# The equivalence between radiance and retrieval assimilation

## a mathematical perspective

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# Outline

# Introduction (1/2)

- The late 1970s saw the first attempts to assimilate temperature retrievals from satellite sounders for numerical weather prediction (NWP).
- Initial results had a modest impact on forecast skill (best over oceans).
- In the 1980's, improvements in NWP models caused reduction of impact of satellite data.
- Problems due to background information contained in retrievals inconsistent with that used in data assimilation: bias.
- Early 1990's: variational assimilation for NWP. Observation operator can be nonlinear: assimilation of satellite radiances.

# Introduction (2/2)

- Radiance assimilation has since proved to obtain excellent results, especially with passive remote sounders of temperature and humidity.
- Simple error structure and effective observation monitoring.
- Problems when NWP model state does not provide sufficiently reliable information.
  - assimilation of cloud-affected radiances in the infrared.
  - atmospheric composition sounding.
- Observation operator represents solution of radiative transfer eq. (not always available for NWP) and has to model characteristics of the instrument. Very high number of channels for high-res sounders.
- Recent interest (e.g., Joiner and Da Silva, 1998; Rodgers 2000, Migliorini et al., 2008) in efficient assimilation of reduced amount of sounding data: e.g. efficient assimilation of retrievals.

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# Characterization of radiance measurements for assimilation (1/2)

- Measured radiance $\mathbf{y}^o \in \mathbb{R}^m$, the true state of a system $\mathbf{x}^t \in \mathbb{R}^n$

$$\mathbf{y}^o_{\mathrm{rad}} = H(\mathbf{x}^t) + \epsilon^o_{\mathrm{rad}} \tag{1}$$

  $H(\mathbf{x}^t)$ observation operator in $\mathbf{x}^t$, $\epsilon^o_{\mathrm{rad}}$ radiance measurement error, assumed Gaussian, unbiased and with covariance $\mathbf{R}_{\mathrm{rad}} \in \mathbb{R}^{m \times m}$.
- Close to $\mathbf{x}_i$ we can write

$$\mathbf{y}^o_{\mathrm{rad}} \simeq H(\mathbf{x}_i) + \mathbf{H}^{(i)}(\mathbf{x}^t - \mathbf{x}_i) + \epsilon^o_{\mathrm{rad}} \tag{2}$$

  where $\mathbf{H}^{(i)} \equiv (\partial H / \partial \mathbf{x})_{\mathbf{x}=\mathbf{x}_i} \in \mathbb{R}^{m \times n}$.
- We define

$$\mathbf{y}^{(i)}_{\mathrm{rad}} \equiv \mathbf{y}^o_{\mathrm{rad}} - H(\mathbf{x}_i) + \mathbf{H}^{(i)}\mathbf{x}_i \simeq \mathbf{H}^{(i)}\mathbf{x}^t + \epsilon^o_{\mathrm{rad}}. \tag{3}$$

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# Characterization of radiance measurements for assimilation (2/2)

- Replace $\mathbf{R}_{\mathrm{rad}} \simeq \mathbf{L}_p \mathbf{\Sigma}_p^2 \mathbf{L}_p^T$, where $\mathbf{L}_p \in \mathbb{R}^{m \times p}$ and $p \leq m$ non-zero (or nonsmall, as compared to machine precision) eigenvalues of $\mathbf{R}_{\mathrm{rad}}$ in $\mathbf{\Sigma}_p^2 \in \mathbb{R}^{p \times p}$.
- Define $\mathbf{y}_{\mathrm{rad}}^{(i)\prime} \equiv \mathbf{\Sigma}_p^{-1} \mathbf{L}_p^T \mathbf{y}_{\mathrm{rad}}^{(i)} \in \mathbb{R}^p$. From Eq. 3 we can write

$$\mathbf{y}_{\mathrm{rad}}^{(i)\prime} \simeq \mathbf{H}_{\mathrm{rad}}^{(i)\prime} \mathbf{x}^t + \epsilon'_{\mathrm{rad}}, \tag{4}$$

