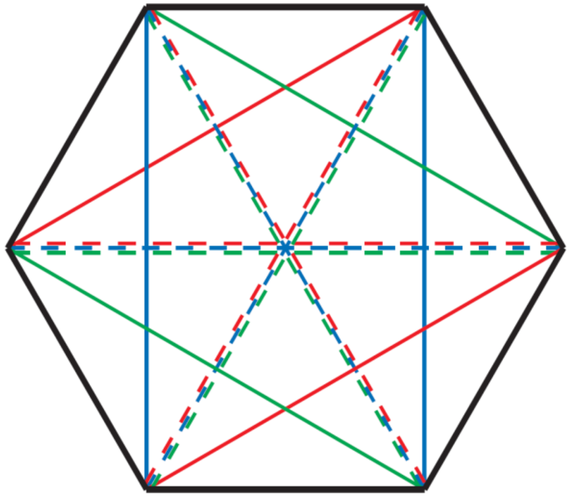
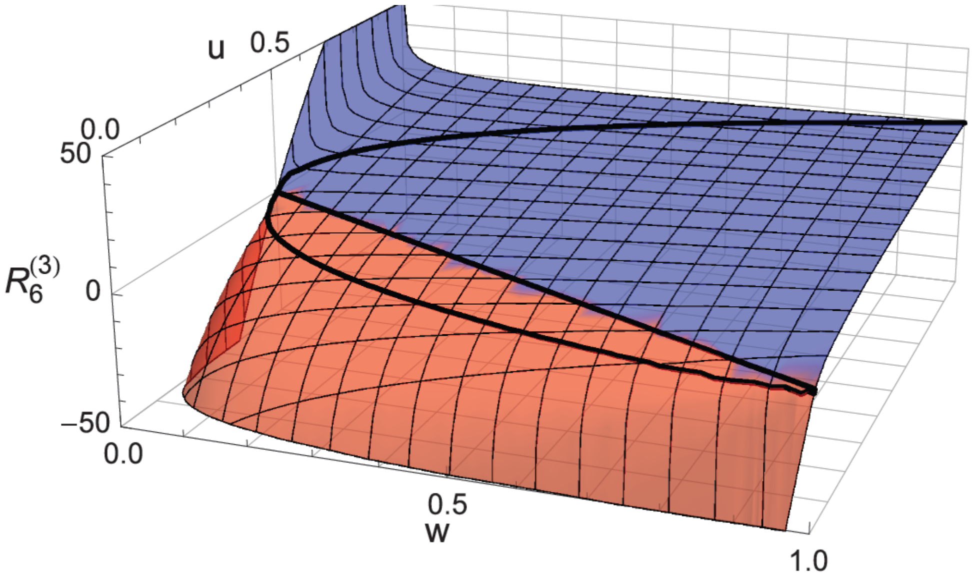
Hexagon Functions and Six-Gluon Scattering in Planar N=4 Super-Yang-Mills





L. Dixon, J. Drummond, M. von Hippel
and J. Pennington, 1307.nnnn

LMS Symposium, Durham

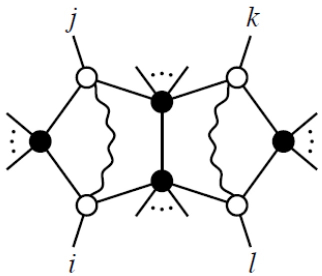
July 8, 2013

All planar N=4 SYM integrands

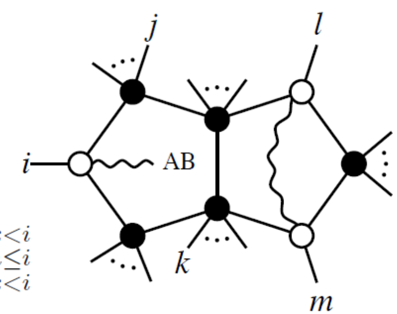
Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka, 1008.2958, 1012.6032

- All-loop BCFW recursion relation for integrand ☺
- Or new approach   Arkani-Hamed et al. 1212.5605
- Manifest Yangian invariance ☺
- Multi-loop integrands in terms of "momentum-twistors" ☺
- Still have to do integrals over the loop momentum ☹

$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$

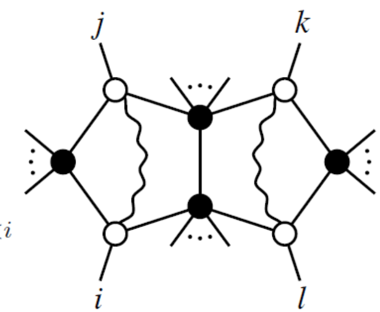


$$\mathcal{A}^{2-\text{loop}}_{\text{NMHV}} = \sum_{\substack{i<j<l<m\leq k<i \\ i<j<k<l<m\leq i \\ i\leq l<m\leq j<k<i}}$$



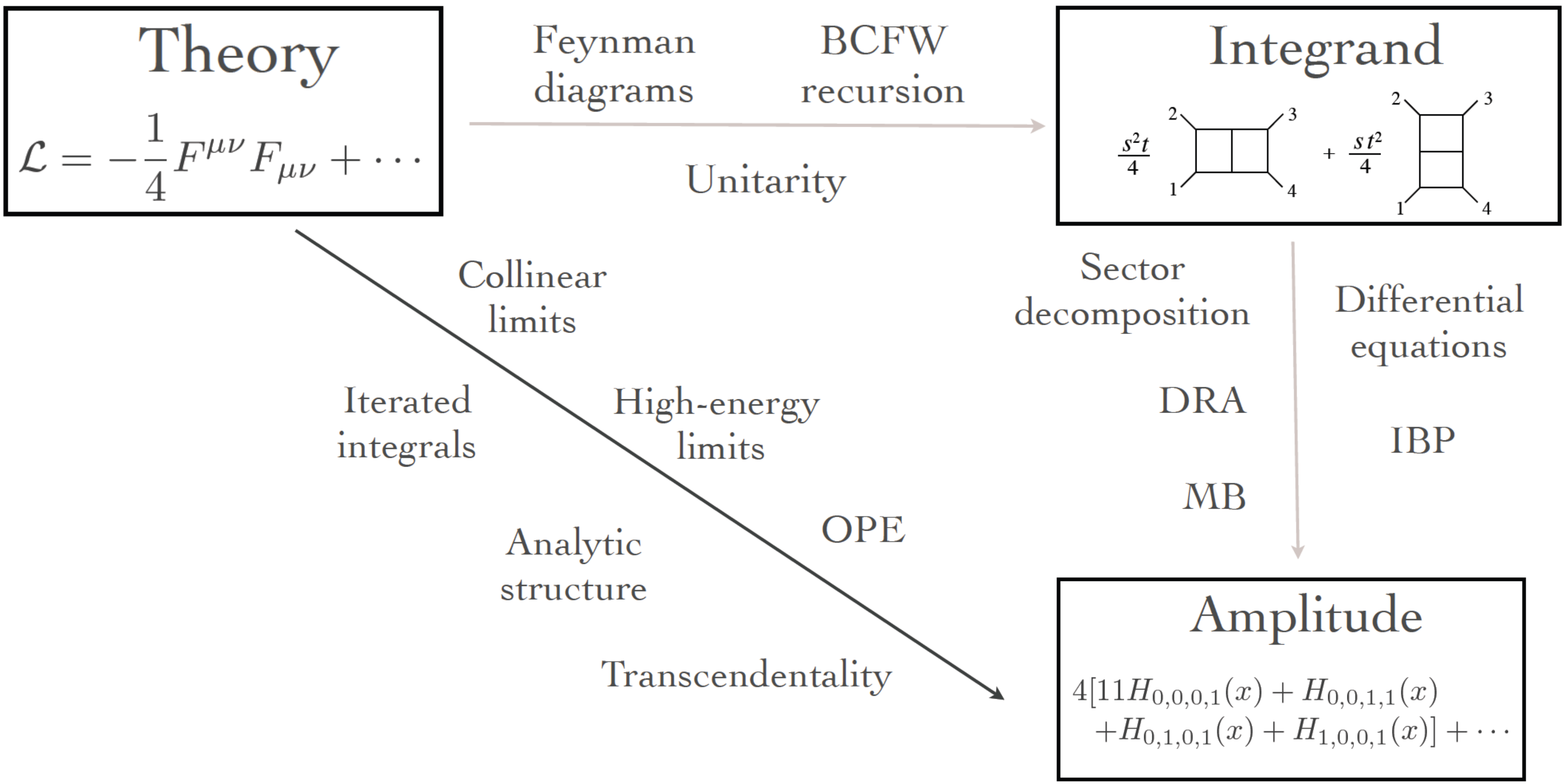
$$\times\, [i, j, j+1, k, k+1]$$

$$+\frac{1}{2}\sum_{i<j<k<l<i}$$



$$\times \left\{ \begin{array}{c} \mathcal{A}^{\text{tree}}_{\text{NMHV}}(j,\ldots,k;\ l,\ldots,i) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(i,\ldots,j) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(k,\ldots,l) \end{array} \right\}$$

Our strategy in brief

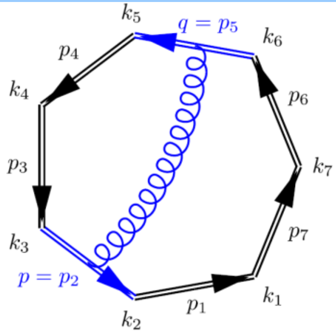


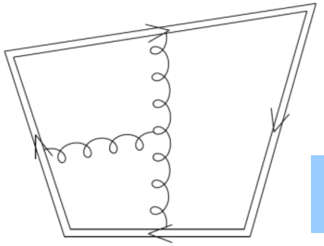
Bootstrapping multi-loop amplitudes

- Make ansatz for functional form of amplitude
- Use “boundary value data” (like near collinear or multi-Regge limits) to fix constants in ansatz
- Also assisted by:
  - dual conformal invariance
  - Wilson loop correspondence
- First amplitude to try is $n = 6$ MHV amplitude in planar N=4 SYM

Wilson loops at weak coupling

Computed for same boundary conditions as scattering amplitude
$k_i = x_i - x_{i+1}$ [inspired by Alday, Maldacena strong coupling result]:





- One loop, *n=4* — Drummond, Korchemsky, Sokatchev, 0707.0243
- One loop, any *n* — Brandhuber, Heslop, Travaglini, 0707.1153
- Two loops, *n=4,5,6* — Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466; Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Wilson-loop VEV **always matches** [MHV] scattering amplitude!

