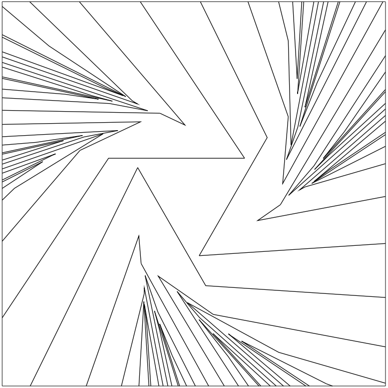

Carefully chosen affine deformation acts properly on $E^{2,1}$.

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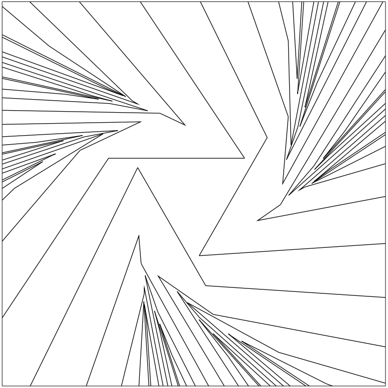

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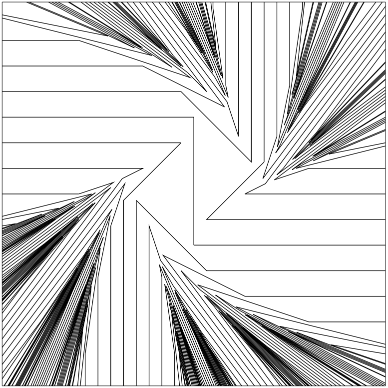
Affine action of level 2 congruence subgroup of $\mathrm{GL}(2, \mathbb{Z})$



Proper affine deformations exist even for *lattices* (Drumm).

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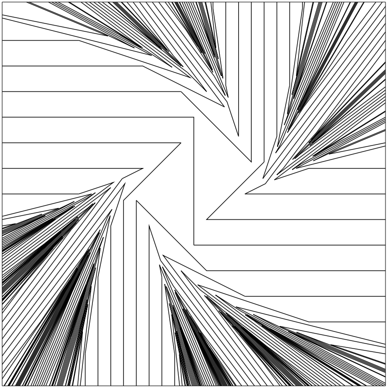

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Classification of Margulis spacetimes

- Mess's theorem (S noncompact) is the only obstruction for the existence of a proper affine deformation:
  - (Drumm 1990) S *noncompact* complete hyperbolic surface with finitely generated $\pi_1(S)$ admits *proper* affine deformation. M^3 is a solid handlebody.
- **Theorem:** (Charette-Drumm-G-Labourie-Margulis) The deformation space of complete affine structures on a solid handlebody Σ of genus 2 consists of four components, one for each topogical type of surface S with $\pi_1(S) \cong \mathbb{Z} \star \mathbb{Z}$. The component corresponding to S is a bundle of open convex cones over the Fricke space $\mathfrak{F}(S)$.

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