Birational geometry and arithmetic

July 2012

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Main idea

Arithmetic properties are governed by global geometric invariants and the properties of the ground field F.

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In some cases, these are effectively computable.

Starting point: Curves over number fields

What do we know about curves over number fields?

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- g ≥ 2: $\#X(F) < \infty$, no effective algorithm to determine $X(F)$ (effective Mordell?)

Curve covers (joint with F. Bogomolov)

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Conjecture

If C, C' are curves of genus ≥ 2 over $\bar{\mathbb{F}}_p$ or $\bar{\mathbb{Q}}$ then $C \Leftrightarrow C'$.

Prototype: hypersurfaces $X_f \subset \mathbb{P}^n$ over $\mathbb{Q}$

Birch 1961

If $n \gg 2^{\deg(f)}$ and X_f is smooth then:

- if there are solutions in $\mathbb{Q}_p$ and in $\mathbb{R}$ then there are solutions in $\mathbb{Q}$
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- a positive proportion of hypersurfaces over $\mathbb{Q}$ have no local obstructions (Poonen-Voloch + Katz, 2003)
- the method works over $\mathbb{F}_q[t]$ as well

Heuristic

- Given: $f \in \mathbb{Z}[x_0, \ldots, x_n]$ homogeneous of degree $\deg(f)$.
- We have $|f(x)| = O(B^{\deg(f)})$, for $\|x\| := \max_j(|x_j|) \leq B$.
- May (?) assume that the probability of $f(x) = 0$ is $B^{-\deg(f)}$.
- There are B^{n+1} "events" with $\|x\| \leq B$.
- We expect B^{n+1-d} solutions with $\|x\| \leq B$.

Hope: reasonable at least when $n + 1 - d \geq 0$.

Theorem

If $\deg(f) \leq n$ then $X_f(\mathbb{C}(t)) \neq \emptyset$.

Proof: Insert $x_j = x_j(t) \in \mathbb{C}[t]$, of degree e, into

$$f = \sum_J f_J x^J = 0, \qquad |J| = \deg(f).$$

This gives a system of $e \cdot \deg(f) + \mathrm{const}$ equations in $(e+1)(n+1)$ variables. This system is solvable for $e \gg 0$, provided $\deg(f) \leq n$.

Existence of rational points

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Over number fields and higher dimensional function fields, there exist local and global obstructions to the existence of rational points.

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Counterexamples:

Iskovskikh 1971: The conic bundle $X \to \mathbb{P}^1$ given by

$$x^2 + y^2 = f(t), \qquad f(t) = (t^2 - 2)(3 - t^2).$$

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The proofs use basic algebraic number theory: quadratic and cubic reciprocity, divisibility of class numbers.

Brauer-Manin obstruction

$$\begin{tikzcd}
 & \mathrm{Br}(X_F) \arrow[r] \arrow[d, "x"'] & \bigoplus_v \mathrm{Br}(X_{F_v}) \arrow[d, "(x_v)_v"] & & \\
0 \arrow[r] & \mathrm{Br}(F) \arrow[r] & \bigoplus_v \mathrm{Br}(F_v) \arrow[rr, "\sum_v \mathrm{inv}_v"] & & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0,
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$$X(F) \subset \overline{X(F)} \subseteq X(\mathbb{A}_F)^{\mathrm{Br}} \subseteq X(\mathbb{A}_F),$$

where

$$X(\mathbb{A}_F)^{\mathrm{Br}} := \cap_{A \in \mathrm{Br}(X)} \{(x_v)_v \in X(\mathbb{A}_F) \mid \sum_v \mathrm{inv}(A(x_v)) = 0\}.$$

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Manin's formulation gives a more systematic approach to identifying the algebraic structure behind the obstruction.

Effectivity of Brauer-Manin obstructions

The obstruction group for geometrically rational surfaces

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Kresch-T. 2010

Let $X \subset \mathbb{P}^n$ be a surface over a number field F. Assume that

- the geometric Picard group of X is torsion free and is generated by finitely many divisors, each with a given set of defining equations
- $\mathrm{Br}(X)$ can be bounded effectively.

Then there is an effective algorithm to compute $X(\mathbb{A}_F)^{\mathrm{Br}}$.