Weak-coupling properties ←→ superconformal invariance for strings in $AdS_5 \times S^5$ under combined bosonic and fermionic T duality symmetry
Berkovits, Maldacena, 0807.3196; Beisert, Ricci, Tseytlin, Wolf, 0807.3228

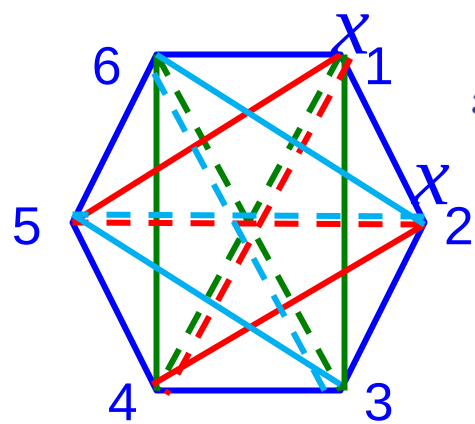
Six-point remainder function R_6

• $n = 6$ first place BDS Ansatz must be modified, due to dual conformal cross ratios

$$u = u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad v = u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} \qquad w = u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}$$

$$\boxed{\mathcal{A}_6^{\mathrm{MHV}}(\epsilon; s_{ij}) \;=\; \mathcal{A}_6^{\mathrm{BDS}}(\epsilon; s_{ij}) \exp[R_6(u_1, u_2, u_3)]}$$

Known function, accounts for infrared divergences (poles in ε), anomalies in dual conformal symmetry, and tree and 1-loop result



$$x_{i,i+1}^2 = 0$$

starts at 2 loops

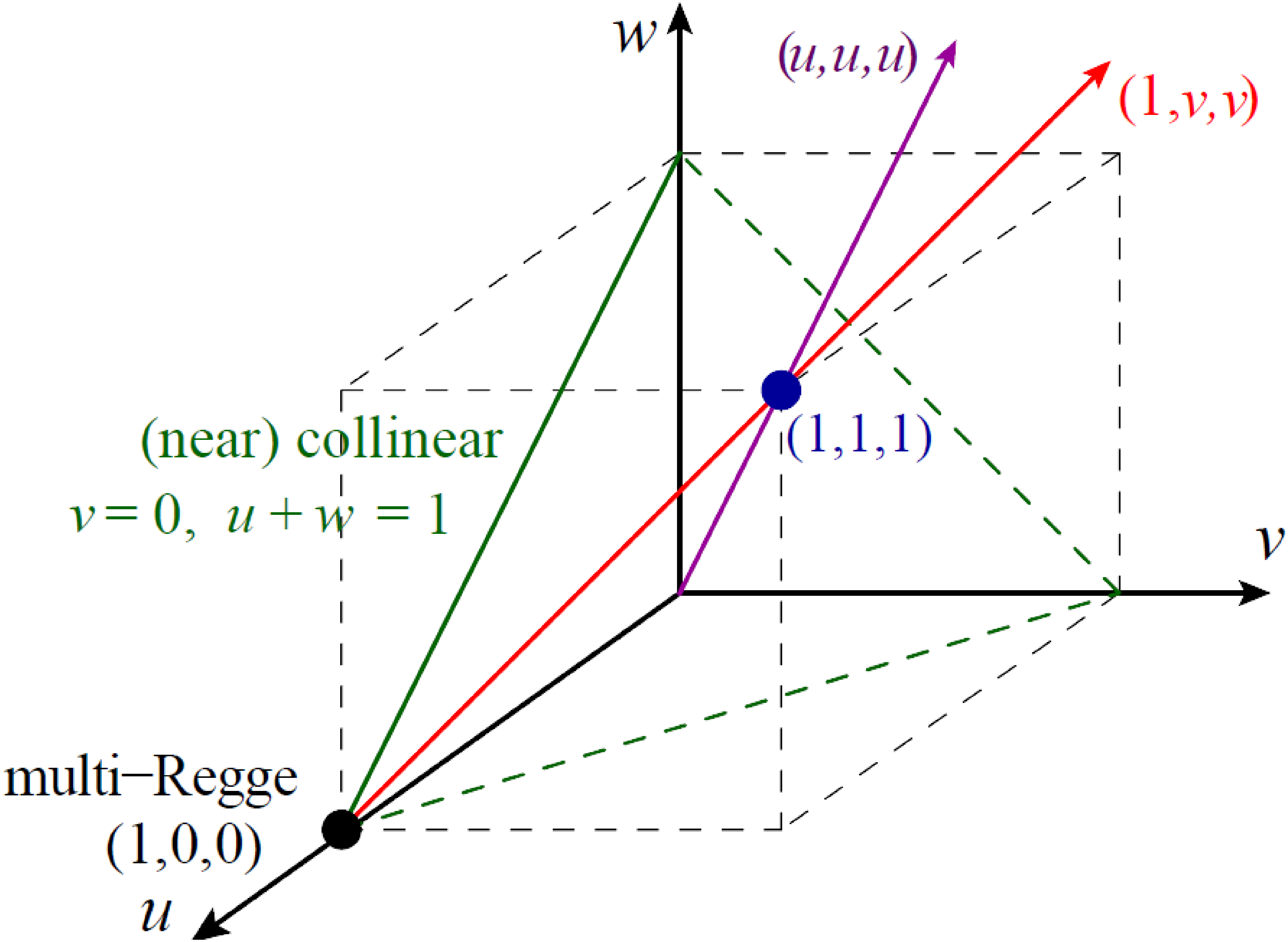
Two loop answer: $R_6^{(2)}(u_1,u_2,u_3)$

• Wilson loop integrals performed by
Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702
17 pages of Goncharov polylogarithms.

• Simplified to classical polylogarithms using symbology
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$



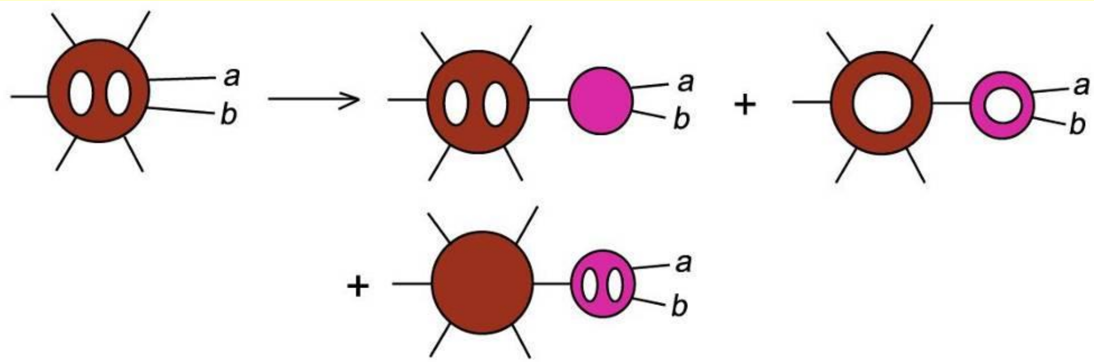
w
(u,u,u)
$(1,v,v)$
(near) collinear
$v=0,\ u+w=1$
$(1,1,1)$
v
multi−Regge
$(1,0,0)$
u

Wilson loop OPEs

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009, 1102.0062

- Remarkably, $R_6^{(2)}(u_1,u_2,u_3)$ can be recovered directly from analytic properties, using "near collinear limit", e.g.

$$v \to 0, \qquad u + w \to 1$$



- Limit controlled by an operator product expansion (OPE)
- **Possible to go to 3 loops, by combining OPE expansion with symbol ansatz** LD, Drummond, Henn, 1108.4461

Here, promote symbol to unique function $R_6^{(3)}(u_1,u_2,u_3)$

Pure functions and symbols

- A pure function $f^{(k)}$ of transcendental degree k is a linear combination of k-fold iterated integrals, with constant (rational) coefficients.
- We can also add terms like $\zeta(p) \times f^{(k-p)}$
- Derivatives of $f^{(k)}$ can be written as

$$d\, f^{(k)} = \sum_r f_r^{(k-1)}\, d\log \phi_r$$

for a finite set of algebraic functions ϕ_r

- Define the symbol $\mathcal{S}$ ({1,1,1,…,1} element of coproduct) recursively in k:

$$\boxed{\mathcal{S}(f^{(k)}) = \sum_r \mathcal{S}(f_r^{(k-1)}) \otimes \phi_r}$$

What entries should **symbol** of R_6 have?

- We assume entries can all be drawn from set:

$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

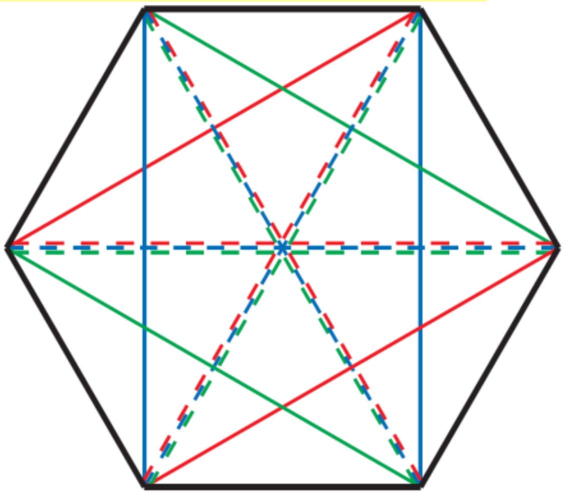
with

$$y_u \equiv \frac{u - z_+}{u - z_-} \qquad + \quad \text{perms}$$

$$z_\pm = \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big]$$

$$\Delta = (1 - u - v - w)^2 - 4uvw$$

arise from 9 projectively invariant combinations of 6 momentum twistors



$S[\,R_6^{(2)}(u,v,w)\,]$ in these variables

GSVV, 1006.5703

$$
\begin{aligned}
-8\,\mathcal{S}[R_6^{(2)}] \;=\;& u\otimes(1-u)\otimes\frac{u}{(1-u)^2}\otimes\frac{u}{1-u} \\
&+2\,(u\otimes v+v\otimes u)\otimes\frac{w}{1-v}\otimes\frac{u}{1-u} \\
&+2\,v\otimes\frac{w}{1-v}\otimes u\otimes\frac{u}{1-u} \\
&+u\otimes(1-u)\otimes y_u y_v y_w\otimes y_u y_v y_w \\
&-2\,u\otimes v\otimes y_w\otimes y_u y_v y_w \\
&+\ 5\ \text{permutations of}\ (u,v,w)
\end{aligned}
$$

First entry

- Always drawn from $\{u, v, w\}$ GMSV, 1102.0062
- Because first entry controls branch-cut location
- Only massless particles

→ all cuts start at origin in $s_{i,i+1}$, $s_{i,i+1,i+2}$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2}$$

Final entry

- Always drawn from

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- Seen in structure of various Feynman integrals

[e.g. Arkani-Hamed et al., 1108.2958]

related to amplitudes

Drummond, Henn, Trnka 1010.3679;
LD, Drummond, Henn, 1104.2787, V. Del Duca et al., 1105.2011,…

- Same condition also from Wilson super-loop approach Caron-Huot, 1105.5606

Generic Constraints

- **Integrability** (must be symbol of some function)
- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**":

every term must have an even number of y_i – 0, 2 or 4

$$\begin{array}{ccc} i\sqrt{\Delta} & \leftrightarrow & -i\sqrt{\Delta} \\ z_+ & \leftrightarrow & z_- \\ y_i & \leftrightarrow & 1/y_i \end{array}$$

- Vanishing in **collinear** limit

$$v \to 0 \qquad u + w \to 1$$

- At 3 loops, these 4 constraints leave 35 free parameters

OPE Constraints

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009; 1102.0062
Basso, Sever, Vieira [BSV], 1303.1396; 1306.2058