  where $\mathbf{H}_{\mathrm{rad}}^{(i)\prime} \equiv \mathbf{\Sigma}_p^{-1} \mathbf{L}_p^T \mathbf{H}^{(i)} \in \mathbb{R}^{p \times n}$ and where the covariance of $\epsilon'_{\mathrm{rad}} \equiv \mathbf{\Sigma}_p^{-1} \mathbf{L}_p^T \epsilon_{\mathrm{rad}}^o$ is the unit matrix $\mathbf{I}_p \in \mathbb{R}^{p \times p}$.
- Also define $\mathbf{y}_{\mathrm{rad}}^{o\prime} \in \mathbb{R}^p$ as $\mathbf{y}_{\mathrm{rad}}^{o\prime} \equiv \mathbf{\Sigma}_p^{-1} \mathbf{L}_p^T \mathbf{y}_{\mathrm{rad}}^o$ and $H'(\mathbf{x}^t) \in \mathbb{R}^p$ as $H'(\mathbf{x}^t) \equiv \mathbf{\Sigma}_p^{-1} \mathbf{L}_p^T H(\mathbf{x}^t)$. Eq. 1 can then be written as

$$\mathbf{y}_{\mathrm{rad}}^{o\prime} = H'(\mathbf{x}^t) + \epsilon'_{\mathrm{rad}}. \tag{5}$$

Equivalence of radiance and retrieval assimilation in the case when the state of the system is well observed.

- Maximum likelihood estimate of $\mathbf{x}^t$ when Eq. 5 is valid is minimum of

$$J_o(\mathbf{x}) = \frac{1}{2}(\mathbf{y}^{o\prime}_{\mathrm{rad}} - H'(\mathbf{x}))^T(\mathbf{y}^{o\prime}_{\mathrm{rad}} - H'(\mathbf{x})). \tag{6}$$

- When number of components of $\mathbf{y}^{(i)\prime}_{\mathrm{rad}}$ is $p \geq n$ and $\mathbf{H}^{(i)\prime}_{\mathrm{rad}}$ is full rank (=$n$) we can instead minimize

$$J_o^{(i)}(\mathbf{x}) = \frac{1}{2}(\mathbf{y}^{(i)\prime}_{\mathrm{rad}} - \mathbf{H}^{(i)\prime}_{\mathrm{rad}}\mathbf{x})^T(\mathbf{y}^{(i)\prime}_{\mathrm{rad}} - \mathbf{H}^{(i)\prime}_{\mathrm{rad}}\mathbf{x}), \tag{7}$$

  The cost function $J_o^{(i)}(\mathbf{x})$ approximates $J_o(\mathbf{x})$ around a small neighbourhood of $\mathbf{x}_i$.

# The over-determined least squares problem

## Assimilation of radiances (2/3)

- We get

$$\mathbf{x}_{i+1} = (\mathbf{H}^{(i)\prime T}_{\rm rad}\mathbf{H}^{(i)\prime}_{\rm rad})^{-1}\mathbf{H}^{(i)\prime^{T}}_{\rm rad}\mathbf{y}^{(i)\prime}_{\rm rad} \tag{8}$$

  Gauss-Newton iteration with positive definite Hessian matrix $\mathbf{H}^{(i)\prime T}_{\rm rad}\mathbf{H}^{(i)\prime}_{\rm rad} \in \mathbb{R}^{n\times n}$.

- At convergence $\mathbf{x}_{i+1} \simeq \mathbf{x}_i \equiv \hat{\mathbf{x}}_{\rm ML}$, $\mathbf{H}^{(i+1)\prime}_{\rm rad} \simeq \mathbf{H}^{(i)\prime}_{\rm rad} \equiv \hat{\mathbf{H}}'_{\rm rad}$, $\mathbf{y}^{(i+1)\prime}_{\rm rad} \simeq \mathbf{y}^{(i)\prime}_{\rm rad} \equiv \hat{\mathbf{y}}'_{\rm rad}$ and Eq. 8 becomes

$$\hat{\mathbf{x}}_{\rm ML} = (\hat{\mathbf{H}}'^{T}_{\rm rad}\hat{\mathbf{H}}'_{\rm rad})^{-1}\hat{\mathbf{H}}'^{T}_{\rm rad}\hat{\mathbf{y}}'_{\rm rad} \tag{9}$$

  $\hat{\mathbf{x}}_{\rm ML}$ is the analysis (3D) or the retrieval (e.g., vertical profile).