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- computations with $\mathrm{Br}(X)[2]$ on general degree two K3 surfaces: Hassett, Varilly-Alvarado
- Finiteness of $\mathrm{Br}(X)/\mathrm{Br}(F)$ for all K3 surfaces over number fields (Skorobogatov-Zarhin 2007)

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Main ingredients:

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- the work of Masser-Wüstholz on the effective Tate conjecture for abelian varieties;
- effective GIT, Matsusaka, Hilbert Nullstellensatz, etc..

Computing the obstruction group

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Bright, Bruin, Flynn, Logan 2007

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$$\mathrm{H}^1(\mathrm{Gal}(\bar{F}/F), \mathrm{Pic}(\bar{X})) = 0.$$

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- Implement an algorithm to compute the BM obstruction and provide more examples of Iskovskikh type.

Uniqueness of the Brauer-Manin obstruction

Conjecture (Colliot-Thélène–Sansuc 1980)

Let X be a smooth projective rationally-connected surface over a number field F, e.g., an intersection of two quadrics in $\mathbb{P}^4$ or a cubic in $\mathbb{P}^3$. Then

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Degree 4 Del Pezzo surfaces admitting a conic bundle $X \to \mathbb{P}^1$.The conjecture is open for general degree 4 Del Pezzo surfaces.

Do we believe this conjecture?

Recall that a general Del Pezzo surface X has points locally, and that

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Elsenhans–Jahnel 2007: Thousands of examples of cubic surfaces over $\mathbb{Q}$ with different Galois actions, the conjecture holds in all cases.

Del Pezzo surfaces over $\mathbb{F}_q(t)$

Theorem (Hassett-T. 2011)

Let k be a finite field with at least $2^2 \cdot 17^4$ elements and X a general Del Pezzo surface of degree 4 over $F = k(t)$ such that its integral model

$$\mathcal{X} \to \mathbb{P}^1$$

is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^4$ of two general forms of bi-degree $(1,2)$. Then $X(F) \neq \emptyset$.

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The idea of proof will follow....

K3 surfaces

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- Let $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a general hypersurface of bidegree $(1, 4)$. Then the K3 surface fibration $\mathcal{X} \to \mathbb{P}^1$ has a Zariski dense set of sections, i.e., such K3 surfaces over $F = \mathbb{C}(t)$ have Zariski dense rational points; more generally, this holds for general pencils of K3 surfaces of degree ≤ 18 (Hassett-T. 2008)

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If X is a K3 surface which is either elliptic or has infinite automorphisms then potential density holds for X.

What about general K3 surfaces, i.e., those with Picard rank one?

- Let $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a general hypersurface of bidegree $(1, 4)$. Then the K3 surface fibration $\mathcal{X} \to \mathbb{P}^1$ has a Zariski dense set of sections, i.e., such K3 surfaces over $F = \mathbb{C}(t)$ have Zariski dense rational points; more generally, this holds for general pencils of K3 surfaces of degree ≤ 18 (Hassett-T. 2008)
- Same holds, if $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^3$ is given by a general form of bidegree $(2, 4)$ (Zhiyuan Li 2011)

Higher dimensions

Potential density holds for:

- all smooth Fano threefolds, with the possible exception of $W_2 \xrightarrow{2:1} \mathbb{P}^3$, ramified in a degree 6 surface S_6 (Harris-T. 1998, Bogomolov-T. 1998)

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- W_2, provided S_6 is singular; similar results in dimension 4 (Chelsov-Park 2004, Cheltsov 2004)
- varieties of lines on general cubic fourfolds (Amerik-Voisin 2008, Amerik-Bogomolov-Rovinski)
- varieties of lines of some special cubic fourfolds, i.e., those not containing a plane and admitting a hyperplane section with 6 ordinary double points in general linear position (Hassett-T. 2008)

Counting rational points

Counting problems depend on:

- a projective embedding $X \hookrightarrow \mathbb{P}^n$;
- a choice of $X^\circ \subset X$;
- a choice of a height function $\mathrm{H} : \mathbb{P}^n(F) \to \mathbb{R}_{>0}$.

