On the S matrices of certain integrable field theories

with O.T. Engelund and R.W. McKeown
also L. Bianchi, B. Hoare and V. Forini

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The $AdS_5 \times S^5$ bosonic Lagrangian

$$
\begin{aligned}
L_{\rm g.f.} = & -\frac{\sqrt{G_{\varphi\varphi}G_{tt}}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \Bigg\{ 1 - \frac{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}}{2} \\
& \times \left[\left(1 + \frac{1}{G_{\varphi\varphi}G_{tt}}\right) \partial_{\mathbf{a}} X \cdot \partial^{\mathbf{a}} X - \left(1 - \frac{1}{G_{\varphi\varphi}G_{tt}}\right) \left(\dot{X}\cdot\dot{X} + \acute{X}\cdot\acute{X}\right) \right] \\
& + \frac{\left[(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}\right]^2}{2 G_{\varphi\varphi} G_{tt}} \left[(\partial_{\mathbf{a}} X \cdot \partial^{\mathbf{a}} X)^2 - (\partial_{\mathbf{a}} X \cdot \partial_{\mathbf{b}} X)^2 \right] \Bigg\}^{1/2} \\
& + \frac{a}{1-a} \, \frac{G_{tt}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \, .
\end{aligned}
$$

$$
X = (Y^m, Z^\mu) \qquad G_{tt} = \left(\frac{1 + \frac{1}{4} Y^m Y_m}{1 - \frac{1}{4} Y^m Y_m} \right)^2 \qquad G_{\varphi\varphi} = \left(\frac{1 - \frac{1}{4} Z^\mu Z_\mu}{1 + \frac{1}{4} Z^\mu Z_\mu} \right)^2
$$

The relevant terms

$$L = \frac{1}{2}(\partial_{\mathbf{a}} X)^2 - \frac{1}{2} X^2 + \frac{1}{4} Z^2 (\partial_{\mathbf{a}} Z)^2 - \frac{1}{4} Y^2 (\partial_{\mathbf{a}} Y)^2 + \frac{1}{4}(Y^2 - Z^2)\left(\dot{X}^2 + \acute{X}^2\right)$$
$$- \frac{1-2a}{8}\left(X^2\right)^2 + \frac{1-2a}{4}(\partial_{\mathbf{a}} X \cdot \partial_{\mathbf{b}} X)^2 - \frac{1-2a}{8}\left[(\partial_{\mathbf{a}} X)^2\right]^2 + \ldots$$

Some 6-point terms

$$\mathcal{L}_6 = +\frac{1}{32\lambda}\left(-Y^2\acute{Y}^4 + Y^4\dot{Y}^2 - Y^2\dot{Y}^4 - \acute{Y}^2(9Y^4 + 2Y^2\dot{Y}^2) + 4Y^2(\dot{Y}\cdot\acute{Y})^2\right)$$
$$+\frac{1}{32\lambda}\left(-Z^2\acute{Z}^4 + Z^4\dot{Z}^2 - Z^2\dot{Z}^4 - \acute{Z}^2(9Z^4 + 2Z^2\dot{Z}^2) + 4Z^2(\dot{Z}\cdot\acute{Z})^2\right) + \ldots$$

No particle production: $A(2 \to 4) = 0 \quad , \quad A(3 \to 3) \neq 0$

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Tree-level S matrix with bosons in initial state

(Klose, McLoughlin, RR, Zarembo)

$$\delta^2(\mathrm{p}) = \frac{1}{\mathrm{p}_1 \times \mathrm{p}_2}(\delta(p_1 - p_3)\delta(p_2 - p_4) + \delta(p_1 - p_4)\delta(p_2 - p_3))$$