- From Eq. 4 at convergence we can write

$$\hat{\mathbf{x}}_{\rm ML} \simeq \mathbf{x}^t + (\hat{\mathbf{H}}'^{T}_{\rm rad}\hat{\mathbf{H}}'_{\rm rad})^{-1}\hat{\mathbf{H}}'^{T}_{\rm rad}\epsilon'_{\rm rad} = \mathbf{x}^t + \epsilon_{\rm ML}. \tag{10}$$

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# The over-determined least squares problem

## Assimilation of radiances (3/3)

- In this approximation, the retrieval error covariance can be written as $\hat{\mathbf{S}}_{\epsilon_{\mathrm{ML}}} = (\hat{\mathbf{H}}'^{T}_{\mathrm{rad}}\hat{\mathbf{H}}'_{\mathrm{rad}})^{-1}$.
- This is justified when $H(\mathbf{x})$ can be replaced with its first-order Taylor expansion about $\hat{\mathbf{x}}_{\mathrm{ML}}$, of radius $\simeq$ retrieval error. This can be checked.
- Let $\hat{\mathbf{H}}'_{\mathrm{rad}} = \mathbf{U\Lambda V}^T$, where $\mathbf{\Lambda} \in \mathbb{R}^{p\times n}$ has $n$ positive singular values
- We can write $\hat{\mathbf{S}}_{\epsilon_{\mathrm{ML}}} = \mathbf{V\Lambda}_n^{-2}\mathbf{V}^T$, where $\mathbf{\Lambda}_n^2 \in \mathbb{R}^{n\times n}$ is diagonal positive definite. We get

$$\begin{aligned} \mathbf{y}'_{\mathrm{ret}} &\equiv \mathbf{\Lambda}_n\mathbf{V}^T\hat{\mathbf{x}}_{\mathrm{ML}} \simeq \mathbf{\Lambda}_n\mathbf{V}^T\mathbf{x}^t + \epsilon'_{\mathrm{ML}} = \\ &= \mathbf{H}'_{\mathrm{ret}}\mathbf{x}^t + \epsilon'_{\mathrm{ML}} \end{aligned} \tag{11}$$

where the covariance of $\epsilon'_{\mathrm{ML}} \equiv \mathbf{\Lambda}_n\mathbf{V}^T\epsilon_{\mathrm{ML}}$ is the identity matrix.

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# The over-determined least squares problem

## Assimilation of maximum likelihood retrievals

- We want to determine ML estimate by assimilating $\mathbf{y}'_{\mathrm{ret}} \in \mathbb{R}^n$ with its rank-$n$ observation operator $\mathbf{H}'_{\mathrm{ret}} \equiv \mathbf{\Lambda}_n \mathbf{V}^T \in \mathbb{R}^{n \times n}$
- The estimate is found by minimizing

$$J_o^{\mathrm{ret}}(\mathbf{x}) = \frac{1}{2}(\mathbf{y}'_{\mathrm{ret}} - \mathbf{H}'_{\mathrm{ret}}\mathbf{x})^T(\mathbf{y}'_{\mathrm{ret}} - \mathbf{H}'_{\mathrm{ret}}\mathbf{x}). \tag{12}$$

- As the rank of $\mathbf{H}'_{\mathrm{ret}}$ is $n$ we have

$$\begin{aligned} \hat{\mathbf{x}}_{\mathrm{ML}}^{\mathrm{ret}} &= (\mathbf{H}'_{\mathrm{ret}})^{-1}\mathbf{y}'_{\mathrm{ret}} = (\mathbf{H}'_{\mathrm{ret}})^{-1}\mathbf{\Lambda}_n \mathbf{V}^T \hat{\mathbf{x}}_{\mathrm{ML}} = \\ &= \hat{\mathbf{x}}_{\mathrm{ML}}. \end{aligned} \tag{13}$$

- This proves the equivalence between radiance and retrieval assimilation for the overdetermined least squares problem, for moderately nonlinear observation operator around $\hat{\mathbf{x}}_{\mathrm{ML}}$

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# The ill-posed or under-determined problem

## Assimilation of radiances (1/2)

- Remote sounding measurements do not provide enough information to constrain all $n$ components of the state vector
- The maximum a posteriori estimate is found by minimizing

$$J(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}^b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}^b)+\frac{1}{2}(\mathbf{y}^{o\prime}_{\mathrm{rad}}-H'(\mathbf{x}))^T(\mathbf{y}^{o\prime}_{\mathrm{rad}}-H'(\mathbf{x})). \quad (14)$$

- As before, we can instead minimize a succession of

$$J^{(i)}(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}^b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}^b)+\frac{1}{2}(\mathbf{y}^{(i)\prime}_{\mathrm{rad}}-\mathbf{H}^{(i)\prime}_{\mathrm{rad}}\mathbf{x})^T(\mathbf{y}^{(i)\prime}_{\mathrm{rad}}-\mathbf{H}^{(i)\prime}_{\mathrm{rad}}\mathbf{x}). \quad (15)$$

