Geometric Structures on Manifolds III: Three-dimensional Margulis spacetimes

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Complete affine 3-manifolds

- Every complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$, where $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ is a discrete subgroup acting freely and properly, is either:
  - Iterated S^1 or $\mathbb{R}$-fibration (Γ solvable); or
  - Complete flat Lorentzian 3-manifold

$$M^3 = \mathsf{E}^{2,1}/\Gamma$$

where Γ is an affine deformation of a a *non-cocompact Fuchsian representation*

$$\Gamma_0 \overset{\rho_0}{\hookrightarrow} \mathsf{SO}(2,1).$$

Notation and terminology

- Denote the Lorentzian affine space $\mathsf{E} := \mathsf{E}^{2,1}$.
- The underlying Lorentzian vector space is $\mathsf{V} := \mathbb{R}^{2,1}$.
  It consists of translations $\mathsf{E} \longrightarrow \mathsf{E}$.
- Denote the discrete embedding of a Fuchsian group Γ_0 by ρ_0; the corresponding hyperbolic surface is $\Sigma := \mathsf{H}^2/\Gamma_0$.
- An affine deformation of ρ_0 will be denoted ρ, and its image $\Gamma = \rho(\Gamma_0)$. Furthermore $\Gamma_0 \cong \mathsf{L}(\Gamma)$.
- If ρ is *proper*, then the quotient is a complete flat Lorentz 3-manifold M^3 with fundamental group $\pi_1(M^3) \cong \Gamma$.

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Affine deformations

- Start with a noncocompact Fuchsian group $\Gamma_0 \subset \mathrm{SO}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$

$$\begin{tikzcd} & \mathrm{Isom(E)} \cong \mathsf{V} \rtimes \mathrm{SO}(2,1) \arrow[d, "\mathsf{L}"] \\ \Gamma_0 \arrow[ur, dashed, "\rho"] \arrow[r, hook, "\rho_0"] & \mathrm{SO}(2,1) \end{tikzcd}$$

  determined by its translational part

$$u \in Z^1(\Gamma_0, \mathsf{V}).$$

- Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.
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The hyperbolic surface

- $\Sigma := \mathrm{H}^2/\Gamma_0$ is *naturally associated* to the complete flat Lorentz 3-manifold.
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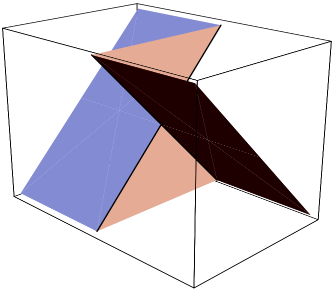
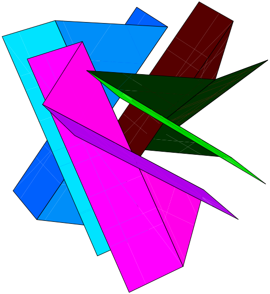
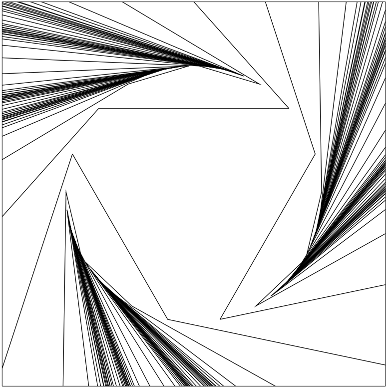
Crooked planes

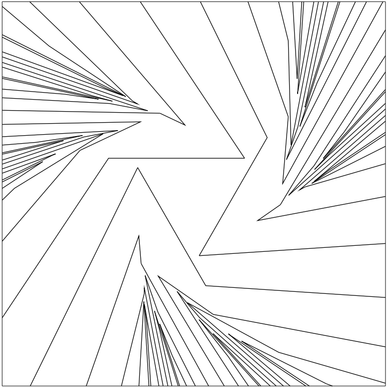




Linear action of ultra-ideal triangle group



Affine deformation of ultraideal triangle group



Carefully chosen affine deformation acts properly on $E^{2,1}$.