- $R_6^{(L)}(u,v,w)$ vanishes in the collinear limit,

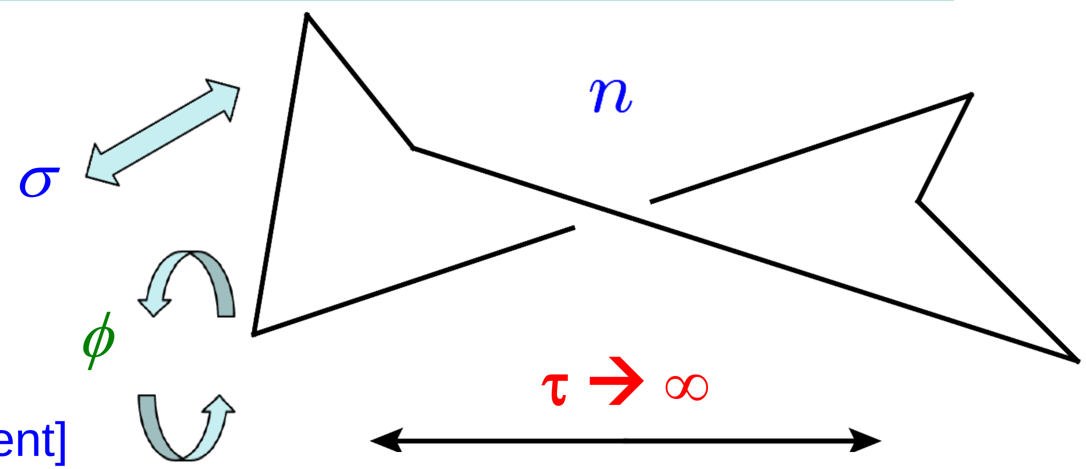
$$v = 1/\cosh^2\tau \to 0 \qquad\qquad \tau \to \infty$$

In **near**-collinear limit, described by an Operator Product Expansion, with generic form

$$R_6^{(L)}(u,v,w) = R_6^{(L)}(\tau,\sigma,\phi) \sim \int dn\ C_n(g)\ \exp[-E_n(g)\tau]$$

$$\begin{aligned} u &= \frac{e^{\sigma}\sinh\tau\tanh\tau}{2(\cosh\sigma\ \cosh\tau + \cos\phi)} \\ v &= \frac{1}{\cosh^2\tau} \\ w &= u\, e^{-2\sigma} \end{aligned}$$

[BSV parametrization a little different]



OPE Constraints (cont.)

- Using conformal invariance, send one long line to ∞, put other one along x^{-}
- Dilatations, boosts, azimuthal rotations preserve configuration.
- σ, ϕ conjugate to twist p, spin m of conformal primary fields (flux tube excitations)
- Expand anomalous dimensions in coupling g^2:

$$E_n(g) = E_n^{(0)} + g^2\, E_n^{(1)} + g^4\, E_n^{(2)} + \ldots$$

$$\exp[-E_n(g)\tau]$$
$$= \exp[-E_n^{(0)}\tau] \times \left[1 - g^2\,\tau\, E_n^{(1)} + g^4 (\frac{1}{2}\tau^2\, [E_n^{(1)}]^2 - \tau\, E_n^{(2)}) + \ldots\right]$$

- **Leading discontinuity τ^{L-1} of $R_6^{(L)}$ needs only one-loop anomalous dimension $E_n^{(1)}$**

OPE Constraints (cont.)

- As $\tau \to \infty$, $v = 1/\cosh^2\tau$ → $\tau^{L-1} \sim [\ln v]^{L-1}$
- Extract this piece from symbol by only keeping terms with L-1 leading v entries

$$\underbrace{v \otimes \ldots \otimes v}_{\text{clip } L-1 \text{ entries}} \otimes \underbrace{\ldots}_{\text{keep } L+1 \text{ entries}}$$

- Powerful constraint: fixes 3 loop symbol up to 2 parameters. But not powerful enough for $L > 3$
- New results of BSV give

$$v^{1/2}\, e^{\pm i\phi}\, [\ln v]^{k}, \quad k = 0,1,2,\ldots L\text{-}1$$

and even

$$v^{1}\, e^{\pm 2i\phi}\, [\ln v]^{k}, \quad k = 0,1,2,\ldots L\text{-}1$$

Constrained Symbol

- Leading discontinuity constraints reduced symbol ansatz to just 2 parameters:

DDH, 1108.4461

$$\mathcal{S}[R_6^{(3)}] = \mathcal{S}[X] + \alpha_1 \mathcal{S}[f_1] + \alpha_2 \mathcal{S}[f_2]$$

- $f_{1,2}$ have no double-v discontinuity, so $\alpha_{1,2}$ couldn't be determined this way.
- Determined soon after using Wilson super-loop integro-differential equation

Caron-Huot, He, 1112.1060

$$\alpha_1 = -3/8 \qquad \alpha_2 = 7/32$$

- Also follow from BSV

Hexagon functions

- Build up a complete description of pure functions $F(u,v,w)$ with correct branch cuts (corresponding to first-entry constraint on symbol) iteratively in the weight n, using $\{n\text{-}1,1\}$ element of the co-product $\Delta_{n\text{-}1,1}(F)$

Duhr, Gangl, Rhodes, 1110.0458

$$\Delta_{n-1,1}(F) \equiv \sum_{i=1}^{3} F^{u_i} \otimes \ln u_i + F^{1-u_i} \otimes \ln(1-u_i) + F^{y_i} \otimes \ln y_i$$

which specifies all first derivatives of F:

$$\frac{\partial F}{\partial u}\bigg|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

$$\sqrt{\Delta}\, y_u \frac{\partial F}{\partial y_u}\bigg|_{y_v,y_w} = (1-u)(1-v-w)F^u - u(1-v)F^v - u(1-w)F^w - u(1-v-w)F^{1-u} + uv\, F^{1-v} + uw\, F^{1-w} + \sqrt{\Delta}\, F^{y_u}\,.$$

Hexagon functions (cont.)

- Coefficients $F^{u_i}, \;\; F^{1-u_i}, \;\; F^{y_i}$ are weight n-1 hexagon functions that can be identified (iteratively) from the symbol of F
- "Beyond-the-symbol" [bts] ambiguities in reconstructing them, proportional to $\zeta(k)$.
- Most ambiguities resolved by equating 2nd order mixed partial derivatives.
- Remaining ones represent freedom to add globally well-defined weight n-k functions multiplied by $\zeta(k)$.

Harmonic Polylogarithms (HPLs)

Remiddi, Vermaseren, hep-ph/9905237

- Describe the y^0 sector of the hexagon functions. Symbol letters: $\{u, 1-u\}$
- Functions defined iteratively by:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

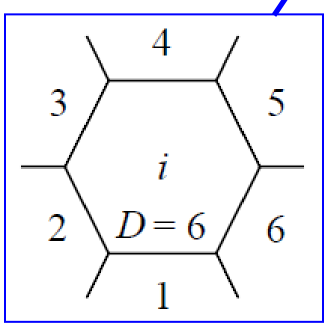
- Here we use argument 1-u.
- Regularity (vanishing) at u=1

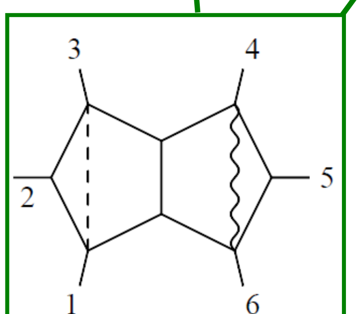
→ last entry of $\vec{w}$ in $H_{\vec{w}}(1-u)$ can't be 0.

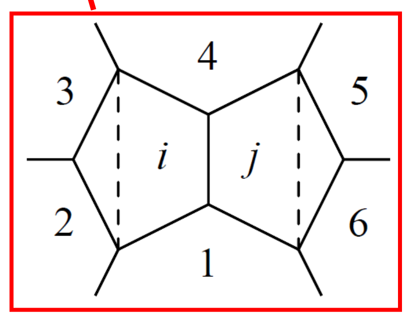
How many hexagon functions?

First entry $\{u, v, w\}$; irreducible (non-product)

Weight	y^0	y^1	y^2	y^3	y^4
1	3 HPLs	-	-	-	-
2	3 HPLs	-	-	-	-
3	6 HPLs	$\tilde{\Phi}_6$	-	-	-
4	9 HPLs	$3 \times F_1$	$3 \times \Omega^{(2)}$	-	-
5	18 HPLs	$G,\ 3 \times K_1$	$5 \times M_1,\ N,\ O,\ 6 \times Q_{\rm ep}$	$3 \times H_1,\ 3 \times J_1$	-
6	27 HPLs	4	27	29	$3 \times \boxed{R_{\rm ep}} + 15$







$R_6^{(3)}(u,v,w)$

$$R_6^{(3)}(u,v,w) \;=\; R_{\rm ep}(u,v,w) + R_{\rm ep}(v,w,u) + R_{\rm ep}(w,u,v) + P_6(u,v,w) + c_1\zeta_6 + c_2(\zeta_3)^2$$

$$P_6 = -\frac{1}{4}(\Omega^{(2)}(u,v,w)\mathrm{Li}_2(1-1/w) + \mathrm{cyc}) - \frac{1}{16}(\tilde{\Phi}_6)^2 + \frac{1}{4}\mathrm{Li}_2(1-1/u)\mathrm{Li}_2(1-1/v)\mathrm{Li}_2(1-1/w)\,.$$

Many relations among coproduct coefficients for $R_{\rm ep}$:

$$R_{\rm ep}^{v} = -R_{\rm ep}^{1-v} = -R_{\rm ep}^{1-u}(u \leftrightarrow v) = R_{\rm ep}^{u}(u \leftrightarrow v)\,, \qquad R_{\rm ep}^{y_v} = R_{\rm ep}^{y_u}\,,$$

$$R_{\rm ep}^{w} = R_{\rm ep}^{1-w} = R_{\rm ep}^{y_w} = 0$$

Only 2 indep. R_{ep} coproduct coefficients

$$R_{\rm ep}^{y_u} = -\frac{1}{32}H_1(u,v,w) - \frac{3}{32}H_1(v,w,u) - \frac{1}{32}H_1(w,u,v) + \frac{3}{128}J_1(u,v,w) + \frac{3}{128}J_1(v,w,u)$$
$$+ \frac{3}{128}J_1(w,u,v) - \frac{1}{8}H_2^u\,\tilde{\Phi}_6 - \frac{1}{8}H_2^v\,\tilde{\Phi}_6 - \frac{1}{32}\ln^2 u\,\tilde{\Phi}_6 + \frac{1}{16}\ln u\,\ln v\,\tilde{\Phi}_6$$
$$- \frac{1}{16}\ln u\,\ln w\,\tilde{\Phi}_6 - \frac{1}{32}\ln^2 v\,\tilde{\Phi}_6 - \frac{1}{16}\ln v\,\ln w\,\tilde{\Phi}_6 + \frac{1}{32}\ln^2 w\,\tilde{\Phi}_6 + \frac{11}{16}\zeta_2\,\tilde{\Phi}_6\,,$$

$$R_{\rm ep}^{u} = -\frac{2}{3}Q_{\rm ep}^u(u,v,w) + \frac{2}{3}Q_{\rm ep}^u(u,w,v) - \frac{2}{3}Q_{\rm ep}^u(v,w,u) - \frac{1}{3}Q_{\rm ep}^u(v,u,w) + Q_{\rm ep}^u(w,v,u)$$
$$+ \frac{1}{32}M_1(u,v,w) - \frac{1}{32}M_1(v,u,w) + \frac{5}{32}\ln u\,\Omega^{(2)}(u,v,w) - \frac{3}{32}\ln u\,\Omega^{(2)}(v,w,u)$$
$$- \frac{1}{32}\ln u\,\Omega^{(2)}(w,u,v) - \frac{5}{32}\ln v\,\Omega^{(2)}(u,v,w) - \frac{1}{32}\ln v\,\Omega^{(2)}(v,w,u) - \frac{3}{32}\ln v\,\Omega^{(2)}(w,u,v)$$
$$+ \frac{1}{8}\ln w\,\Omega^{(2)}(u,v,w) + \frac{1}{16}\ln w\,\Omega^{(2)}(v,w,u) + \frac{1}{8}\ln w\,\Omega^{(2)}(w,u,v) + R_{\rm ep,\,rat}^{u}\,,$$