- The MAP estimate $\hat{\mathbf{x}}_{\mathrm{MAP}} \in \mathbb{R}^n$ can be written as

$$\hat{\mathbf{x}}_{\mathrm{MAP}} = \mathbf{x}^b + \mathbf{K}(\hat{\mathbf{y}}'_{\mathrm{rad}} - \hat{\mathbf{H}}'_{\mathrm{rad}}\mathbf{x}^b) \quad (16)$$

with

$$\mathbf{K} \equiv \mathbf{B}\hat{\mathbf{H}}'^T_{\mathrm{rad}}(\hat{\mathbf{H}}'_{\mathrm{rad}}\mathbf{B}\hat{\mathbf{H}}'^T_{\mathrm{rad}} + \mathbf{I}_p)^{-1} \quad (17)$$

- $\mathbf{K} \in \mathbb{R}^{n\times p}$ is the *Kalman gain*, where $p < n$ for underdetermined problems
- Let us now define $\mathbf{S} \in \mathbb{R}^{p\times n}$ as the signal-to-noise matrix, of rank $r \leq \min(p, n) = p$, given by $\mathbf{S} \equiv \hat{\mathbf{H}}'_{\mathrm{rad}}\mathbf{B}^{1/2} = \mathbf{U}_r\mathbf{\Lambda}_r\mathbf{V}_r^T$.
- It is possible to show that $\mathrm{rank}(\mathbf{K}) = r$ and that

$$\mathbf{K} = \mathbf{B}^{1/2}\mathbf{V}_r\mathbf{\Lambda}_r(\mathbf{\Lambda}_r^2 + \mathbf{I}_r)^{-1}\mathbf{U}_r^T. \quad (18)$$

- When $H(\mathbf{x})$ can be replaced with its first-order Taylor expansion about $\hat{\mathbf{x}}_{\mathrm{MAP}}$ over a region of the state space where the posterior probability is significant we can write

$$\hat{\mathbf{x}}_{\mathrm{MAP}} \simeq \mathbf{x}^b + \mathbf{K}\hat{\mathbf{H}}'_{\mathrm{rad}}(\mathbf{x}^t - \mathbf{x}^b) + \mathbf{K}\epsilon'_{\mathrm{rad}} \quad (19)$$

- and calculate the covariance $\hat{\mathbf{P}}_{\epsilon_{\mathrm{MAP}}}$ of $\epsilon_{\mathrm{MAP}} \equiv \hat{\mathbf{x}}_{\mathrm{MAP}} - \mathbf{x}^t$.

# The ill-posed or under-determined problem

## Assimilation of MAP retrievals (1/2)

- Assume that the observation operator for the retrieval is approximately linear around a neighbourhood of $\hat{\mathbf{x}}_{\mathrm{MAP}}$ of radius comparable to the estimation error.
- $$\mathbf{y}_{\mathrm{ret}} \equiv \hat{\mathbf{x}}_{\mathrm{MAP}} - \mathbf{x}^b + \mathbf{K}\hat{\mathbf{H}}'_{\mathrm{rad}}\mathbf{x}^b \in \mathbb{R}^n \tag{20}$$
- from Eqs. 18 and 19 we can write

$$\begin{aligned} \mathbf{y}_{\mathrm{ret}} &\simeq \mathbf{K}\hat{\mathbf{H}}'_{\mathrm{rad}}\mathbf{x}^t + \epsilon_{\mathrm{ret}} \\ &= \mathbf{KSB}^{-1/2}\mathbf{x}^t + \epsilon_{\mathrm{ret}} \\ &= \mathbf{B}^{1/2}\mathbf{V}_r\mathbf{\Lambda}_r^2(\mathbf{\Lambda}_r^2 + \mathbf{I}_r)^{-1}\mathbf{V}_r^T\mathbf{B}^{-1/2}\mathbf{x}^t + \epsilon_{\mathrm{ret}} \end{aligned} \tag{21}$$

  where $\epsilon_{\mathrm{ret}} = \mathbf{K}\epsilon'_{\mathrm{rad}}$, with covariance equal to $\mathbf{KK}^T = \mathbf{B}^{1/2}\mathbf{V}_r\mathbf{\Lambda}_r^2(\mathbf{\Lambda}_r^2 + \mathbf{I}_r)^{-2}\mathbf{V}_r^T\mathbf{B}^{1/2}$.
- We now define $\mathbf{y}'_{\mathrm{ret}} \in \mathbb{R}^r$ as we can write

$$\mathbf{y}'_{\mathrm{ret}} \equiv \mathbf{\Lambda}_r^{-1}(\mathbf{\Lambda}_r^2 + \mathbf{I}_r)\mathbf{V}_r^T\mathbf{B}^{-1/2}\mathbf{y}_{\mathrm{ret}} \tag{22}$$