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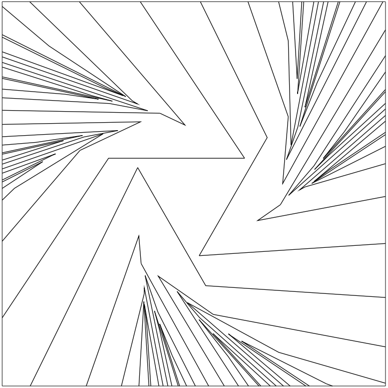

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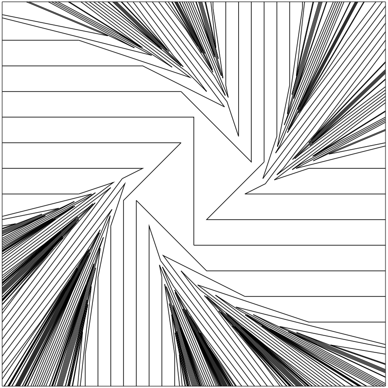
Affine action of level 2 congruence subgroup of $\mathrm{GL}(2, \mathbb{Z})$



An arithmetic example

- For $i = 1, 2, 3$ choose three positive integers μ_1, μ_2, μ_3. Then the subgroup Γ of $\mathrm{Sp}(4, \mathbb{Z})$ generated by

$$\begin{bmatrix} -1 & -2 & \mu_1+\mu_2-\mu_3 & 0 \\ 0 & -1 & 2\mu_1 & -\mu_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -\mu_2 & -2\mu_2 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

  is a proper affine deformation of a rank two free group.
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The deformation space

- The deformation space of hyperbolic structures is the *Fricke space*
$$\mathfrak{F}(S) \approx [0,\infty)^b \times (0,\infty)^{b-3\chi(\Sigma)}.$$
where $\partial\Sigma$ has b components.
- Thus the space of affine deformations of Γ_0 is the product
$$\mathfrak{F}(S) \times H^1(\Gamma_0, \mathsf{V})$$
- *Similarity classes* of (nontrivial) affine deformations of Γ_0 form the projective space $\mathsf{P}H^1(\Gamma_0, \mathsf{V})$
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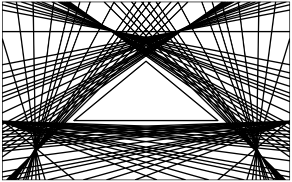
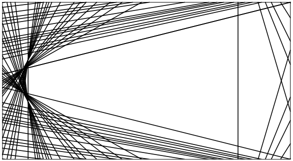
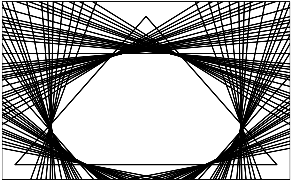
Deformation spaces for surfaces with $\chi(\Sigma)$



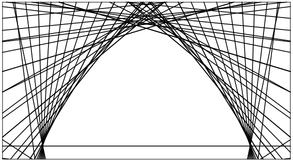
(c) Three-holed sphere



(d) Two-holed $\mathbb{R}P^2$



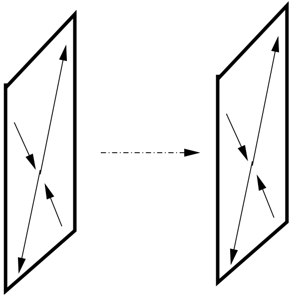
(e) One-holed torus



(f) One-holed Klein bottle

Example: Cyclic groups

- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1) \subset GL(3,\mathbb{R})$. A fundamental domain is the *slab* bounded by two parallel planes.



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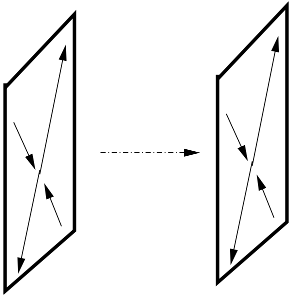

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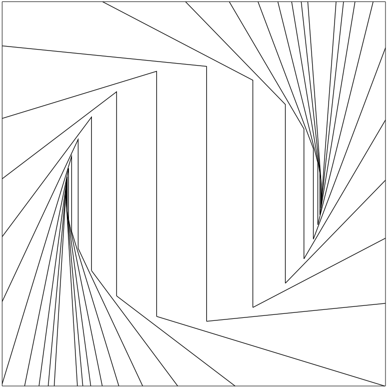
Images of crooked planes under a linear cyclic group



The resulting tessellation for a linear boost.