$$\begin{aligned}
\mathbb{T}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle = & + \tfrac{1}{2}\tfrac{(p-p')^2}{\varepsilon' p - \varepsilon p'}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle + \tfrac{pp'}{\varepsilon' p - \varepsilon p'}\Big(|Y_{a\dot{b}}Y'_{b\dot{a}}\rangle + |Y_{b\dot{a}}Y'_{a\dot{b}}\rangle\Big) \\
& - \tfrac{pp'}{\varepsilon' p - \varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\Big(\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|\Psi_{a\dot{\alpha}}\Psi'_{b\dot{\beta}}\rangle + \epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha\dot{a}}\Upsilon'_{\beta\dot{b}}\rangle\Big) \\
\mathbb{T}|Z_{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle = & - \tfrac{1}{2}\tfrac{(p-p')^2}{\varepsilon' p - \varepsilon p'}|Z_{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle - \tfrac{pp'}{\varepsilon' p - \varepsilon p'}\Big(|Z_{\alpha\dot{\beta}}Z'_{\beta\dot{\alpha}}\rangle + |Z_{\beta\dot{\alpha}}Z'_{\alpha\dot{\beta}}\rangle\Big) \\
& + \tfrac{pp'}{\varepsilon' p - \varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\Big(\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{a}\dot{b}}|\Upsilon_{\alpha\dot{a}}\Upsilon'_{\beta\dot{b}}\rangle + \epsilon_{\alpha\beta}\epsilon^{ab}|\Psi_{a\dot{\alpha}}\Psi'_{b\dot{\beta}}\rangle\Big) \\
\mathbb{T}|Y_{a\dot{a}}Z'_{\alpha\dot{\alpha}}\rangle = & - \tfrac{1}{2}\tfrac{p^2-p'^2}{\varepsilon' p - \varepsilon p'}|Y_{a\dot{a}}Z'_{\alpha\dot{\alpha}}\rangle + \tfrac{pp'}{\varepsilon' p - \varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}\left(|\Upsilon_{\alpha\dot{a}}\Psi'_{a\dot{\alpha}}\rangle - |\Psi_{a\dot{\alpha}}\Upsilon'_{\alpha\dot{a}}\rangle\right) \\
\mathbb{T}|Z_{\alpha\dot{\alpha}}Y'_{a\dot{a}}\rangle = & + \tfrac{1}{2}\tfrac{p^2-p'^2}{\varepsilon' p - \varepsilon p'}|Z_{\alpha\dot{\alpha}}Y'_{a\dot{a}}\rangle - \tfrac{pp'}{\varepsilon' p - \varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}\left(|\Psi_{a\dot{\alpha}}\Upsilon'_{\alpha\dot{a}}\rangle - |\Upsilon_{\alpha\dot{a}}\Psi'_{a\dot{\alpha}}\rangle\right)
\end{aligned}$$

- Useful identities

$$\varepsilon' p - \varepsilon p' = \sinh(\theta - \theta')$$

$$(p - p')\cosh\tfrac{\theta-\theta'}{2} = (\varepsilon + \varepsilon')\sinh\tfrac{\theta-\theta'}{2}$$

$$\sinh\tfrac{\theta}{2} = \tfrac{1}{2}\sqrt{\varepsilon + p} - \tfrac{1}{2}\sqrt{\varepsilon - p}$$

$$\cosh\tfrac{\theta}{2} = \tfrac{1}{2}\sqrt{\varepsilon + p} + \tfrac{1}{2}\sqrt{\varepsilon - p}$$

$$\sinh\tfrac{\theta-\theta'}{2} = \tfrac{1}{2}\sqrt{(\varepsilon + p)(\varepsilon' - p')} - \tfrac{1}{2}\sqrt{(\varepsilon - p)(\varepsilon' + p')}$$

$$\cosh\tfrac{\theta-\theta'}{2} = \tfrac{1}{2}\sqrt{(\varepsilon + p)(\varepsilon' - p')} + \tfrac{1}{2}\sqrt{(\varepsilon - p)(\varepsilon' + p')}$$

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Factorized tree-level S matrix: $\mathbb{S} = \mathrm{S} \otimes \mathrm{S} \quad \Rightarrow \quad \mathbb{T} = \mathbb{I} \otimes \mathrm{T} + \mathrm{T} \otimes \mathbb{I}$