2 pages of 1-d HPLs

Similar (but shorter) expressions for lower degree functions

Integrating the coproducts

- Can express in terms of multiple polylog's $G(\vec{w};1)$, with w_i drawn from $\{0,\ 1/y_i\ ,\ 1/(y_i\, y_j),\ 1/(y_1\, y_2\, y_3)\ \}$
- **Alternatively:**
- Coproducts define coupled set of first-order PDEs
- Integrate them numerically from base point $(1,1,1)$

[only need initial value at one point]

- Or solve PDEs analytically in special limits, especially:

1. Near-collinear limit
2. Multi-regge limit

Integration contours in (u,v,w)

$$F(u,v,w) = -\sqrt{\Delta}\int_1^u \frac{du_t}{v_t[u(1-w)+(w-u)u_t]}\frac{\partial F}{\partial \ln y_v}(u_t,v_t,w_t)$$

base point $(u,v,w) = (1,1,1)$

$y_u\, y_v\, y_w = 1$

$$v_t = 1 - \frac{(1-v)\,u_t\,(1-u_t)}{u\,(1-w)+(w-u)\,u_t}$$

$$w_t = \frac{(1-u)\,w\,u_t}{u\,(1-w)+(w-u)\,u_t}\,.$$

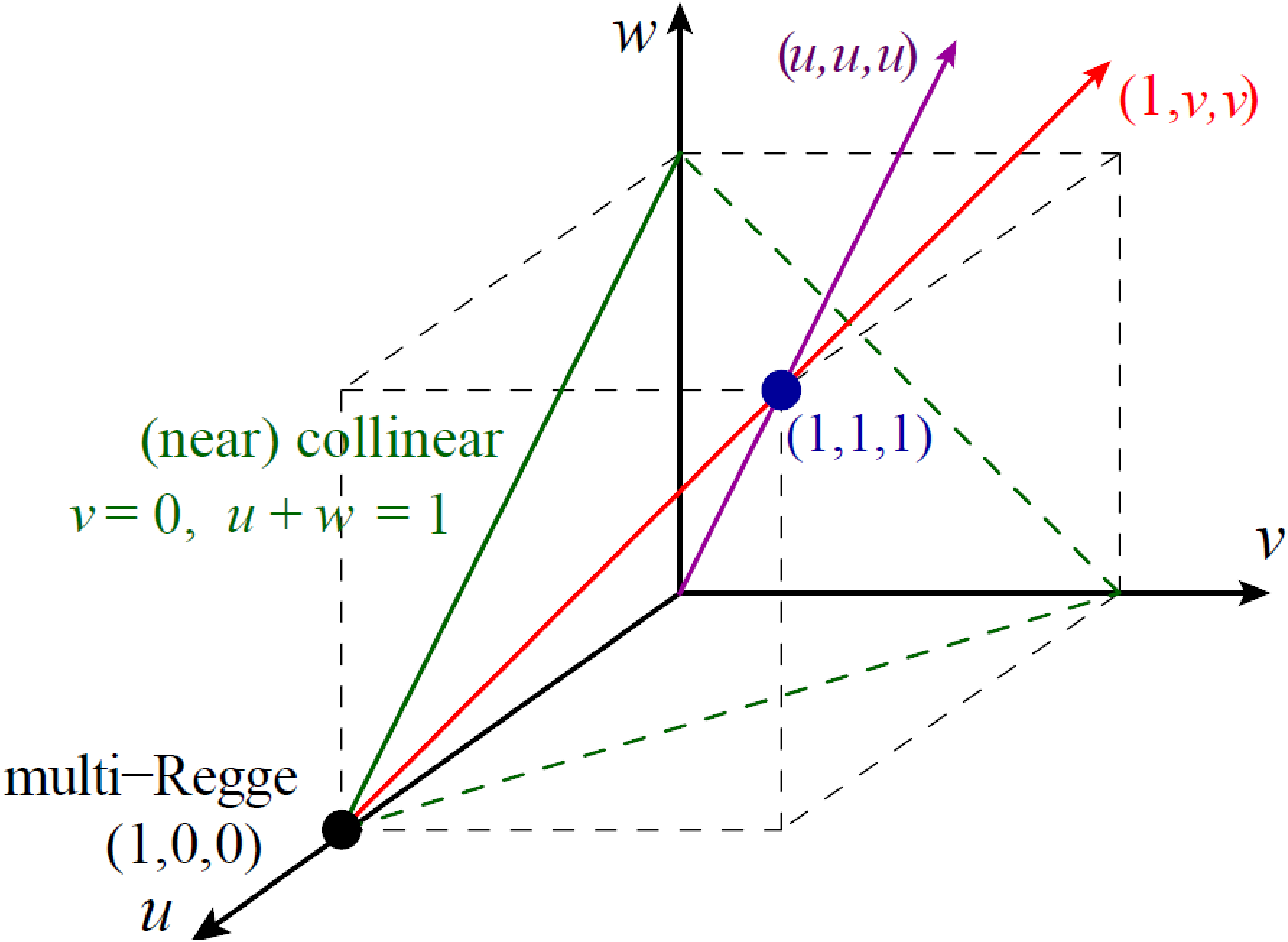
$$F(u,v,w) = F(1,0,0) + \sqrt{\Delta}\int_1^u \frac{du_t}{(1-v_t)[uw+(1-u-w)u_t]}\frac{\partial F}{\partial \ln(y_u/y_w)}(u_t,v_t,w_t)$$

base point $(u,v,w) = (1,0,0)$

$y_u = 1$

$$v_t = \frac{vu_t(1-u_t)}{uw+(1-u-w)u_t}$$

$$w_t = \frac{uw(1-u_t)}{uw+(1-u-w)u_t}$$



w
(u,u,u)
$(1,v,v)$
(near) collinear
$v = 0, \ u + w = 1$
$(1,1,1)$
v
multi−Regge
$(1,0,0)$
u

Fixing all the constants

- 11 bts constants (plus $\alpha_{1,2}$) before analyzing limits
- Vanishing of collinear limit $v \to 0$ fixes everything, except α_2 and 1 bts constant
- Near-collinear limit,

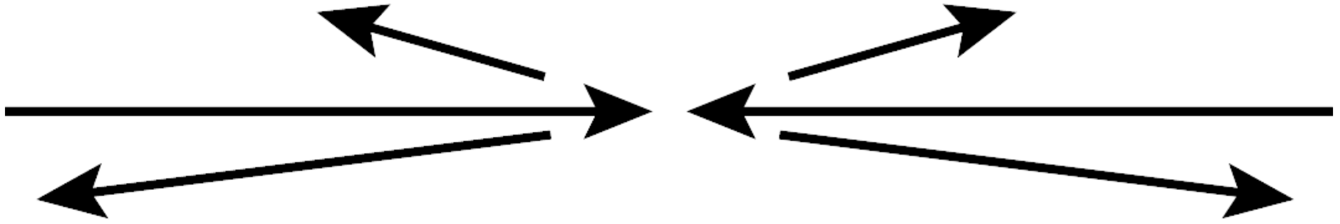
$$v^{1/2}\, e^{\pm i\phi}\ [\ln v]^{k}\,,\ k = 0,1$$

fixes last 2 constants

(α_2 agrees with Caron-Huot+He and BSV)

Multi-Regge limit

• Minkowski kinematics, large rapidity separations between the 4 final-state gluons:



• Properties of planar N=4 SYM amplitude in this limit studied extensively at weak coupling:
Bartels, Lipatov, Sabio Vera, 0802.2065, 0807.0894; Lipatov, 1008.1015; Lipatov, Prygarin, 1008.1016, 1011.2673; Bartels, Lipatov, Prygarin, 1012.3178, 1104.4709; LD, Drummond, Henn, 1108.4461; Fadin, Lipatov, 1111.0782; LD, Duhr, Pennington, 1207.0186

• Factorization and exponentiation in this limit provides additional source of "boundary data" for bootstrapping!

Physical 2→4 multi-Regge limit

- Euclidean MRK limit vanishes
- To get nonzero result for physical region, first let

$u \to u\, e^{-2\pi i}$, then $u \to 1$, $v, w \to 0$

different w, sorry!

$$\frac{v}{1-u} \to \frac{1}{(1+w)(1+w^*)} \qquad \frac{w}{1-u} \to \frac{ww^*}{(1+w)(1+w^*)}$$

$$\boxed{R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \left[g_r^{(L)}(w,w^*) + 2\pi i\, h_r^{(L)}(w,w^*) \right]}$$

$g_{L-1}^{(L)}$ (**LLA**) and $g_{L-2}^{(L)}$ (**NLLA**) well understood

Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186;
Pennington, 1209.5357

Put **LLA, NLLA** results into bootstrap;
extract **N^{k}LLA**, $k > 1$

NNLLA impact factor now fixed

Result from DDP, 1207.0186 still had
3 beyond-the-symbol ambiguities

$$\Phi^{(2)}_{\rm Reg}(\nu,n) = \frac{1}{2}\left[\Phi^{(1)}_{\rm Reg}(\nu,n)\right]^2 - E^{(1)}_{\nu,n}\,E_{\nu,n} + \frac{1}{8}\,[D_\nu E_{\nu,n}]^2 + \frac{5\pi^2}{16}\,E^2_{\nu,n} - \frac{1}{2}\,\zeta_3\,E_{\nu,n} + \frac{5}{64}\,N^4 + \frac{5}{16}\,N^2\,V^2 - \frac{5\pi^2}{64}\,N^2 - \frac{\pi^2}{4}\,V^2 + \frac{17\pi^4}{360} + d_1\,\zeta_3\,E_{\nu,n} - d_2\,\frac{\pi^2}{6}\,\left[12\,E^2_{\nu,n} + N^2\right] + \gamma''\,\frac{\pi^2}{6}\left[E^2_{\nu,n} - \frac{1}{4}\,N^2\right].$$

Now all 3 are fixed:

$\gamma'' = -5/4$ $\qquad d_1 = 1/2$ $\qquad d_2 = 3/32$

Simple slice: $(u,u,1) \leftarrow\rightarrow (1,v,v)$

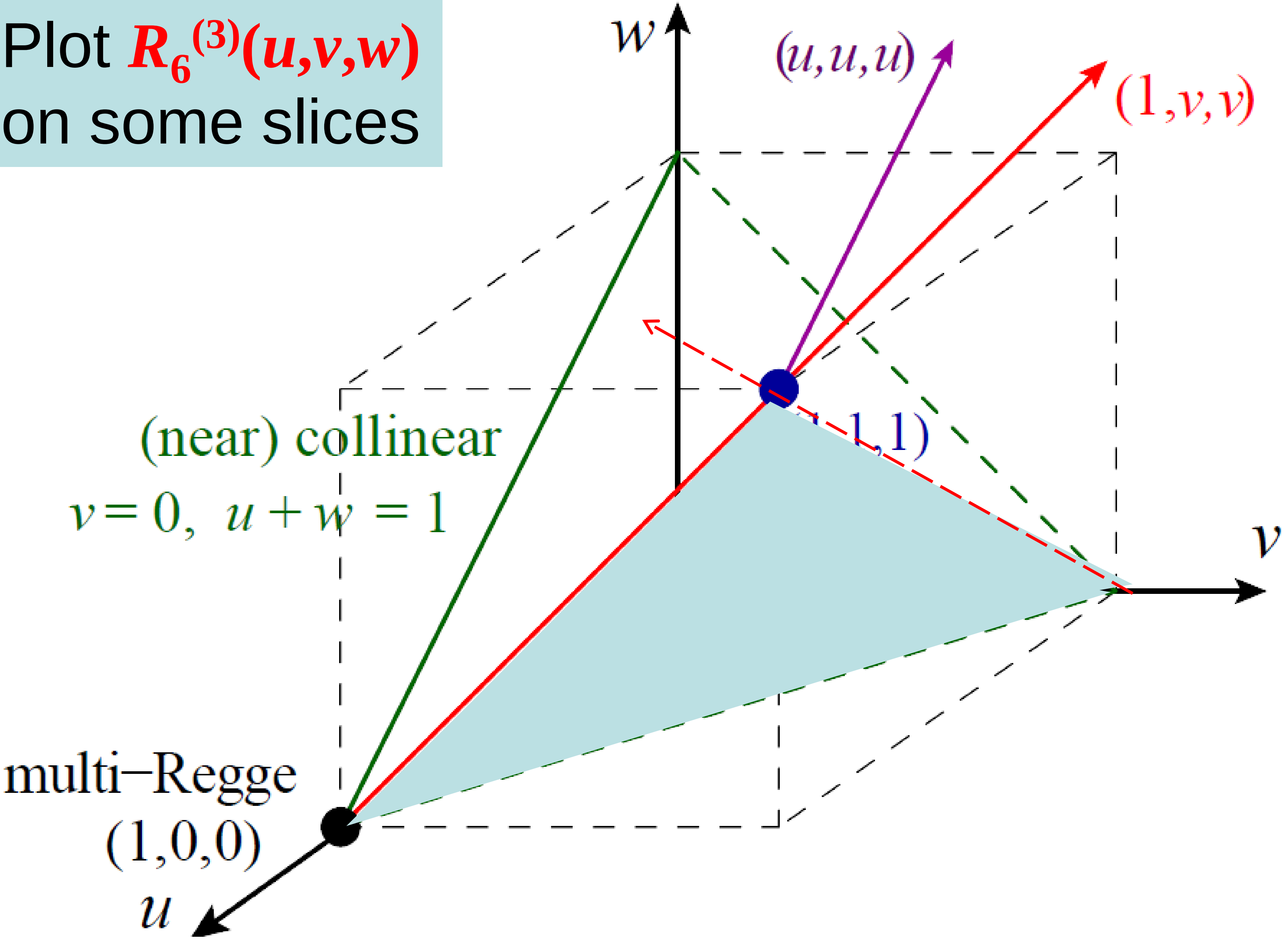
Collapses to 1d HPLs: $H^u_{3,2,1} \equiv H_{0,0,1,0,1,1}(1-u)$, etc.

$$
\begin{aligned}
R_6^{(3)}(u,u,1) = & -3\,H_6^u + 2\,H_{5,1}^u - 9\,H_{4,1,1}^u - 2\,H_{3,2,1}^u + 6\,H_{3,1,1,1}^u - 15\,H_{2,1,1,1,1}^u \\
& - \frac{1}{4}\,(H_3^u)^2 - \frac{1}{2}\,H_3^u\,H_{2,1}^u + \frac{3}{4}\,(H_{2,1}^u)^2 - \frac{5}{12}\,(H_2^u)^3 + \frac{1}{2}\,H_2^u\left[3\,(H_4^u + H_{2,1,1}^u) + H_{3,1}^u\right] \\
& - H_1^u\,(3\,H_5^u - 2\,H_{4,1}^u + 9\,H_{3,1,1}^u + 2\,H_{2,2,1}^u - 6\,H_{2,1,1,1}^u - H_2^u\,H_3^u) \\
& - \frac{1}{4}\,(H_1^u)^2\left[3\,(H_4^u + H_{2,1,1}^u) - 5\,H_{3,1}^u + \frac{1}{2}\,(H_2^u)^2\right] \\
& - \zeta_2\left[H_4^u + H_{31}^u + 3\,H_{211}^u + H_1^u\,(H_3^u + H_{21}^u) - (H_1^u)^2\,H_2^u - \frac{3}{2}\,(H_2^u)^2\right] \\
& - \zeta_4\left[(H_1^u)^2 + 2\,H_2^u\right] + \frac{413}{24}\,\zeta_6 + (\zeta_3)^2\,.
\end{aligned}
$$

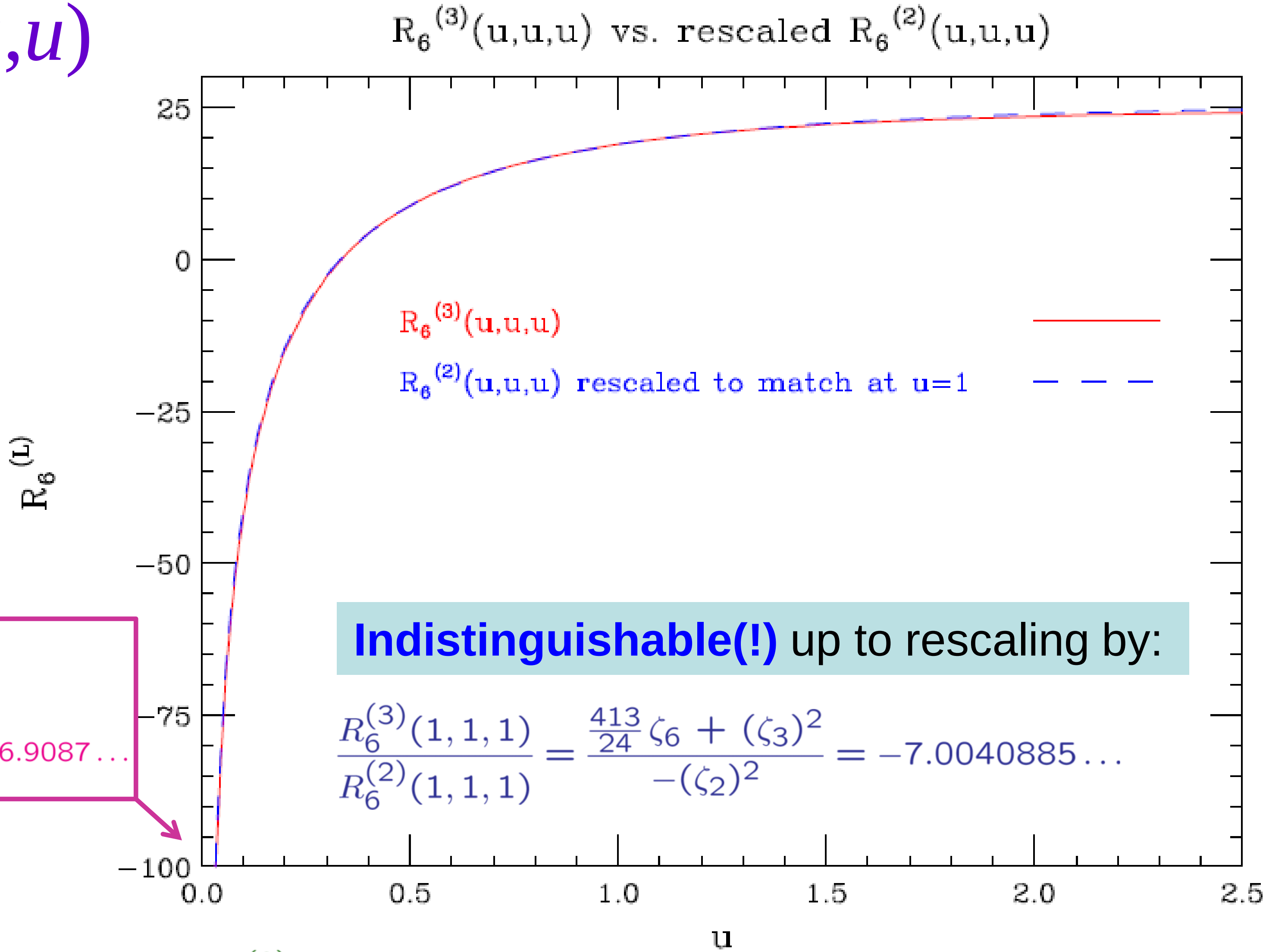
Includes base point $(1,1,1)$:

$$R_6^{(3)}(1,1,1) = \frac{413}{24}\,\zeta_6 + (\zeta_3)^2$$

Plot $R_6^{(3)}(u,v,w)$ on some slices



(u,u,u)



Indistinguishable(!) up to rescaling by:

$$\frac{R_6^{(3)}(1,1,1)}{R_6^{(2)}(1,1,1)} = \frac{\frac{413}{24}\zeta_6 + (\zeta_3)^2}{-(\zeta_2)^2} = -7.0040885\ldots$$