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- It follows that Eq. 21 can be written as
$$\mathbf{y}'_{\mathrm{ret}} \simeq \mathbf{\Lambda}_r \mathbf{V}_r^T \mathbf{B}^{-1/2} \mathbf{x}^t + \epsilon'_{\mathrm{ret}} \equiv \mathbf{H}'_{\mathrm{ret}} \mathbf{x}^t + \epsilon'_{\mathrm{ret}} \tag{23}$$
where the covariance of $\epsilon'_{\mathrm{ret}} \equiv \mathbf{\Lambda}_r^{-1}(\mathbf{\Lambda}_r^2 + \mathbf{I}_r)\mathbf{V}_r^T \mathbf{B}^{-1/2} \epsilon_{\mathrm{ret}}$ is equal to the identity matrix $\mathbf{I}_r \in \mathbb{R}^{r \times r}$.
- From the previous definitions it follows that we can also write
$$\mathbf{y}'_{\mathrm{ret}} = \mathbf{U}_r^T \hat{\mathbf{y}}'_{\mathrm{rad}}. \tag{24}$$
$$\mathbf{H}'_{\mathrm{ret}} = \mathbf{U}_r^T \mathbf{S} \mathbf{B}^{-1/2} = \mathbf{U}_r^T \hat{\mathbf{H}}'_{\mathrm{rad}}. \tag{25}$$
so that both $\mathbf{y}'_{\mathrm{ret}}$ and $\mathbf{H}'_{\mathrm{ret}}$ can also be calculated from quantities in radiance space.
- From Eq. 23 it follows that the covariance of $\mathbf{y}'_{\mathrm{ret}}$ results equal to $\mathbf{\Lambda}_r^2 + \mathbf{I}_r$. Each component of $\mathbf{y}'_{\mathrm{ret}}$ then provides an information content given by $(1/2)\ln(1 + \lambda_i^2)$.
- The transformed retrieval $\mathbf{y}'_{\mathrm{ret}}$ can be assimilated by means of its observation operator $\mathbf{H}'_{\mathrm{ret}}$.

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- We want now to assimilate $\mathbf{y}'_{\text{ret}}$ in the case when the prior information used for data assimilation is the same as that used to determine the retrieval, by minimizing

$$J^{\text{ret}}(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}^b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}^b)+\frac{1}{2}(\mathbf{y}'_{\text{ret}}-\mathbf{H}'_{\text{ret}}\mathbf{x})^T(\mathbf{y}'_{\text{ret}}-\mathbf{H}'_{\text{ret}}\mathbf{x}) \quad (26)$$

- We get

$$\hat{\mathbf{x}}^{\text{ret}}_{\text{MAP}} = \mathbf{x}_b + \mathbf{K}_{\text{ret}}(\mathbf{y}'_{\text{ret}} - \mathbf{H}'_{\text{ret}}\mathbf{x}_b), \quad (27)$$

- where $\mathbf{K}_{\text{ret}} \equiv \mathbf{B}\mathbf{H}'^T_{\text{ret}}(\mathbf{H}'_{\text{ret}}\mathbf{B}\mathbf{H}'^T_{\text{ret}} + \mathbf{I}_r)^{-1}$ can be written as (see Eqs. 17, 18 and 25)

$$\begin{aligned}
\mathbf{K}_{\text{ret}} &= \mathbf{B}\hat{\mathbf{H}}'^T_{\text{rad}}\mathbf{U}_r(\mathbf{U}_r^T\hat{\mathbf{H}}'_{\text{rad}}\mathbf{B}\hat{\mathbf{H}}'^T_{\text{rad}}\mathbf{U}_r + \mathbf{I}_r)^{-1} \quad (28)\\
&= \mathbf{B}^{1/2}\mathbf{S}^T\mathbf{U}_r(\mathbf{U}_r^T\mathbf{S}\mathbf{S}^T\mathbf{U}_r + \mathbf{I}_r)^{-1}\\
&= \mathbf{B}^{1/2}\mathbf{V}_r\mathbf{\Lambda}_r(\mathbf{\Lambda}_r^2 + \mathbf{I}_r)^{-1}\\
&= \mathbf{K}\mathbf{U}_r.
\end{aligned}$$