Klose, McLoughlin, RR, Zarembo

$$\mathrm{T}_{ab}^{cd} = \mathrm{A}\,\delta_a^c \delta_b^d + \mathrm{B}\,\delta_a^d \delta_b^c \,, \qquad \mathrm{T}_{ab}^{\gamma\delta} = \mathrm{C}\,\epsilon_{ab}\epsilon^{\gamma\delta} \,,$$

$$\mathrm{T}_{\alpha\beta}^{\gamma\delta} = \mathrm{D}\,\delta_\alpha^\gamma \delta_\beta^\delta + \mathrm{E}\,\delta_\alpha^\delta \delta_\beta^\gamma \,, \qquad \mathrm{T}_{\alpha\beta}^{cd} = \mathrm{F}\,\epsilon_{\alpha\beta}\epsilon^{cd} \,,$$

$$\mathrm{T}_{a\beta}^{c\delta} = \mathrm{G}\,\delta_a^c \delta_\beta^\delta \,, \qquad \mathrm{T}_{\alpha b}^{\gamma d} = \mathrm{L}\,\delta_\alpha^\gamma \delta_b^d \,,$$

$$\mathrm{T}_{a\beta}^{\gamma d} = \mathrm{H}\,\delta_a^d \delta_\beta^\gamma \,, \qquad \mathrm{T}_{\alpha b}^{c\delta} = \mathrm{K}\,\delta_\alpha^\delta \delta_b^c \,.$$

$$\mathrm{A}(p,p') = \frac{1}{4}\left[(1-2a)\,(\varepsilon' p - \varepsilon p') + \frac{(p-p')^2}{\varepsilon' p - \varepsilon p'}\right] \,,$$

$$\mathrm{B}(p,p') = -\mathrm{E}(p,p') = \frac{pp'}{\varepsilon' p - \varepsilon p'} \,,$$

$$\mathrm{C}(p,p') = \mathrm{F}(p,p') = \frac{1}{2}\frac{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}\,(\varepsilon' p - \varepsilon p' + p' - p)}{\varepsilon' p - \varepsilon p'} \,,$$

$$\varepsilon = \sqrt{1+p^2}$$

$$\varepsilon' = \sqrt{1+p'^2}$$

$$\mathrm{D}(p,p') = \frac{1}{4}\left[(1-2a)\,(\varepsilon' p - \varepsilon p') - \frac{(p-p')^2}{\varepsilon' p - \varepsilon p'}\right] \,,$$

$$\mathrm{G}(p,p') = \mathrm{L}(p',p) = \frac{1}{4}\left[(1-2a)\,(\varepsilon' p - \varepsilon p') - \frac{p^2 - p'^2}{\varepsilon' p - \varepsilon p'}\right] \,,$$

$$\mathrm{H}(p,p') = \mathrm{K}(p,p') = \frac{1}{2}\,\frac{pp'}{\varepsilon' p - \varepsilon p'}\,\frac{(\varepsilon+1)\,(\varepsilon'+1) - pp'}{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}} \,.$$

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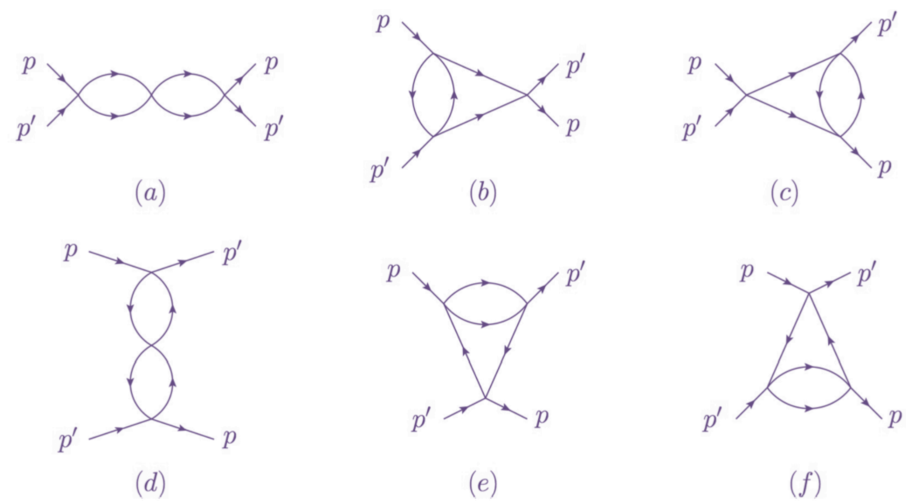
Two-loop S matrix: logarithmic terms