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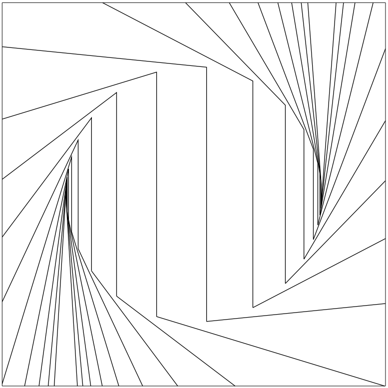

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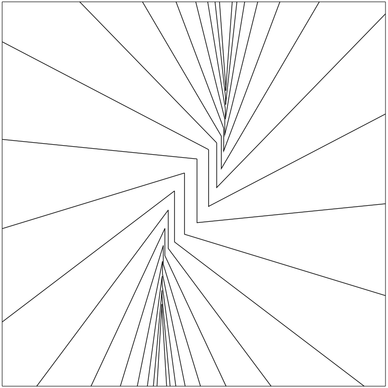

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- — which is now proper on *all* of $E^{2,1}$.

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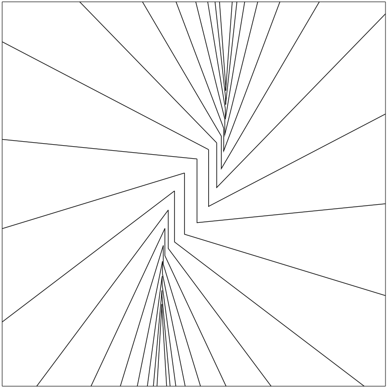

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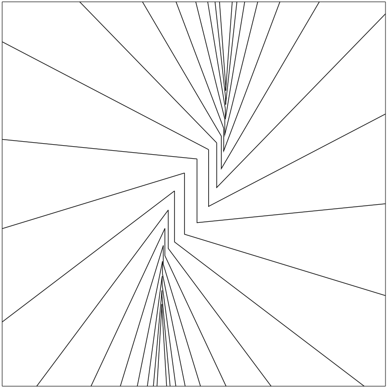

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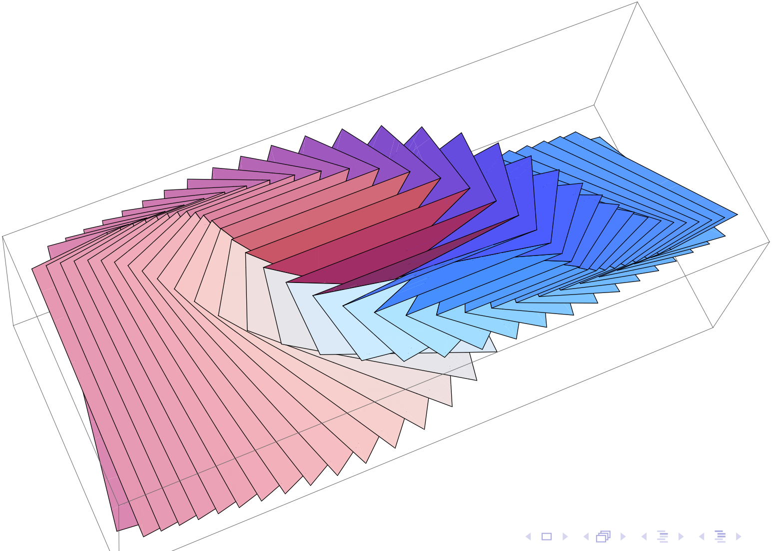
A foliation by crooked planes



Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathsf{E}^{2,1}/\Gamma$ is a *closed geodesic*. Like surfaces, most loops are freely homotopic to (unique) closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

  - $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ in Σ^2
  - $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ in M^3.

- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric.
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$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

  - $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ in Σ^2
  - $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ in M^3.
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric.
  - γ translates along C_γ by $\alpha(\gamma)$.

Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(\mathsf{L},u)} \mathsf{Isom}(\mathsf{E}^{2,1})$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathsf{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- (Margulis 1983) If ρ acts properly, either
  - $\alpha_u(\gamma) > 0 \ \forall \gamma \neq 1$, or
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- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

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Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
  - The Lorentzian vector space $\mathbb{R}^{2,1}$ corresponds to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Killing form, and the action of $\mathrm{O}(2,1)$ is the adjoint representation.
  - This Lie algebra comprises the *Killing vector fields*, infinitesimal isometries, of H^2.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0, \mathsf{V})$.

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Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathsf{V})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\bigg|_{t=0} \ell_t(\gamma)$$

- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

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Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
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The Crooked Plane Conjecture

- Conjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
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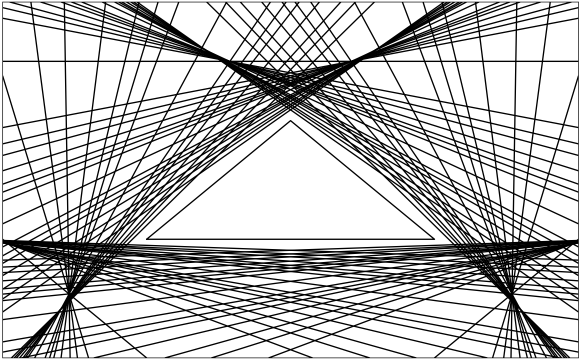
Functionals $\alpha(\gamma)$ when $\Sigma \approx$ three-holed sphere



Charette-Drumm-Margulis functionals of $\partial\Sigma$ completely describe deformation space as $(0,\infty)^3$.

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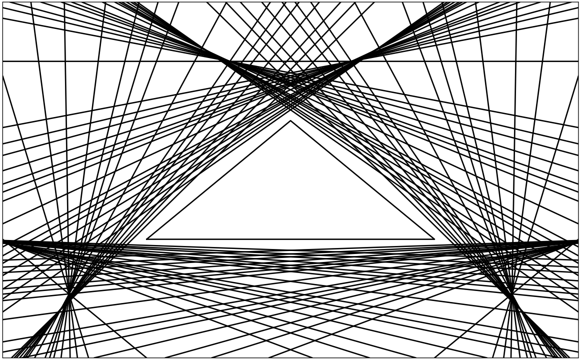

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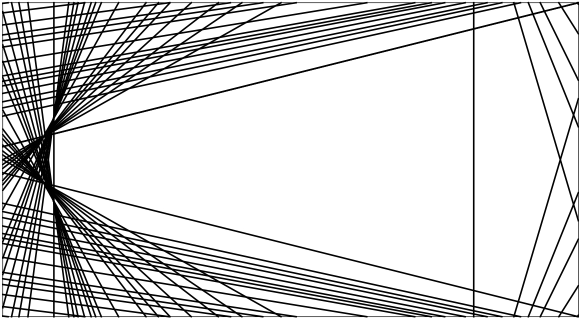
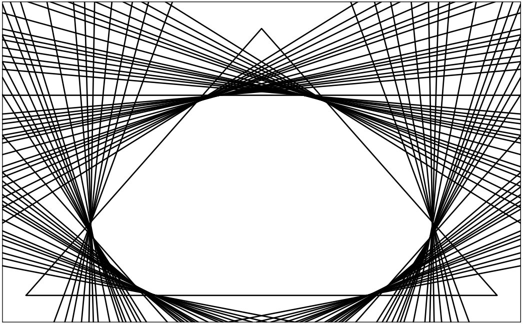


Deformation space is quadrilateral bounded by the four lines defined by CDM-functionals of $\partial\Sigma$ and the two orientation-reversing interior simple loops.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed torus



- Properness region bounded by infinitely many intervals, each corresponding to simple loop.
- ∂-points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).

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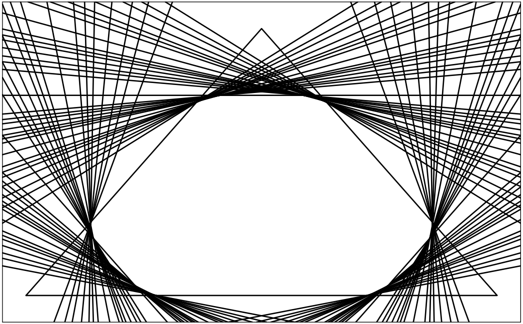

