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Leading Singularities and Off-Shell Conformal Integrals

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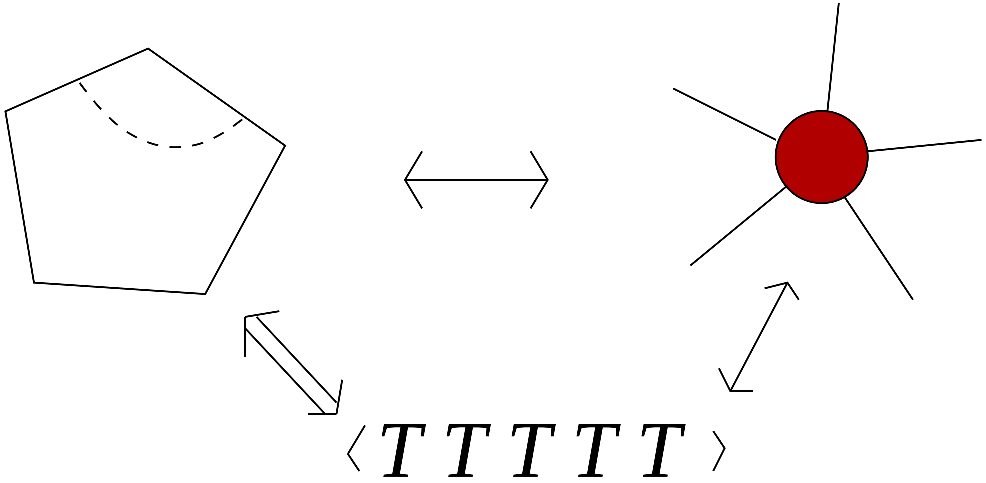
Introduction

- Polygonal **Wilson loops** with light-like edges are **dual** to on-shell **amplitudes** in $\mathcal{N} = 4$ SYM.

  New: n-point functions of the **stress tensor** $\mathcal{T}$ generate **both** in a **light-cone limit**.
  [Alday, Eden, Heslop, Korchemsky, Maldacena, Sokatchev]



- **Three-loop four-point correlator** combining older work on **supersymmetry constraints** [Eden, Howe, Petkou, Sokatchev, Schubert, West] with the **"triality"**. No Feynman graphs!
- Beyond ladders, **two unknown conformal three-loop integrals** ("easy" **E** and "hard" **H**).
- **Here**: Evaluate E, H using **leading singularities** and the **symbol**.
- **General technique** not limited to $\mathcal{N} = 4$.

The off-shell four-point function

Quantum corrections take a **factorised form**: [Eden, Petkou, Schubert, Sokatchev (2000)]

$$G_4(1,2,3,4) = G_4^{(0)} + \frac{2\,(N_c^2-1)}{(4\pi^2)^4}\, R(1,2,3,4)\ \left[aF^{(1)} + a^2F^{(2)} + a^3F^{(3)} + O(a^4)\right],$$

where (**y**'s keep track of indices of the **internal symmetry** $SU(4)$)

$$\begin{aligned} R(1,2,3,4) = {} & \frac{y_{12}^2y_{23}^2y_{34}^2y_{14}^2}{x_{12}^2x_{23}^2x_{34}^2x_{14}^2}(x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2) + \frac{y_{12}^2y_{13}^2y_{24}^2y_{34}^2}{x_{12}^2x_{13}^2x_{24}^2x_{34}^2}(x_{14}^2x_{23}^2 - x_{12}^2x_{34}^2 - x_{13}^2x_{24}^2) \\ & + \frac{y_{13}^2y_{14}^2y_{23}^2y_{24}^2}{x_{13}^2x_{14}^2x_{23}^2x_{24}^2}(x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2 - x_{13}^2x_{24}^2) + \frac{y_{12}^4y_{34}^4}{x_{12}^2x_{34}^2} + \frac{y_{13}^4y_{24}^4}{x_{13}^2x_{24}^2} + \frac{y_{14}^4y_{23}^4}{x_{14}^2x_{23}^2} \end{aligned}$$

Integrands at one and two loops:

[Eden, Schubert, Sokatchev (2000)], [Bianchi, Kovacs, Rossi, Stanev (2000)]

$$I_4^{(1)} \propto \frac{1}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}\,, \qquad I_4^{(2)} \propto \frac{x_{12}^2x_{34}^2x_{56}^2 \ + \text{(14 terms)}}{(x_{15}^2x_{25}^2x_{35}^2x_{45}^2)(x_{16}^2x_{26}^2x_{36}^2x_{46}^2)x_{56}^2}$$