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Main problem

$$\mathrm{N}(X^\circ(F), \mathrm{B}) = \#\{x \in X^\circ(F) \mid \mathrm{H}(x) \leq \mathrm{B}\} \overset{?}{\sim} c \cdot \mathrm{B}^a \log(\mathrm{B})^{b-1}$$

The geometric framework

Conjecture (Manin 1989)

Let $X \subset \mathbb{P}^n$ be a smooth projective Fano variety over a number field F, in its anticanonical embedding.

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$$\mathsf{N}(X^\circ(F), \mathsf{B}) \sim c \cdot \mathsf{B} \log(\mathsf{B})^{b-1}, \quad \mathsf{B} \to \infty,$$

where $b = \mathrm{rk}\,\mathrm{Pic}(X)$.

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We do not know, in general, whether or not $X(F)$ is Zariski dense, even after a finite extension of F. Potential density of rational points has been proved for some families of Fano varieties, but is still open, e.g., for hypersurfaces $X_d \subset \mathbb{P}^d$, with $d \geq 5$.

The function field case / Batyrev 1987

Let $F = \mathbb{F}_q(B)$ be a global function field and X/F a smooth Fano variety. Let

$$\pi \colon \mathcal{X} \to B$$

be a model. A point $x \in X(F)$ gives rise to a section $\tilde{x}$ of π. Let $\mathcal{L}$ be a very ample line bundle on $\mathcal{X}$. The height zeta function takes the form

$$\begin{aligned} \mathrm{Z}(s) &= \sum_{\tilde{x}} q^{-(\mathcal{L},\tilde{x})s} \\ &= \sum_{d} \mathcal{M}_d(\mathbb{F}_q) q^{-ds}, \end{aligned}$$

where $d = (\mathcal{L}, \tilde{x})$ and $\mathcal{M}_d$ is the space of sections of degree d.

The function field case / Batyrev 1987

The dimension of $\mathcal{M}_d$ can be estimated, provided $\tilde{x}$ is unobstructed:

$$\dim \mathcal{M}_d \sim (-K_{\mathcal{X}}, \tilde{x}), \qquad \tilde{x} \in \mathcal{M}_d.$$

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Heuristic assumption:

$$\mathcal{M}_d(\mathbb{F}_q) = q^{\dim(M_d)}$$

leads to a modified zeta function

$$\mathsf{Z}_{\mathrm{mod}}(s) = \sum q^{-(\mathcal{L},\tilde{x})s+(-K_{\mathcal{X}},\tilde{x})},$$

its analytic properties are governed by the ratio of the linear forms

$$(-K_{\mathcal{X}}, \cdot) \text{ and } (\mathcal{L}, \cdot)$$

The Batyrev–Manin conjecture

$$\mathsf{N}(X^\circ, \mathcal{L}, \mathsf{B}) = c \cdot \mathsf{B}^{a(L)} \cdot \log(\mathsf{B})^{b(L)-1}(1 + o(1)), \quad \mathsf{B} \to \infty$$

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- $c(-\mathcal{K}_X) = \alpha(X) \cdot \beta(X) \cdot \tau(\mathcal{K}_X)$ – "volume" of the effective cone, nontrivial part of the Brauer group $\mathrm{Br}(X)/\mathrm{Br}(F)$, Peyre's Tamagawa type number,

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- $c(\mathcal{L}) = \sum_y c(\mathcal{L}|_{X_y})$, where $X \to Y$ is a “Mori fiber space” – $\mathcal{L}$-primitive fibrations of Batyrev–T.

Justification

G. Segal 1979

- $\mathrm{Cont}_d(\mathbb{S}^2, \mathbb{P}^n(\mathbb{C})) \sim \mathrm{Hol}_d(\mathbb{S}^2, \mathbb{P}^n(\mathbb{C}))$, $d \to \infty$

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Basic idea

$\mathcal{M}_d(\mathbb{F}_q) \sim q^{\dim(M_d)}$, for $d \to \infty$, provided the homology stabilizes.

Ellenberg–Venkatesh–Westerland 2009

Effective stabilization of homology of Hurwitz spaces

There exist A, B, D such that

$$\dim \mathrm{H}_d(\mathrm{Hur}^c_{G,n}) = \dim \mathrm{H}_d(\mathrm{Hur}^c_{G,n+D}),$$

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Applications in the context of height zeta functions?