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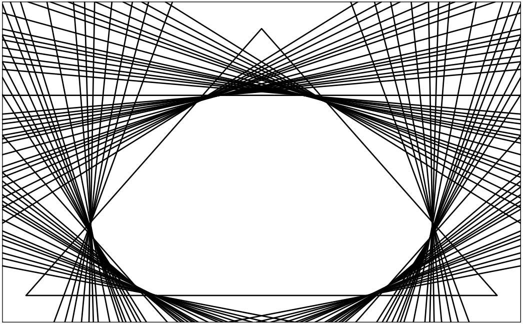

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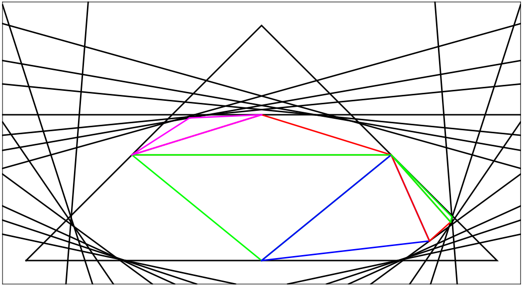
Realizing an ideal triangulation by crooked planes



- Properness region tiled by triangles.
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- Flip of ideal triangulation $\longleftrightarrow$ moving to adjacent triangle.

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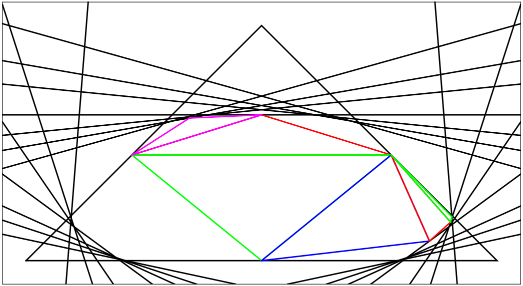

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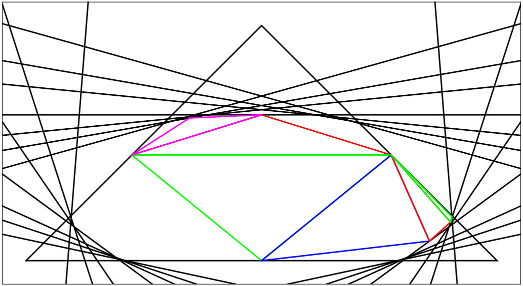

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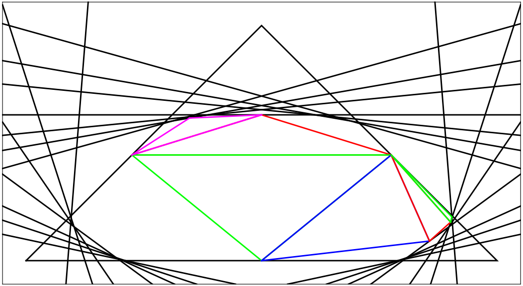

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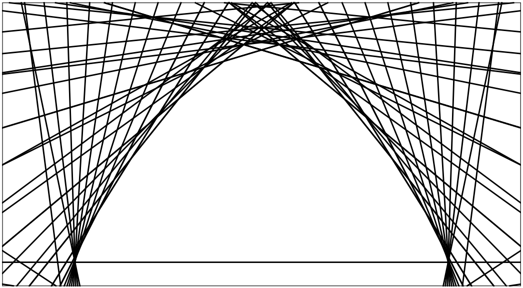
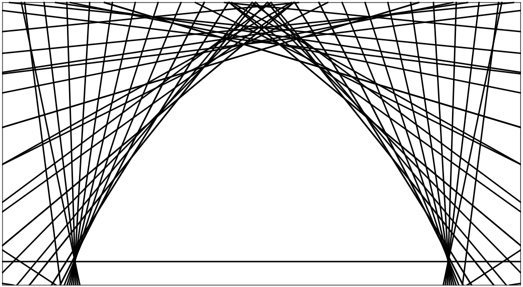
- Flip of ideal triangulation $\longleftrightarrow$ moving to adjacent triangle.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed Klein bottle



Properness region bounded by infinitely many intervals, each defined by CDM-invariants of simple orientation-reversing loops, arranged cyclically, and the one orientation-preserving interior simple loop.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed Klein bottle



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