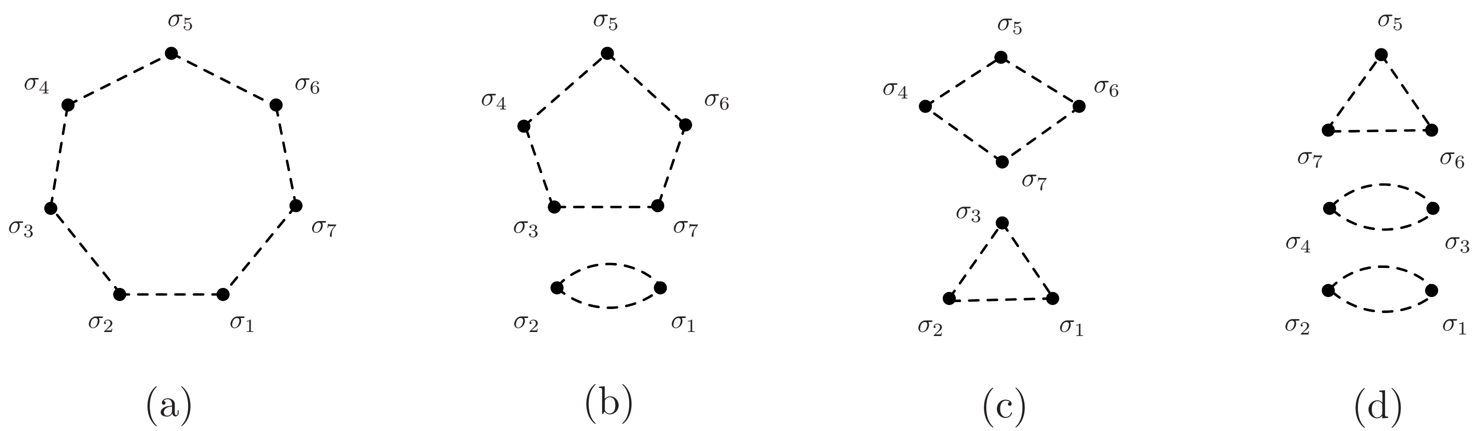
- Numerator of $I_4^{(l)}$ must have $\mathbf{S_{l+4}}$ **symmetry**. [Eden, Heslop, Korchemsky, Sokatchev (2011)]

Three loop ansatz:

$$I_4^{(3)} \propto \frac{P^{(3)}}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2)(x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2)(x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2) x_{56}^2 x_{57}^2 x_{67}^2}$$

$P^{(3)}(x_{ij}^2)$ should be **$\mathbf{S_7}$ symmetric** and it should have conformal **weight -2** at every point. Options:

(a) heptagon:	$x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 \; + \; S_7$ permutations
(b) 2-gon × pentagon:	$(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \; + \; S_7$ permutations
(c) triangle × square:	$(x_{12}^2 x_{23}^2 x_{31}^2)(x_{45}^2 x_{56}^2 x_{67}^2 x_{74}^2) \; + \; S_7$ permutations
(d) 2-gon × 2-gon × triangle:	$(x_{12}^4)(x_{34}^4)(x_{56}^2 x_{67}^2 x_{75}^2) \; + \; S_7$ permutations



Coefficients fixed by **comparing to amplitudes** [Eden, Heslop, Korchemsky, Sokatchev (2011)]

The four-point correlator up to three loops

Functions in the **quantum part**:

$$F^{(1)} = g(1,2,3,4)\,,$$

$$\begin{aligned}F^{(2)} &= h(1,2;3,4) + h(3,4;1,2) + h(2,3;1,4) + h(1,4;2,3)\\ &\quad + h(1,3;2,4) + h(2,4;1,3) + \frac{1}{2}\big(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2\big)\big[\,g(1,2,3,4)\big]^{\,2}\,,\end{aligned}$$

$$\begin{aligned}F^{(3)} &= \big[\,L(1,3;2,4) + 5 \text{ perms}\,\big] + \big[\,T(1,3;2,4) + 11 \text{ perms}\,\big]\\ &\quad + \big[\,E(2;1,3;4) + 11 \text{ perms}\,\big] + \tfrac{1}{2}\big[\,H(1,3;2,4) + 11 \text{ perms}\,\big]\\ &\quad + \big[\,(g\times h)(1,3;2,4) + 5 \text{ perms}\,\big]\end{aligned}$$