Engelund, McKeown, RR

$$i\mathrm{T}^{(2)} = \frac{1}{4}C_a\tilde{I}_a + \frac{1}{2}C_b\tilde{I}_b + \frac{1}{2}C_c\tilde{I}_c + \frac{1}{4}C_d\tilde{I}_d + \frac{1}{2}C_e\tilde{I}_e + \frac{1}{2}C_f\tilde{I}_f$$

$$+ \frac{1}{2}C_{s,\mathrm{extra}}\,\tilde{I}_s + \frac{1}{2}C_{u,\mathrm{extra}}\,\tilde{I}_u \quad + \text{ rational}$$



$$(C_a)^{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} (i\mathrm{T}^{(0)})^{CD'}_{GH'} (C_s)^{GH'}_{AB'}$$

$$(C_b)^{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} (i\mathrm{T}^{(0)})^{CD'}_{GH'} (C_u)^{GH'}_{AB'}$$

$$(C_c)^{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} (C_s)^{CD'}_{GH'} (i\mathrm{T}^{(0)})^{GH'}_{AB'}$$

$$(C_d)^{CD'}_{AB'} = (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (i\mathrm{T}^{(0)})^{CH'}_{GB'} (C_u)^{GD'}_{AH'}$$

$$(C_e)^{CD'}_{AB'} = (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (i\mathrm{T}^{(0)})^{CH'}_{GB'} (C_s)^{GD'}_{AH'}$$

$$(C_f)^{CD'}_{AB'} = (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (C_s)^{CH'}_{GB'} (i\mathrm{T}^{(0)})^{GD'}_{AH'}$$

$$J = \mathrm{p} \times \mathrm{p}' = \varepsilon p' - \varepsilon' p$$

One-loop integrals

Klose, McLoughlin, Minahan, Zarembo

$$I_s = \frac{1}{J_s}\left(-\frac{i}{\pi}\ln\frac{p'_-}{p_-} - 1\right) \qquad I_u = \frac{1}{J_u}\left(+\frac{i}{\pi}\ln\frac{p'_-}{p_-} + 0\right)$$

Two-loop integrals

Klose, McLoughlin, Minahan, Zarembo

$$I_a = \left(\frac{1}{J_s}\left(-\frac{i}{\pi}\ln\frac{p'_-}{p_-} - 1\right)\right)^2$$

$$I_d = \left(\frac{1}{J_u}\left(+\frac{i}{\pi}\ln\frac{p'_-}{p_-} + 0\right)\right)^2$$

$$J = \mathrm{p} \times \mathrm{p}' = \varepsilon p' - \varepsilon' p$$

$$I_b = \frac{1}{16\pi^2}\left(\frac{4}{J_u^2}\ln^2\frac{p'_-}{p_-} + \left(-\frac{8i\pi}{J_u^2} + \frac{2}{J_u}\right)\ln\frac{p'_-}{p_-} + \text{rational}\right)$$

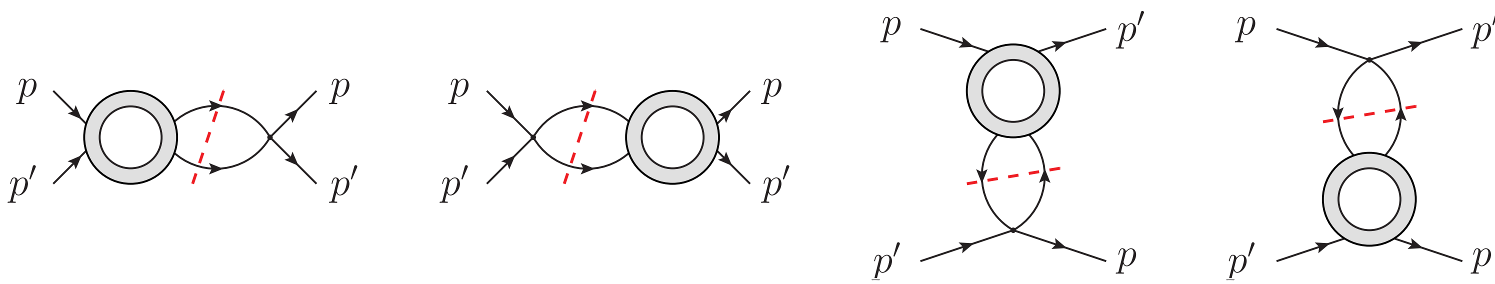
$$I_c = \frac{1}{16\pi^2}\left(\frac{4}{J_u^2}\ln^2\frac{p'_-}{p_-} + \left(-\frac{8i\pi}{J_u^2} + \frac{2}{J_u}\right)\ln\frac{p'_-}{p_-} + \text{rational}\right)$$