# The ill-posed or under-determined problem

## Assimilation of MAP retrievals with the same prior information (2/2)

- From Eqs. 16, 22, 25, 27 and 28 it follows that the analysis $\hat{\mathbf{x}}^{\text{ret}}_{\text{MAP}}$ can be written as

$$\begin{aligned} \hat{\mathbf{x}}^{\text{ret}}_{\text{MAP}} &= \mathbf{x}^b + \mathbf{K}\mathbf{U}_r(\mathbf{U}_r^T\hat{\mathbf{y}}'_{\text{rad}} - \mathbf{U}_r^T\hat{\mathbf{H}}'_{\text{rad}}\mathbf{x}^b) \\ &= \mathbf{x}^b + \mathbf{K}\mathbf{U}_r\mathbf{U}_r^T(\hat{\mathbf{y}}'_{\text{rad}} - \hat{\mathbf{H}}'_{\text{rad}}\mathbf{x}^b) \\ &= \mathbf{x}^b + \mathbf{K}(\hat{\mathbf{y}}'_{\text{rad}} - \hat{\mathbf{H}}'_{\text{rad}}\mathbf{x}^b) = \hat{\mathbf{x}}_{\text{MAP}}, \end{aligned} \tag{29}$$

  where we have used the equivalence $\mathbf{K} = \mathbf{K}\mathbf{U}_r\mathbf{U}_r^T$ that follows from Eq. 18.

- This proves the equivalence between assimilating radiances and retrievals in the case when the prior information used first to determine and then to assimilate the retrieval are the same.

- We want to assimilate a succession of radiance measurements $\mathbf{y}^{(i)\prime}_{\mathrm{rad}}$ with $\mathbf{x}^*_b$ and $\mathbf{B}^*$.
- The resulting analysis $\hat{\mathbf{x}}^*_{\mathrm{MAP}}$ can be written as (see Eq. 16)

$$\hat{\mathbf{x}}^*_{\mathrm{MAP}} = \mathbf{x}^*_b + \mathbf{K}^*(\hat{\mathbf{y}}^{*\prime}_{\mathrm{rad}} - \hat{\mathbf{H}}^{*\prime}_{\mathrm{rad}}\mathbf{x}^*_b) \tag{30}$$

- $\hat{\mathbf{y}}^{*\prime}_{\mathrm{rad}}$ and $\hat{\mathbf{H}}^{*\prime}_{\mathrm{rad}}$ differ from $\hat{\mathbf{y}}^{\prime}_{\mathrm{rad}}$ and $\hat{\mathbf{H}}^{\prime}_{\mathrm{rad}}$, respectively, for the different value of the retrieval used as linearization point of $H(\mathbf{x})$. Note that, in general, the rank of $\hat{\mathbf{H}}^{*\prime}_{\mathrm{rad}}$ is $s \neq r$. From Eq. 4 we can write

$$\begin{aligned}\hat{\mathbf{x}}^*_{\mathrm{MAP}} &\simeq \mathbf{x}^*_b + \mathbf{K}^*\hat{\mathbf{H}}^{*\prime}_{\mathrm{rad}}(\mathbf{x}_t - \mathbf{x}^*_b) + \mathbf{K}^*\epsilon^{\prime}_{\mathrm{rad}} \\ &= \mathbf{x}^*_b + \mathbf{K}^*\mathbf{S}^*\mathbf{B}^{*-1/2}(\mathbf{x}_t - \mathbf{x}^*_b) + \mathbf{K}^*\epsilon^{\prime}_{\mathrm{rad}}\end{aligned} \tag{31}$$

with $\mathbf{S}^* \equiv \hat{\mathbf{H}}^{*\prime}_{\mathrm{rad}}\mathbf{B}^{*1/2} = \mathbf{U}^*_s\mathbf{\Lambda}^*_s\mathbf{V}^{*T}_s$.

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- From Eqs. 17 we can write

$$
\begin{aligned}
\mathbf{K}^* &\equiv \mathbf{B}^* \hat{\mathbf{H}}^{*\prime T}_{\mathrm{rad}} (\hat{\mathbf{H}}^{*\prime}_{\mathrm{rad}} \mathbf{B}^* \hat{\mathbf{H}}^{*\prime T}_{\mathrm{rad}} + \mathbf{I}_p)^{-1} \\
&= \mathbf{B}^{*1/2} \mathbf{S}^{*T} (\mathbf{S}^* \mathbf{S}^{*T} + \mathbf{I}_p)^{-1} \\
&= \mathbf{B}^{*1/2} \mathbf{V}^*_s \mathbf{\Lambda}^*_s (\mathbf{\Lambda}^{*2}_s + \mathbf{I}_s)^{-1} \mathbf{U}^{*T}_s \\
&= \mathbf{B}^{*1/2} \mathbf{S}^{*T} \mathbf{U}^*_s (\mathbf{U}^{*T}_s \mathbf{S}^* \mathbf{S}^{*T} \mathbf{U}^*_s + \mathbf{I}_s)^{-1} \mathbf{U}^{*T}_s .
\end{aligned}
\tag{32}
$$