List of integrals:

$$g(1,2,3,4) = \frac{1}{\pi^2}\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,,$$

$$h(1,2;3,4) = \frac{x_{34}^2}{\pi^4}\int \frac{d^4x_5\, d^4x_6}{(x_{15}^2 x_{35}^2 x_{45}^2)\, x_{56}^2\, (x_{26}^2 x_{36}^2 x_{46}^2)}$$

At **third order**:

$$(g \times h)(1,2;3,4) = \frac{x_{12}^2 x_{34}^4}{\pi^6} \int \frac{d^4x_5 d^4x_6 d^4x_7}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2)(x_{16}^2 x_{36}^2 x_{46}^2)(x_{27}^2 x_{37}^2 x_{47}^2) x_{67}^2},$$

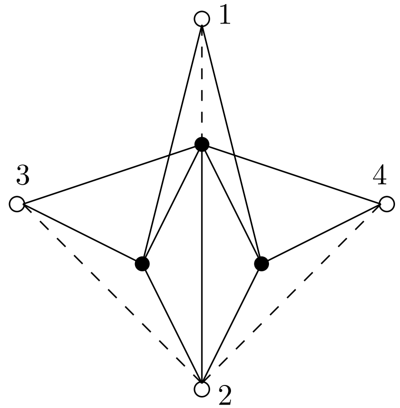
$$L(1,2;3,4) = \frac{x_{34}^4}{\pi^6} \int \frac{d^4x_5\, d^4x_6\, d^4x_7}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{36}^2 x_{46}^2) x_{67}^2 (x_{27}^2 x_{37}^2 x_{47}^2)},$$

$$T(1,2;3,4) = \frac{x_{34}^2}{\pi^6} \int \frac{d^4x_5 d^4x_6 d^4x_7\ x_{17}^2}{(x_{15}^2 x_{35}^2)(x_{16}^2 x_{46}^2)(x_{37}^2 x_{27}^2 x_{47}^2) x_{56}^2 x_{57}^2 x_{67}^2},$$

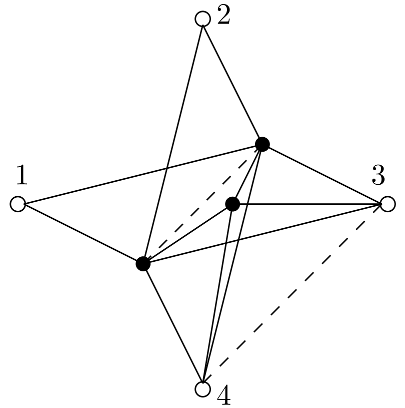
$$E(1;3,4;2) = \frac{x_{23}^2 x_{24}^2}{\pi^6} \int \frac{d^4x_5\, d^4x_6\, d^4x_7\ x_{16}^2}{(x_{15}^2 x_{25}^2 x_{35}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{47}^2)},$$

$$H(1,2;3,4) = \frac{x_{34}^2}{\pi^6} \int \frac{d^4x_5\, d^4x_6\, d^4x_7\ x_{57}^2}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2)}$$

- Flip identity implies $\mathbf{T} \propto \mathbf{L}$ (not valid on shell)
- **E**, **H** previously **unknown**

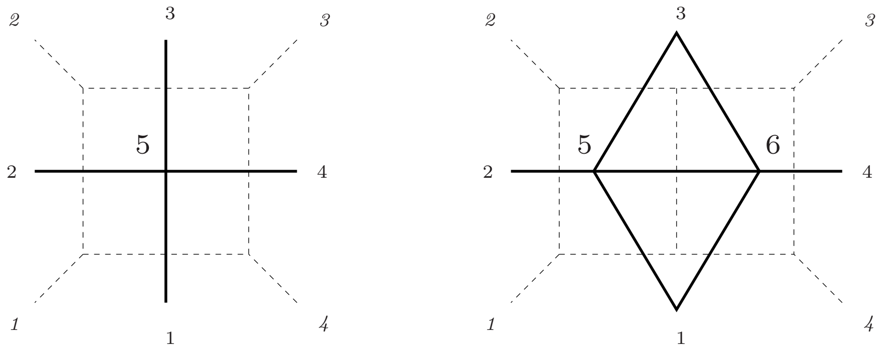


$E_{12;34}$



$H_{12;34}$

The one- and two-loop boxes **g** and **h**:



$g(1,2,3,4)$ $h(1,2,3;1,3,4)$

$$g(1,3,2,4) = \frac{1}{x_{13}^2 x_{24}^2}\,\Phi^{(1)}(u,v)\,,$$
$$h(1,3;2,4) = \frac{1}{x_{13}^2 x_{24}^2}\,\Phi^{(2)}(u,v)$$

The L-loop box integrals are explicitly known: [Davydychev, Ussyukina (1993)]

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = x\bar{x}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-x)(1-\bar{x})\,, \qquad \lambda(u,v) = \sqrt{(1-v+u)^2 - 4u} = x - \bar{x}\,,$$

$$\Phi^{(L)}(u,v) = -\frac{(-1)^{L+1}}{\lambda(u,v)} \sum_{r=0}^{L} \frac{(-1)^r (2L-r)!}{r!(L-r)!L!} \log^r(u)\left(\mathrm{Li}_{2L-r}(x) - \mathrm{Li}_{2L-r}(\bar{x})\right)\,.$$

- **"Rational" factor** $1/\lambda$ times **pure function. Symbol** with letters $\{x,\ 1-x,\ \bar{x},\ 1-\bar{x}\}$.

Asymptotic Expansions

Example: the one-loop box

$$g_{1234} = \frac{1}{\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \frac{1}{x_{13}^2 x_{24}^2} \Phi^{(1)}(u,v)$$

where

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = x\,\bar{x}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-x)(1-\bar{x})\,.$$

Limit: $x_{12}, x_{34} \to 0$ or $\bar{x} \to 0 \Rightarrow u, Y = 1 - v$ small. Work at $O(u^0)$.

Conformal invariance: The four-point integral can be uniquely reconstructed from the limit

$$\lim_{x_4 \to \infty} x_4^2\, g_{1234} = \frac{1}{\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2} = \frac{1}{x_{13}^2} \Phi^{(1)}(u,v)\,, \qquad u \to \frac{x_{12}^2}{x_{13}^2}\,, \quad v \to \frac{x_{23}^2}{x_{13}^2}\,.$$

Put $x_1 = 0$, $x_2 = p_1 << p_2 = x_3$. Thus

$$u = \frac{p_1^2}{p_2^2} \to 0 \qquad v = \frac{(p_1 - p_2)^2}{p_2^2} = 1 - \frac{2\,p_1.p_2}{p_2^2} + u = 1 - Y + O(u)\,.$$

Some properties of Gegenbauer polynomials:

$$\frac{1}{(x-y)^2} = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{|y|^n}{|x|^n}\,C_n(\hat{x}.\hat{y}) \quad : \quad |x|>|y|\,,$$

$$\int d\hat{x}\;C_n(\hat{x}.\hat{y})\;C_m(\hat{x}.\hat{z}) = \frac{2\pi^2\,\delta_{nm}}{n+1}\,C_n(\hat{y}.\hat{z})\,,$$

$$\frac{|p_1|^n}{|p_2|^n}\,C_n(\hat{p}_1.\hat{p}_2) = \sum_{m=0}^{\left[\frac{n}{2}\right]} C_n^{n-2m}\,u^m\,Y^{n-2m}\,, \quad C_n^n = 1\,.$$

Let $|p_1| < |p_2|$. It follows

$$\frac{1}{\pi^2}\int\frac{d^4k}{k^2\,(k-p_1)^2\,(k-p_2)^2}$$

$$= \sum_{n=0}^{\infty}\frac{C_n(\hat{p}_1.\hat{p}_2)}{n+1}\left[\frac{1}{p_1^2\,p_2^2}\int_0^{p_1^2} d(k^2)\left(\frac{k^2}{|p_1||p_2|}\right)^n + \frac{1}{p_2^2}\int_{p_1^2}^{p_2^2}\frac{d(k^2)}{k^2}\left(\frac{|p_1|}{|p_2|}\right)^n + \int_{p_2^2}^{\infty}\frac{d(k^2)}{k^4}\left(\frac{|p_1||p_2|}{k^2}\right)^n\right]$$

$$= \frac{1}{p_2^2}\sum_{n=1}^{\infty}\frac{Y^{n-1}}{n}\left[-\log(u)+\frac{2}{n}\right] + O(u)$$

- Integration region split according to validity of the expansions, **finite** at all stages