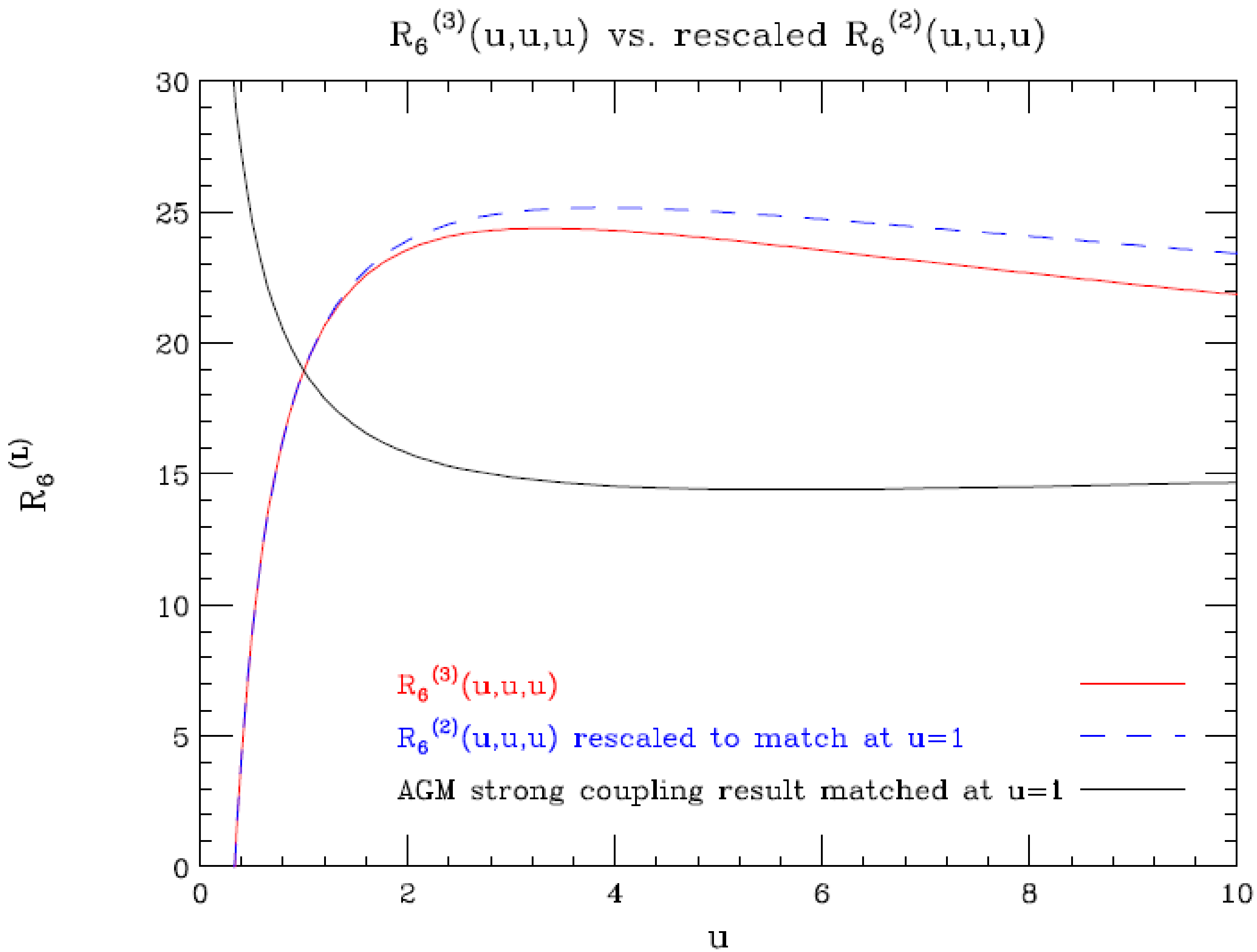
ratio ~

$$-7\frac{\pi^2}{10} = -6.9087\ldots$$

cf. cusp ratio:

$$\frac{\gamma_K^{(3)}}{\gamma_K^{(2)}} = \frac{22\,\zeta_4}{-4\,\zeta_2} = -3.61885\ldots$$

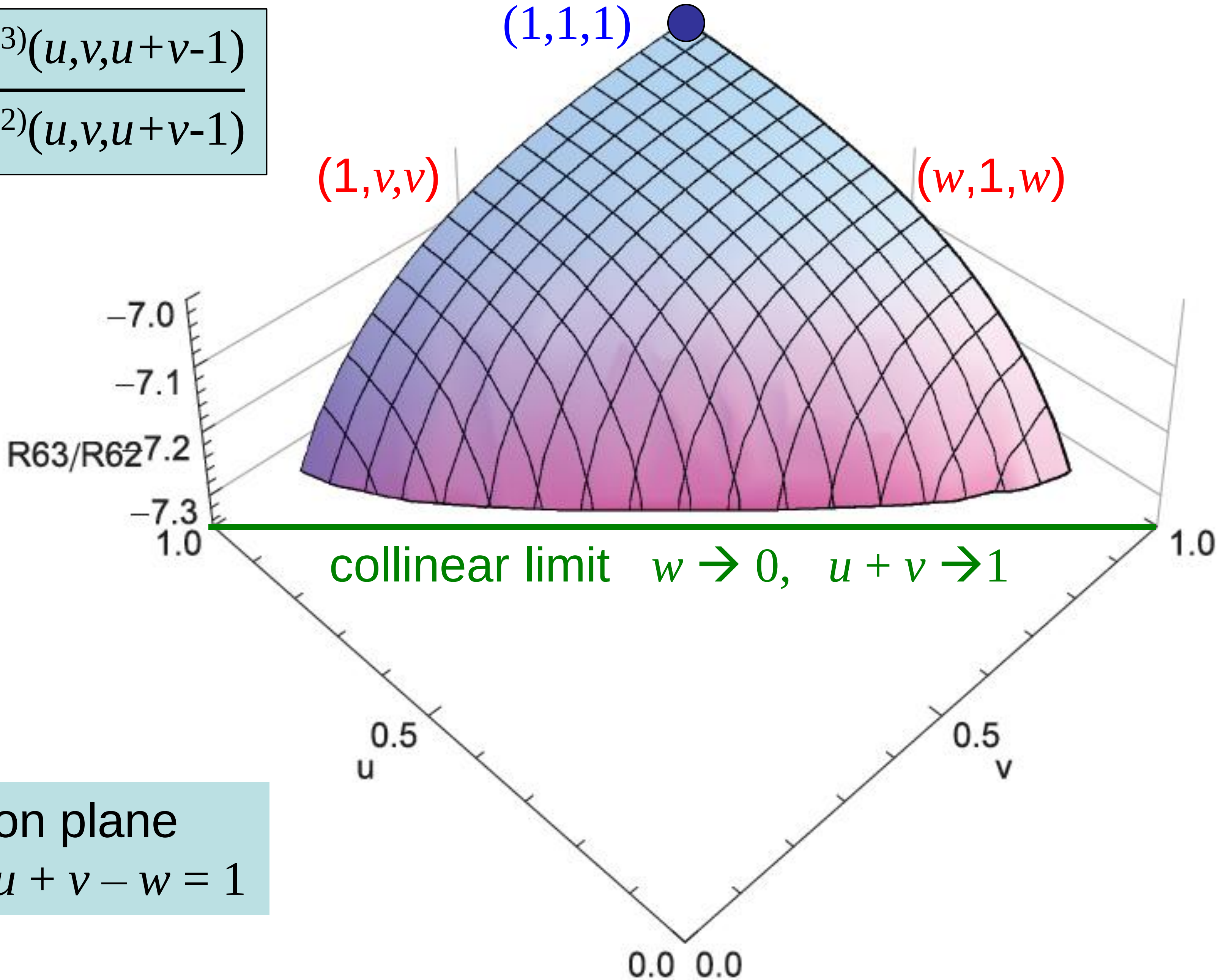
Proportionality ceases at large u



ratio ~ -1.23

0911.4708

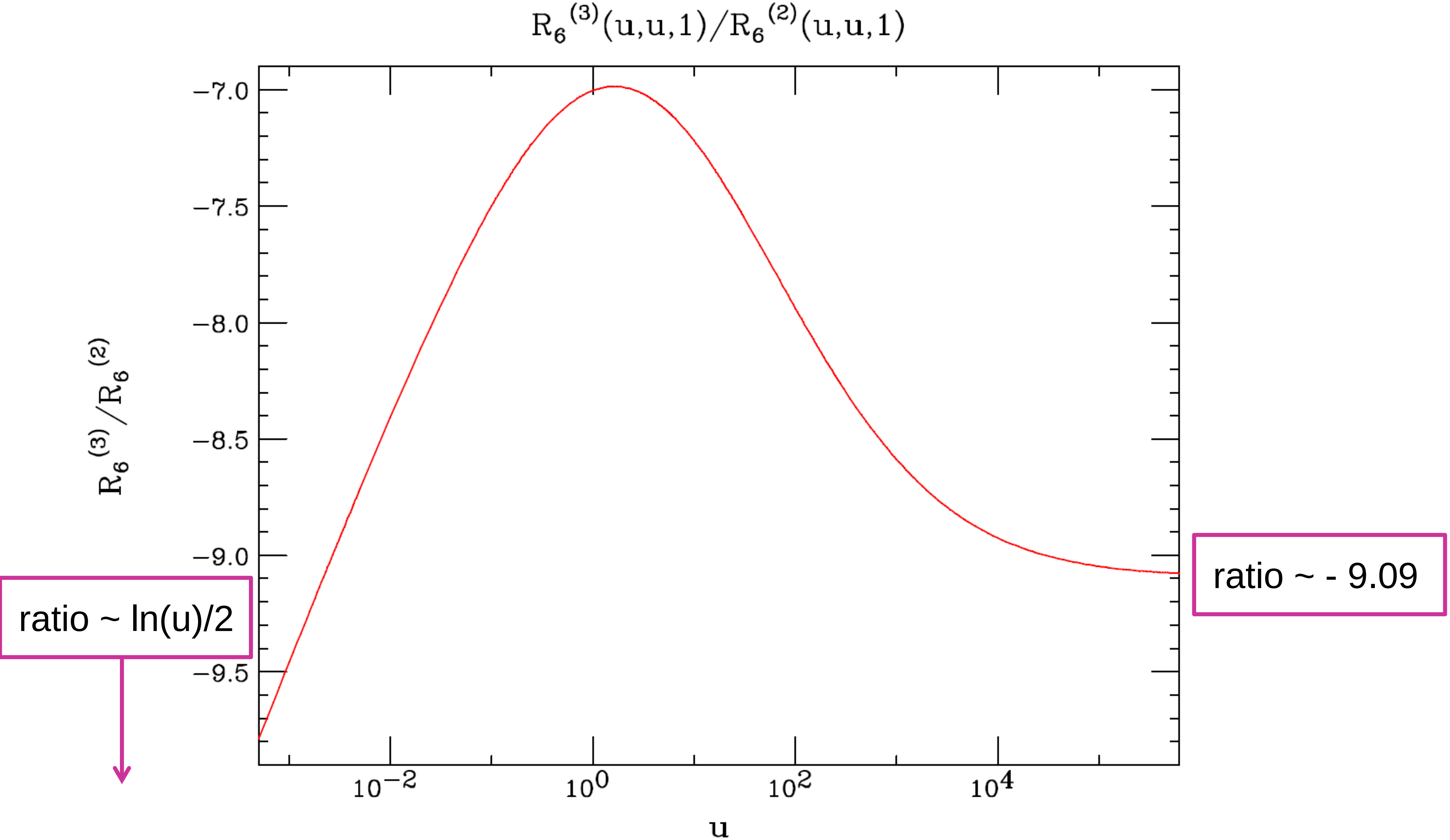
$$\frac{R_6^{(3)}(u,v,u+v-1)}{R_6^{(2)}(u,v,u+v-1)}$$