Results over $\mathbb{Q}$

Extensive numerical computations confirming Manin's conjecture, and its refinements, for Del Pezzo surfaces, hypersurfaces of small degree in dimension 3 and 4.

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Caution: counterexamples to Manin's conjecture for cubic surface bundles over $\mathbb{P}^1$ (Batyrev-T. 1996). These are compactifications of affine spaces.

Points of smallest height

Legendre: If $ax^2 + by^2 = cz^2$ is solvable mod p, for all p, then it is solvable in $\mathbb{Z}$.

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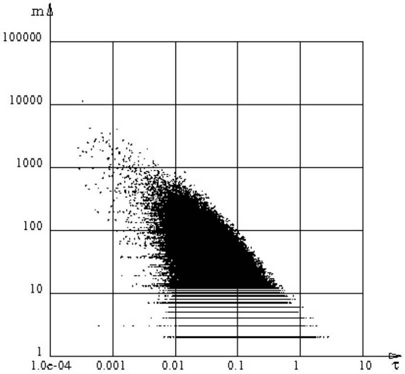
In particular, Manin's and Peyre's conjecture suggest that

$$\mathsf{H}_{\min} \leq \frac{1}{\tau}$$

Points of smallest height

There are extensive numerical data for smallest points on Del Pezzo surfaces, Fano threefolds. E.g.,



Elsenhans-Jahnel 2010

How are all of these related? (Hassett-T.)

Let X be a Del Pezzo surface over $F = \mathbb{F}_q(t)$ and

$$\pi : \mathcal{X} \to \mathbb{P}^1.$$

its integral model. Fix a height, and consider the spaces $\mathcal{M}_d$ of sections of π of height d (degree of the section).

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- for $d \geq d_0$, $\mathcal{M}_d$ fibers over $\mathrm{IJ}(\mathcal{X})$, with general fiber a rationally connected variety

Del Pezzo surfaces over $\mathbb{F}_q(t)$

We consider fibrations

$$\pi : \mathcal{X} \to \mathbb{P}^1,$$

with general fiber a degree-four Del Pezzo surface and with square-free discriminant. In this situation, we have an embedding

$$\mathcal{X} \subset \mathbb{P}(\pi_* \omega_\pi^{-1}).$$

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We have

$$\pi_* \omega_\pi^{-1} = \oplus_{i=1}^5 \mathcal{O}_{\mathbb{P}^1}(-a_i),$$

with

$$a_1 \le a_2 \le a_3 \le a_4 \le a_5,$$

occurring cases are discussed by Shramov (2006), in his investigations of rationality properties of such fibrations.

We assume that $\pi_*\omega_\pi^{-1}$ is generic, i.e., $a_5 - a_1 \leq 1$; we can realize

$$\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^d, \quad d = 4, 5, \ldots, 8,$$

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Theorem (Hassett-T. 2011)

Let k be a finite field with at least $2^2 \cdot 17^4$ elements and X a general Del Pezzo surface of degree 4 over $F = k(t)$ such that its integral model

$$\mathcal{X} \to \mathbb{P}^1$$

is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^4$ of two general forms of bi-degree $(1, 2)$. Then $X(F) \neq \emptyset$.

Idea of proof

Write $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^4$ as a complete intersection

$$P_1 s + Q_1 t = P_2 s + Q_2 t = 0,$$

where P_i, Q_i are quadrics in $\mathbb{P}^4$.

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where P_i, Q_i are quadrics in $\mathbb{P}^4$. The projection

$$\pi : \mathcal{X} \to \mathbb{P}^1$$

has 16 constant sections corresponding to solutions $y_1, \ldots, y_{16}$ of

$$P_1 = Q_1 = P_2 = Q_2 = 0.$$

Idea of proof

Projection onto the second factor gives a (nonrational) singular quartic threefold $\mathcal{Y}$:

$$P_1 Q_2 - Q_1 P_2 = 0,$$

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with nodes at $y_1, \ldots, y_{16}$. The projection $\mathcal{X} \to \mathcal{Y}$ is a small resolution of the singularities of $\mathcal{Y}$. We analyse lines in the smooth locus of $\mathcal{Y}$.

Main observation

There exists an irreducible curve (of genus 289) of lines $\mathfrak{l} \subset \mathcal{Y}$, giving sections of π.