- Consider now the retrieval $\mathbf{y}'_{\mathrm{ret}}$ defined in Eq. 22 and estimated by using prior information $\mathbf{x}_b$ and $\mathbf{B}$. We want to assimilate $\mathbf{y}'_{\mathrm{ret}}$ with its observation operator $\mathbf{H}'_{\mathrm{ret}}$ by finding the state $\hat{\mathbf{x}}^{\mathrm{ret}*}_{\mathrm{MAP}}$ that minimizes $J^{\mathrm{ret}}(\mathbf{x}_t)$ (see Eq. 26), in the case when the prior information used to constrain $\mathbf{y}'_{\mathrm{ret}}$ is $\mathbf{x}^*_b$ and $\mathbf{B}^*$.
- We need now to show that $\hat{\mathbf{x}}^{\mathrm{ret}*}_{\mathrm{MAP}} \simeq \hat{\mathbf{x}}^*_{\mathrm{MAP}}$.

- From Eq. 27 it follows that $\hat{\mathbf{x}}^{\mathrm{ret}*}_{\mathrm{MAP}}$ can be written as

$$\hat{\mathbf{x}}^{\mathrm{ret}*}_{\mathrm{MAP}} = \mathbf{x}^*_b + \mathbf{K}^{\star}_{\mathrm{ret}}(\mathbf{y}'_{\mathrm{ret}} - \mathbf{H}'_{\mathrm{ret}}\mathbf{x}^*_b) \tag{33}$$

  where, from Eqs. 17 and 25, $\mathbf{K}^{\star}_{\mathrm{ret}} \in \mathbb{R}^{n\times r}$ can be expressed as

$$\begin{aligned} \mathbf{K}^{\star}_{\mathrm{ret}} &\equiv \mathbf{B}^*\mathbf{H}'^T_{\mathrm{ret}}(\mathbf{H}'_{\mathrm{ret}}\mathbf{B}^*\mathbf{H}'^T_{\mathrm{ret}} + \mathbf{I}_r)^{-1} \\ &= \mathbf{B}^*\hat{\mathbf{H}}'^T_{\mathrm{rad}}\mathbf{U}_r(\mathbf{U}_r^T\hat{\mathbf{H}}'_{\mathrm{rad}}\mathbf{B}^*\hat{\mathbf{H}}'^T_{\mathrm{rad}}\mathbf{U}_r + \mathbf{I}_r)^{-1} \\ &= \mathbf{B}^{*1/2}\mathbf{S}^{\star T}\mathbf{U}_r(\mathbf{U}_r^T\mathbf{S}^{\star}\mathbf{S}^{\star T}\mathbf{U}_r + \mathbf{I}_r)^{-1} \end{aligned} \tag{34}$$

  with $\mathbf{S}^{\star} \equiv \hat{\mathbf{H}}'_{\mathrm{rad}}\mathbf{B}^{*1/2} \in \mathbb{R}^{p\times n}$.

- In analogy with Eq. 28, let us now find the conditions when it is possible to write $\mathbf{K}^{\star}_{\mathrm{ret}} = \mathbf{K}^*\mathbf{U}^*_s$. A comparison between Eqs. 32 and 34 shows that $\mathbf{K}^{\star}_{\mathrm{ret}} = \mathbf{K}^*\mathbf{U}^*_s$ when $s = r$ and $\mathbf{U}_r^T\mathbf{S}^{\star} = \mathbf{U}_r^{*T}\mathbf{S}^*$.