$$I_e = \frac{1}{16\pi^2}\left(\frac{4}{J_u^2}\ln^2\frac{p'_-}{p_-} - \frac{2}{J_u}\ln\frac{p'_-}{p_-} + \text{rational}\right)$$

$$I_f = \frac{1}{16\pi^2}\left(\frac{4}{J_u^2}\ln^2\frac{p'_-}{p_-} - \frac{2}{J_u}\ln\frac{p'_-}{p_-} + \text{rational}\right)$$

► All double-logarithms cancel out in the two-loop S matrix

Coefficients of the 1-loop integrals; subleading logarithms



$$\frac{1}{J_s}(C_{s,\text{extra}})^{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} \left((i\mathrm{T}^{(0)})^{CD'}_{GH'} (i\tilde{\mathrm{T}}^{(1)})^{GH'}_{AB'} + (i\tilde{\mathrm{T}}^{(1)})^{CD'}_{GH'} (i\mathrm{T}^{(0)})^{GH'}_{AB'} \right) + \frac{(C_a)^{CD'}_{AB'}}{J_s^2} - \frac{1}{2}\left(\frac{(C_e)^{CD'}_{AB'}}{J_s} + \frac{(C_f)^{CD'}_{AB'}}{J_s} \right) I_t$$

$$\frac{1}{J_u}(C_{u,\text{extra}})^{CD'}_{AB'} = (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} \left((i\mathrm{T}^{(0)})^{CH'}_{GB'} (i\tilde{\mathrm{T}}^{(1)})^{GD'}_{AH'} + (i\tilde{\mathrm{T}}^{(1)})^{CH'}_{GB'} (i\mathrm{T}^{(0)})^{GD'}_{AH'} \right) + \frac{1}{2}\left(\frac{(C_e)^{CD'}_{AB'}}{J_s J_u} + \frac{(C_f)^{CD'}_{AB'}}{J_u J_s} \right) - \frac{1}{2}\left(\frac{(C_e)^{CD'}_{AB'}}{J_s} + \frac{(C_f)^{CD'}_{AB'}}{J_s} \right) I_t$$

Coefficients of the 1-loop integrals; subleading logarithms

$$\frac{1}{J_s}(C_{s,\text{extra}})^{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} \left((i\mathrm{T}^{(0)})^{CD'}_{GH'} (i\tilde{\mathrm{T}}^{(1)})^{GH'}_{AB'} + (i\tilde{\mathrm{T}}^{(1)})^{CD'}_{GH'} (i\mathrm{T}^{(0)})^{GH'}_{AB'} \right) + \frac{(C_a)^{CD'}_{AB'}}{J_s^2} - \frac{1}{2}\left(\frac{(C_e)^{CD'}_{AB'}}{J_s} + \frac{(C_f)^{CD'}_{AB'}}{J_s} \right) I_t$$

$$\frac{1}{J_u}(C_{u,\text{extra}})^{CD'}_{AB'} = (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} \left((i\mathrm{T}^{(0)})^{CH'}_{GB'} (i\tilde{\mathrm{T}}^{(1)})^{GD'}_{AH'} + (i\tilde{\mathrm{T}}^{(1)})^{CH'}_{GB'} (i\mathrm{T}^{(0)})^{GD'}_{AH'} \right) + \frac{1}{2}\left(\frac{(C_e)^{CD'}_{AB'}}{J_s J_u} + \frac{(C_f)^{CD'}_{AB'}}{J_u J_s} \right) - \frac{1}{2}\left(\frac{(C_e)^{CD'}_{AB'}}{J_s} + \frac{(C_f)^{CD'}_{AB'}}{J_s} \right) I_t$$

Piece together the two-loop S matrix

$$i\mathrm{T}^{(2)} = -\frac{1}{2}\left(-\frac{1}{\pi}\frac{p^2 p'^2(\varepsilon\varepsilon' - pp')}{(\varepsilon' p - \varepsilon p')^2} \ln\frac{p'_-}{p_-} \right) \mathrm{T}^{(0)} + \text{rational} = -\frac{1}{2}\hat{\theta}^{(1)}_{12}\mathrm{T}^{(0)} + \text{rational}$$