on plane

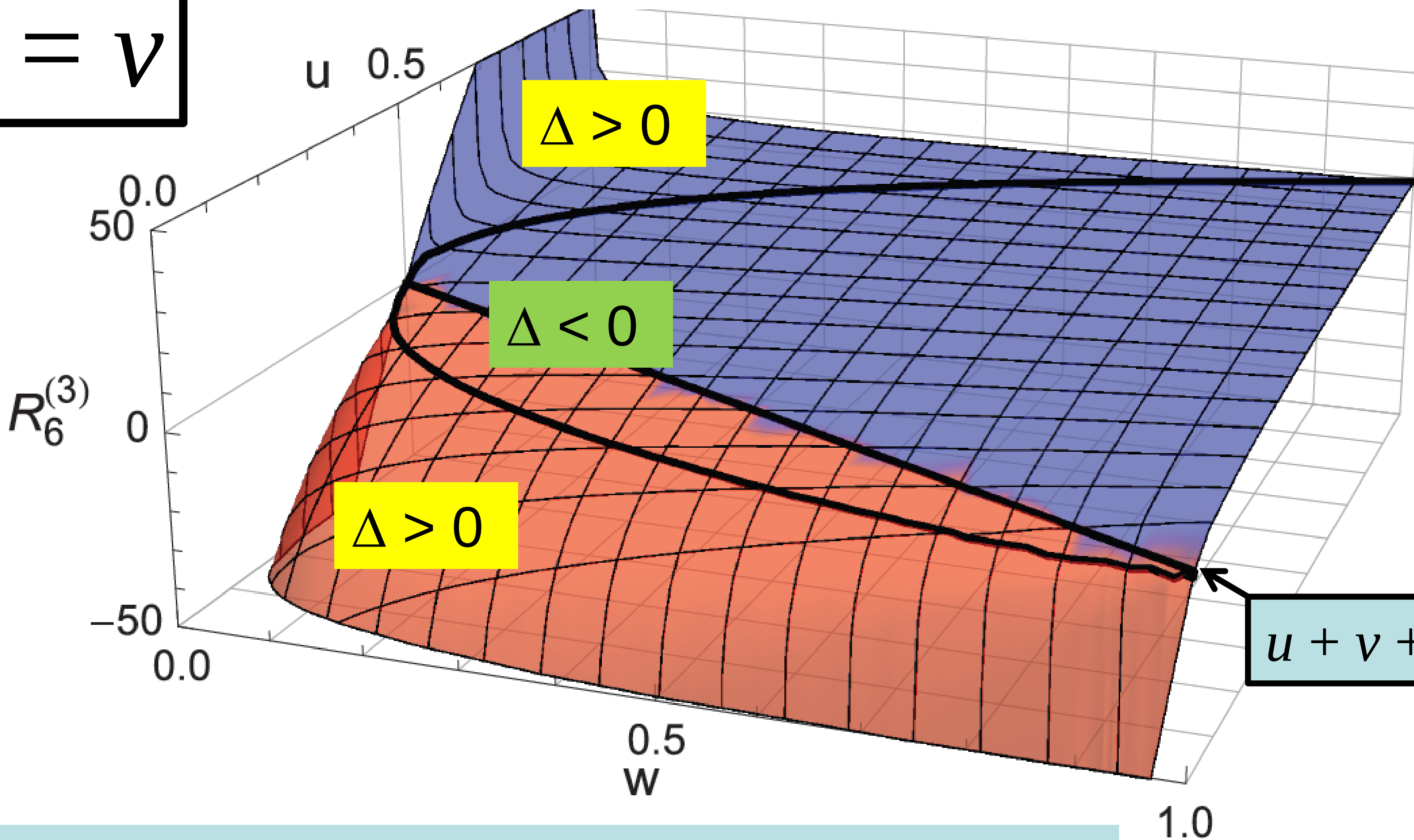
$u + v - w = 1$

Ratio for $(u,u,1) \leftarrow\rightarrow (1,v,v) \leftarrow \rightarrow (w,1,w)$



Sign is stable within $\Delta > 0$ regions

$u = v$

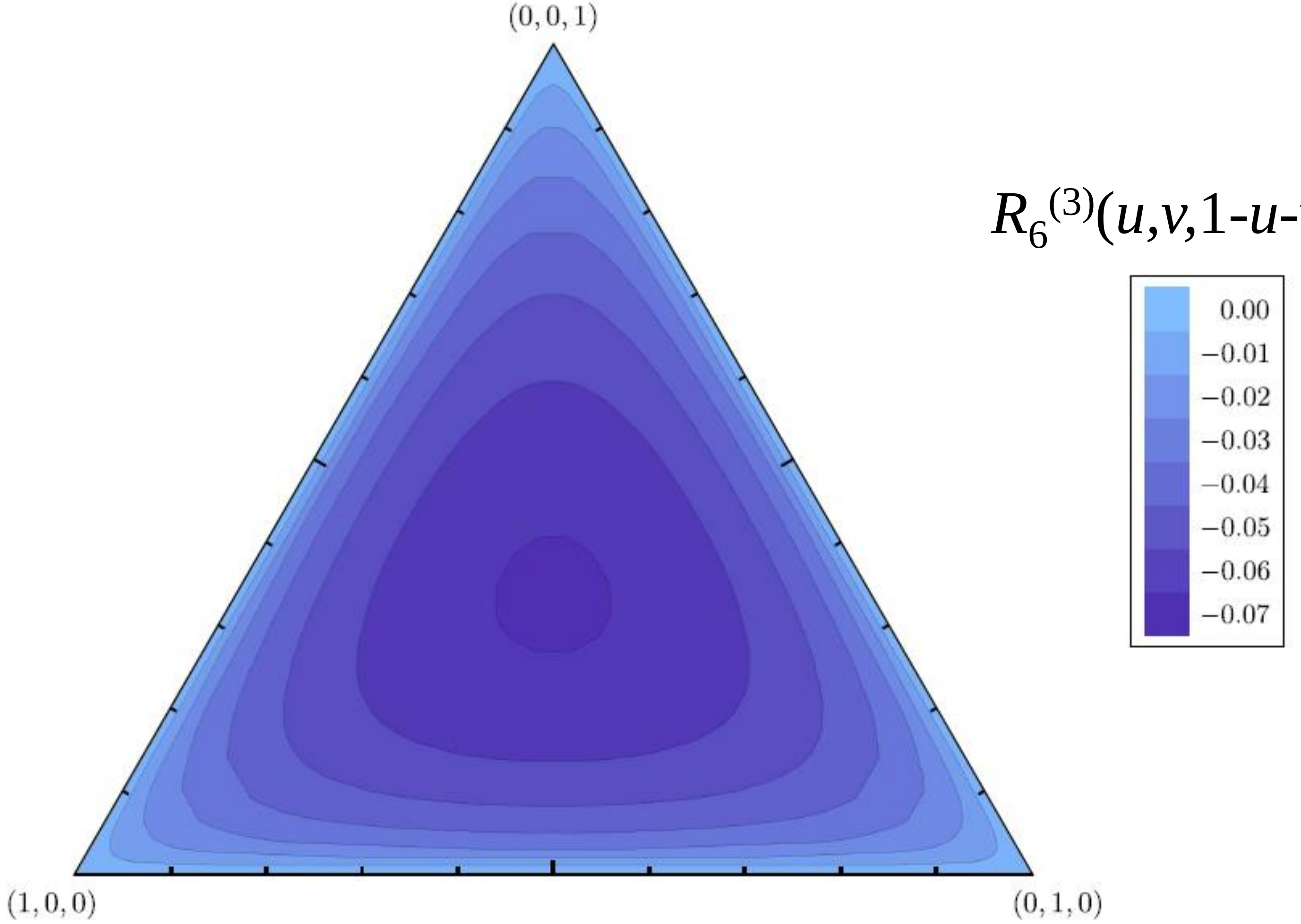


$u + v + w = 1$

relation to positive Grassmannian? Arkani-Hamed et al

Almost vanishes on $\boxed{u + v + w = 1}$

$R_6^{(3)}(u,v,1\text{-}u\text{-}v)$



Four loops

LD, Duhr, Drummond, Pennington, in progress

- In the course of 1207.0186, we "determined" the 4 loop remainder-function symbol.
- But still had 113 undetermined constants ☹
- Consistency with LLA and NLLA multi-Regge limits → 81 constants 😐
- Consistency with BSV's $v^{1/2}\, e^{\pm i\phi}$ → 4 constants ☺
- Adding BSV's $v^{1}\, e^{\pm 2i\phi}$ → 0 constants!! ☺☺

[Thanks to BSV for supplying this info!]

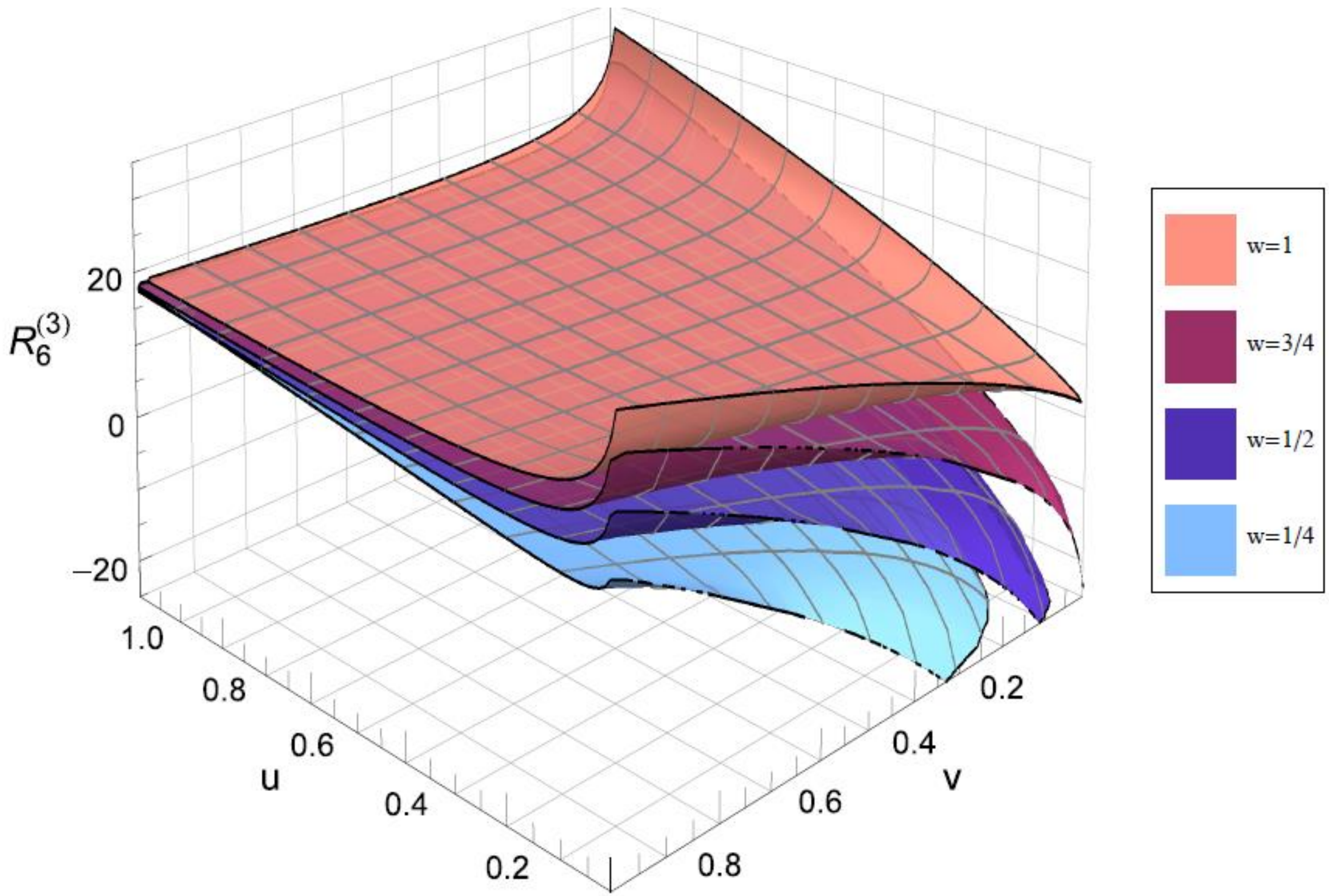
- Next step: Fix bts constants, after defining functions globally…

Conclusions

- Ansatz for function space allows determination of integrated planar N=4 SYM amplitudes over full kinematical phase space
- No need to know any integrands at all
- Important additional inputs from boundary data: near-collinear and/or multi-Regge limits
- First constrained symbol, then promoted symbol to a function via the {n-1,1} coproduct, working in space of hexagon functions.
- Would be very nice to have a more systematic construction of this space!