# The ill-posed or under-determined problem

Assimilation of MAP retrievals with different prior information (4/5)

- Therefore, by assuming $\mathbf{U}_r^T\mathbf{S}^\star = \mathbf{U}_r^{*T}\mathbf{S}^*$, from Eqs. 4, 24, 25, 33 and 34 we can write

$$\begin{aligned}
\hat{\mathbf{x}}_{\text{MAP}}^{\text{ret}*} &= \mathbf{x}_b^* + \mathbf{K}^*\mathbf{U}_r^*\mathbf{U}_r^T(\hat{\mathbf{y}}'_{\text{rad}} - \hat{\mathbf{H}}'_{\text{rad}}\mathbf{x}_b^*) \\
&\simeq \mathbf{x}_b^* + \mathbf{K}^*\mathbf{U}_r^*\mathbf{U}_r^T\mathbf{S}^\star\mathbf{B}^{*-1/2}(\mathbf{x}_t - \mathbf{x}_b^*) + \mathbf{K}^*\mathbf{U}_r^*\mathbf{U}_r^T\epsilon'_{\text{rad}} \\
&= \mathbf{x}_b^* + \mathbf{K}^*\mathbf{S}^*\mathbf{B}^{*-1/2}(\mathbf{x}_t - \mathbf{x}_b^*) + \mathbf{K}^*\mathbf{U}_r^*\mathbf{U}_r^T\epsilon'_{\text{rad}}
\end{aligned} \tag{35}$$

  where $\mathbf{K}^*\mathbf{U}_r^*\mathbf{U}_r^{*T} = \mathbf{K}^*$.

- From Eqs. 31 and 35 it follows that the condition $\mathbf{U}_r^T\mathbf{S}^\star = \mathbf{U}_r^{*T}\mathbf{S}^*$ implies that $\hat{\mathbf{x}}_{\text{MAP}}^{\text{ret}*} \simeq \hat{\mathbf{x}}_{\text{MAP}}^*$ within retrieval noise.
- Now, by noting that $\mathbf{S}^\star$ can in general also be written as $\mathbf{S}^\star = \mathbf{S}\mathbf{B}^{-1/2}\mathbf{B}^{*1/2}$, it follows that $\hat{\mathbf{x}}_{\text{MAP}}^{\text{ret}*} \simeq \hat{\mathbf{x}}_{\text{MAP}}^*$ holds when $\mathbf{U}_r^{*T}\mathbf{S}^*\mathbf{B}^{*-1/2} = \mathbf{U}_r^T\mathbf{S}\mathbf{B}^{-1/2}$, that is, when $\mathbf{H}'_{\text{ret}} \equiv \mathbf{\Lambda}_r\mathbf{V}_r^T\mathbf{B}^{-1/2} = \mathbf{\Lambda}_r^*\mathbf{V}_r^{*T}\mathbf{B}^{*-1/2}$.

# The ill-posed or under-determined problem

## Assimilation of MAP retrievals with different prior information (5/5)

- This means that $\hat{\mathbf{x}}^{\mathrm{ret}*}_{\mathrm{MAP}} \simeq \hat{\mathbf{x}}^{*}_{\mathrm{MAP}}$ holds when the covariance of $\mathbf{H}'_{\mathrm{ret}}\mathbf{x}_t$, in the case when the covariance of $\mathbf{x}_t$ is $\mathbf{B}$, is equal to the covariance of $\mathbf{H}'_{\mathrm{ret}}\mathbf{x}_t$, in the case when the covariance of $\mathbf{x}_t$ is $\mathbf{B}^*$, i.e., when $\mathbf{\Lambda}^* = \mathbf{\Lambda}$.
- The equivalence is satisfied when the difference between $\hat{\mathbf{x}}^{*}_{\mathrm{MAP}}$ and $\hat{\mathbf{x}}_{\mathrm{MAP}}$ – arising from the use of a different prior constraint – preserves the information content of the measurements, defined in terms of the diagonal elements of $\mathbf{\Lambda}_r$.
- Note that $\mathbf{\Lambda}^* = \mathbf{\Lambda}$ does not necessarily implies that $\mathbf{B}^* = \mathbf{B}$, as the covariance of the components of the state $\mathbf{x}_t$ which lie in the null space of $\mathbf{H}'_{\mathrm{ret}}$, in the case when the covariance of $\mathbf{x}_t$ is $\mathbf{B}^*$, do not alter the information content of that the same measurements have in the case when the covariance of $\mathbf{x}_t$ is $\mathbf{B}$.

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# Conclusions

- Conditions for equivalence between assimilation of radiances and retrievals generated from the same set of measurements:
- Observation operator approximately linear about the retrievals, in region comparable to retrieval error.
- Any prior information used should not underrepresent the variability of the state so as to preserve the information content of the measurements.
- When posterior density is multimodal, it may be beneficial to perform the retrieval before assimilation, using a more sophisticated minimization algorithm.
- See Migliorini, 2011, On the equivalence between radiance and retrieval assimilation, MWR, in press, doi: 10.1175/MWR-D-10-05047.1