► logarithms exponentiate

$$x_{1,2} = \frac{1+\varepsilon_{1,2}}{p_{1,2}}$$

$$\varepsilon_{1,2} = \sqrt{1+p_{1,2}^2}$$

- Integrate to complete phase argument:

$$\begin{aligned}\chi^{(1)}(x_1,x_2) \;=\; & -\frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1}-1/\sqrt{x_2}}{\sqrt{x_1}-\sqrt{x_2}} - \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1}+1/\sqrt{x_2}}{\sqrt{x_1}+\sqrt{x_2}} \\ & + \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1}+1/\sqrt{x_2}}{\sqrt{x_1}-\sqrt{x_2}} + \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1}-1/\sqrt{x_2}}{\sqrt{x_1}+\sqrt{x_2}} + \dots\end{aligned}$$

Recall:

$$\begin{aligned}\theta_{12} \;=\; & \chi(x_1^+,x_2^+) - \chi(x_1^+,x_2^-) - \chi(x_1^-,x_2^+) + \chi(x_1^-,x_2^-) \\ - \; & \chi(x_2^+,x_1^+) + \chi(x_2^+,x_1^-) + \chi(x_2^-,x_1^+) - \chi(x_2^-,x_1^-)\end{aligned}$$

Further corrections change the functional form of $x(p)$

$$x^{\pm} = x \pm \frac{i}{2g}(1+\varepsilon) + \dots$$

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A summary and more

- Recovered known results

- Minimal changes needed for other integrable models of interest:
  $AdS_4 \times CP^3,\ AdS_3 \times S^3 \times S^3 \times S^1,\ AdS_3 \times S^3 \times T^4,\ AdS_2 \times S^2 \times T^6$
  truncation of massless states becomes important; new results e.g.

$AdS_3 \times S^3 \times S^3 \times S^1$

$$iT^{(1)}_{LL} = i\left(\frac{1}{2}\left(-\frac{1}{\pi}\frac{p^2(p')^2(\mathbf{p}\cdot\mathbf{p}' + mm')}{2(\varepsilon' p - p'\varepsilon)^2}\left(\ln\left|\frac{p'_-}{p_-}\right| - \ln\left|\frac{m'}{m}\right|\right)\right)\mathbb{1} + \text{rational}\right)$$

$$iT^{(1)}_{LR} = i\left(\frac{1}{2}\left(-\frac{1}{\pi}\frac{p^2(p')^2(\mathbf{p}\cdot\mathbf{p}' - mm')}{2(\varepsilon' p - p'\varepsilon)^2}\left(\ln\left|\frac{p'_-}{p_-}\right| - \ln\left|\frac{m'}{m}\right|\right)\right)\mathbb{1} + \text{rational}\right)$$

$(AdS_3 \times S^3)_q \times T^4$

$$iT^{(1)}_{LL} = i\Big(\frac{1}{2}\Big(-\frac{1}{\pi}\frac{p^2(p')^2\left(\varepsilon_+\varepsilon'_+ + pp' + \sqrt{(1+2pq)(1+2p'q)}\right)}{2(p-p')^2}\ln\left|\frac{\varepsilon'_+ - p' - q}{\varepsilon_+ - p - q}\right|\Big)\mathbb{1} + \text{rat.}\Big)$$

$$iT^{(1)}_{LR} = i\Big(\frac{1}{2}\Big(-\frac{1}{\pi}\frac{p^2(p')^2\left(\varepsilon_+\varepsilon'_- + pp' - \sqrt{(1+2pq)(1-2p'q)}\right)}{2(p+p')^2}\ln\left|\frac{\varepsilon'_- - p' + q}{\varepsilon_+ - p - q}\right|\Big)\mathbb{1} + \text{rat.}\Big)$$

A summary and more

- Check higher-loop factorization; does this constrain/fix rational terms?
- Better handle on rational terms, especially those proportional to $\mathbb{I}$
- Related -- better handle on regularization
- Corrections to 2d dispersion relation; (contribute also to S matrix)
  2d form factors of fundamental fields are important — Engelund, in progress
- S matrices around other vacua;
  - relation to 4d S matrices if vacuum is long folded string — Basso, Sever, Vieira
  - further tests of integrability results