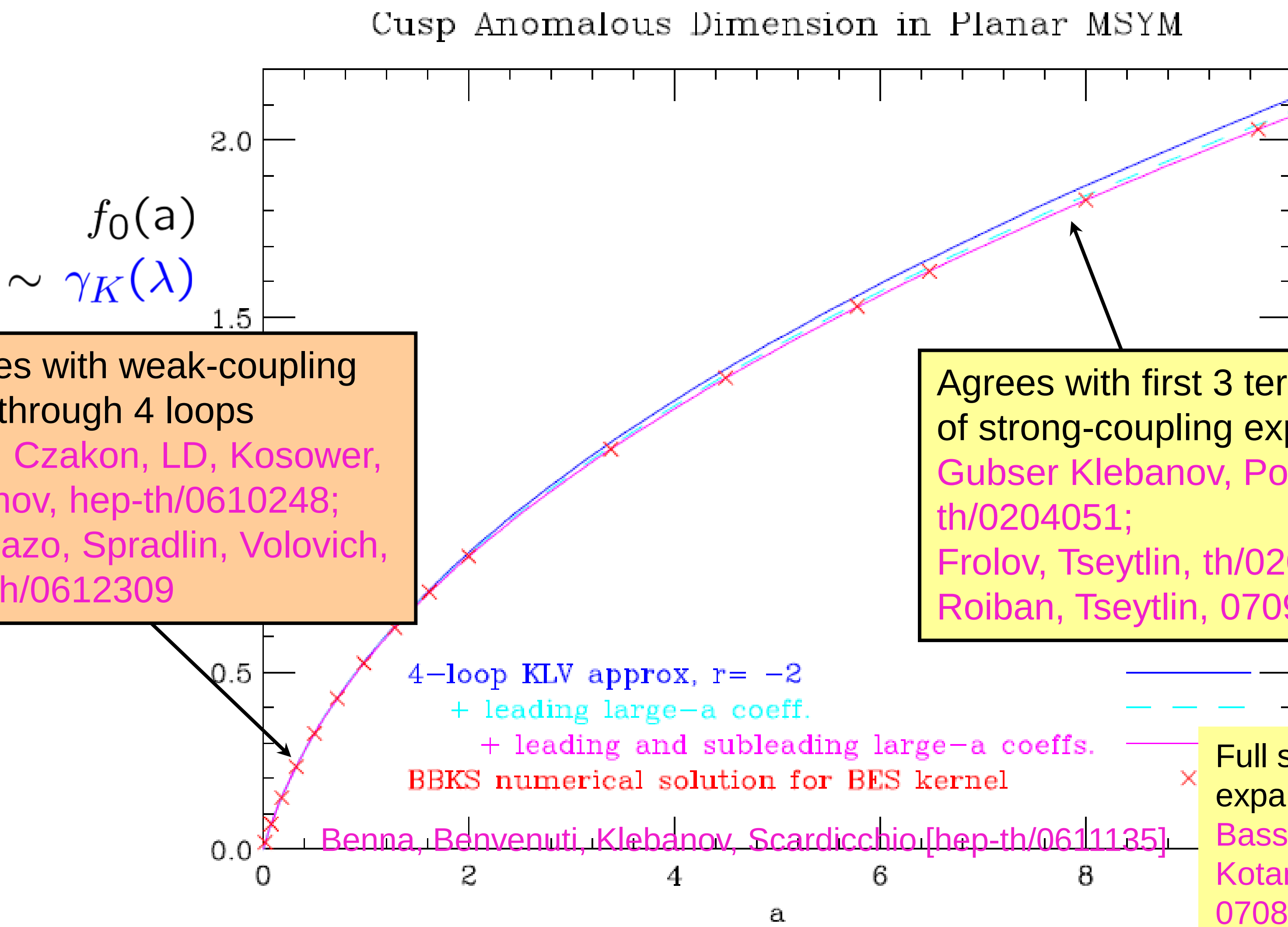
Extra Slides

Slices of constant *w*



$\gamma_K(\lambda)$ to all orders

Beisert, Eden, Staudacher [hep-th/0610251]



Agrees with weak-coupling data through 4 loops
Bern, Czakon, LD, Kosower, Smirnov, hep-th/0610248;
Cachazo, Spradlin, Volovich, hep-th/0612309

Agrees with first 3 terms of strong-coupling expansion
Gubser Klebanov, Polyakov, th/0204051;
Frolov, Tseytlin, th/0204226;
Roiban, Tseytlin, 0709.0681 [th]

Full strong-coupling expansion
Basso, Korchemsky, Kotański, 0708.3933 [th]

Integrability and planar N=4 SYM

- Anomalous dimensions for excitations of the GKP string, defined by $\mathrm{Tr}[X_1 \mathcal{D}^{+j} X_1]$ $\quad j \to \infty$

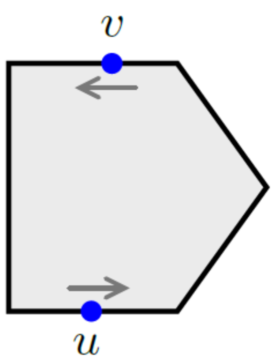
which also corresponds to excitations of a light-like Wilson line

Basso, 1010.5237

- And scattering of these excitations $S(u,v)$
- And the related pentagon transition

$P(u|v)$ for Wilson loops ...,

Basso, Sever, Vieira [BSV], 1303.1396; 1306.2058



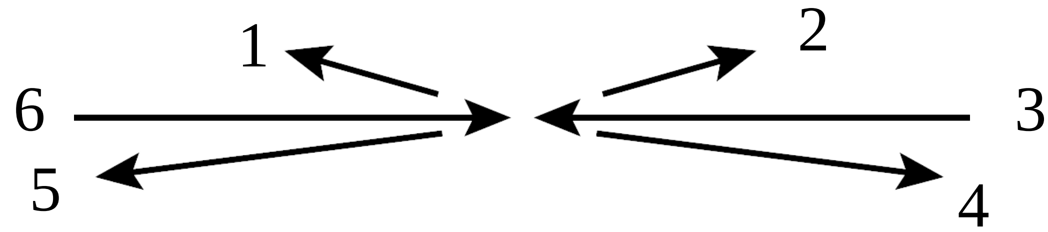
$P(u|v)$

Multi-Regge kinematics

$$u = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2} \to 1$$

$$\frac{v}{1-u} \to x$$

$$\frac{w}{1-u} \to y$$



Very nice change of variables
[LP, 1011.2673] is to (w, w^*) :

$$x = \frac{1}{(1+w)(1+w^*)}$$

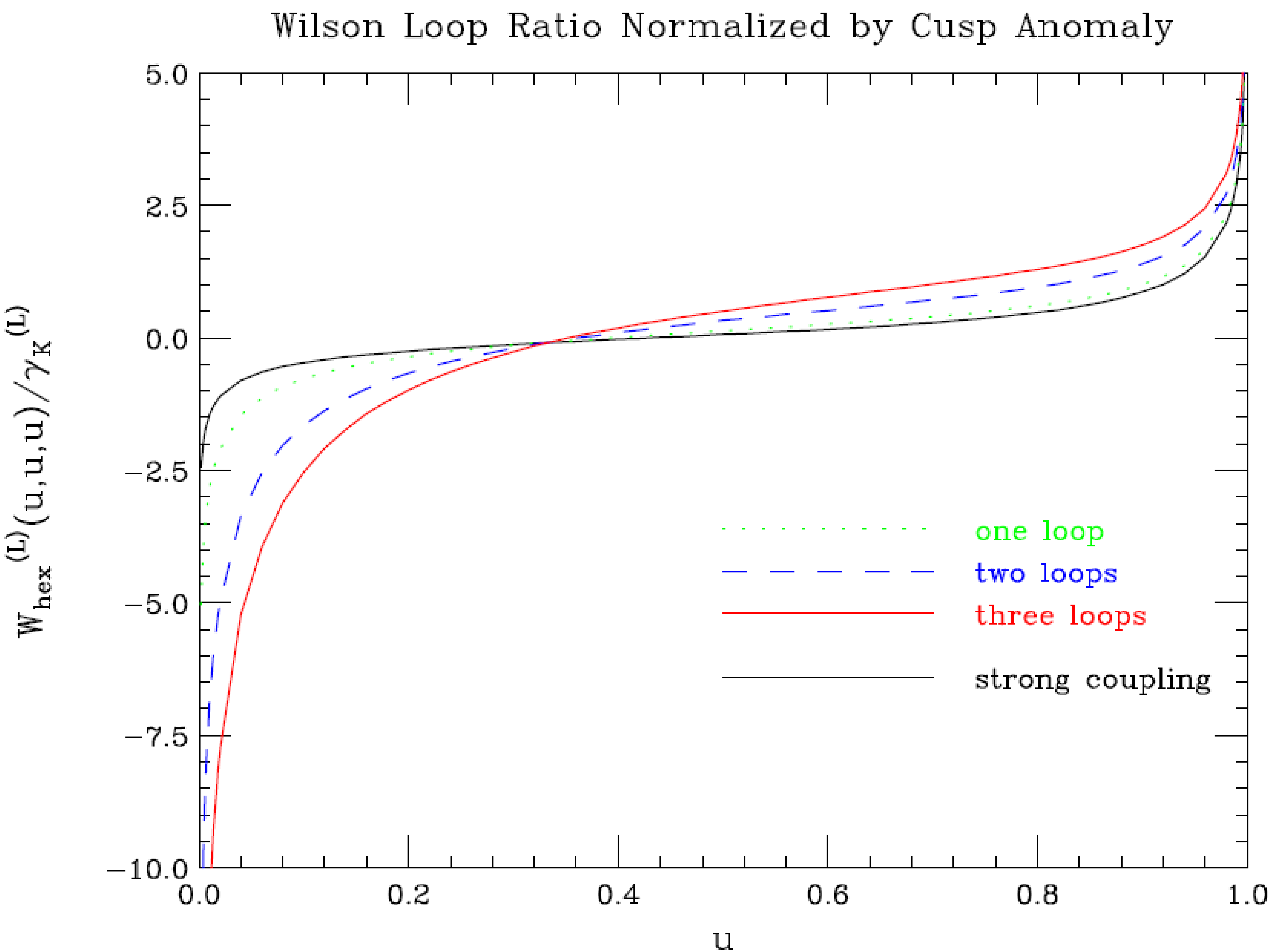
$$y = \frac{ww^*}{(1+w)(1+w^*)}$$

$$y_u \to 1$$

$$y_v \to \frac{1+w^*}{1+w}$$

$$y_w \to \frac{(1+w)w^*}{w(1+w^*)}$$

2 symmetries: conjugation $w \leftrightarrow w^*$
and inversion $w \leftrightarrow 1/w, \ w^* \leftrightarrow 1/w^*$


Wilson Loop Ratio Normalized by Cusp Anomaly
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0.0
0.2
0.4
0.6
0.8
1.0
u
$W_{hex}^{(L)}(u,u,u)/\gamma_K^{(L)}$
one loop
two loops
three loops
strong coupling