Expansion by regions

One-loop bubble:

$$\int \frac{d^D k \; k^{\mu_1} \ldots k^{\mu_n}}{(k^2)^\alpha \, ((k-q)^2)^\beta} \;=\; G(\alpha,\beta,n) \, \frac{q^{\mu_1} \ldots q^{\mu_n}}{(q^2)^{\alpha+\beta-D/2}} \;+\; \text{traces}\,, \qquad D \;=\; 4 - 2\,\epsilon$$

$$G(\alpha,\beta,n) \;=\; \frac{\Gamma(\alpha+\beta-D/2)\Gamma(D/2-\alpha+n)\Gamma(D/2-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta+n)}$$

Reconsider

$$I \;=\; \frac{1}{\pi^2} \int \frac{d^D k}{k^2\,(k-p_1)^2\,(k-p_2)^2}\,, \qquad |p_1| << |p_2|.$$

Hard region:

$$k \sim O(p_2)\,, \qquad \frac{1}{(k-p_1)^2} \;=\; \frac{1}{k^2} \sum_{n=0}^{\infty} \left(\frac{2\,k.p_1}{k^2} \right)^n + O(p_1)^2$$

$$I \;\to\; \int \frac{d^D k}{(k^2)^2 (k-p_2)^2} \sum_{n=0}^{\infty} \left(\frac{2\,k.p_1}{k^2} \right)^n + O(p_1^2) \;=\; \frac{1}{(p_2^2)^{1+\epsilon}} \sum_{n=0}^{\infty} G(2+n,1,n)\; Y^n + O(u)\,.$$

- Extend integration domain down to $|k| = 0$ and regularise **infrared divergence**

Soft region:

$$k \sim O(p_1)\,, \qquad \frac{1}{(k-p_2)^2} = \frac{1}{p_2^2}\sum_{n=0}^{\infty}\left(\frac{2\,k.p_2}{p_2^2}\right)^n + \ldots$$

$$I \;\rightarrow\; \frac{1}{p_2^2}\int \frac{d^Dk}{k^2(k-p_1)^2}\sum_{n=0}^{\infty}\left(\frac{2\,k.p_2}{p_2^2}\right)^n + O(u) \;=\; \frac{1}{p_2^2\,(p_1^2)^{\epsilon}}\sum_{n=0}^{\infty} G(1,1,n)\;Y^n + O(u)\,.$$

- Extend integration domain up to $|k| = \infty$ and regularise **ultraviolet divergence**

Now

$$G(1,1,n)|_{\epsilon^{-1}} \;=\; \frac{1}{(n+1)\,\epsilon} \;=\; -G(2+n,1,n)|_{\epsilon^{-1}}\,, \quad G(1,1,n)\;+\;G(2+n,1,n) \;=\; \frac{2}{(n+1)^2} + O(\epsilon)$$

so that

$$\text{Hard} + \text{Soft} \;=\; \frac{1}{p_2^2}\sum_{n=1}^{\infty}\frac{Y^{n-1}}{n}\left[-\log(u) + \frac{2}{n}\right] + O(u)\,.$$

- At l loops 2^l regions: k_i hard or soft.
- All momenta hard or all soft: l loop bubble. Else product of $l-n$ and n loop bubbles. **IBP** problem!
- 3 loops: `Mincer`. 4 loops: up to Y^{15} by `FIRE` and `LiteRed`.

- Conformal four-point integrals generally have **three distinct limits**.
- Asymptotic **expansions** of the l-loop **ladders**:

$$\sum_{n=1}^{\infty} Y^{n-1} \frac{1}{n^{m_1}} S_{m_2}(n) \; \log^{m_3}(u) \, \zeta(m_4) + O(u) \,, \qquad m_1+m_2+m_3+m_4 \;=\; 2\,l \,, \qquad m_4 \;=\; 2\,k+1 \qquad (*)$$

$S_{m_2}(n)$: **harmonic sums** of weight m_2

- **Uniform transcendentality**

- **E**, **H**: **fits** like $(*)$ on **expansion by regions** [Eden (2012)]
- In terms of **HPL**s (two examples):

$$x_{13}^4 \, x_{24}^4 \, H_{12;34} \;=\; \frac{4 \log u}{Y^2} \Big(H_{1,1,2,1} - H_{1,1,1,2} - 6\,\zeta_3\, H_{1,1} \Big) - \frac{2}{Y^2} \Big(4\, H_{2,1,2,1} - 4\, H_{2,1,1,2}$$
$$+ \; 4\, H_{1,1,3,1} - H_{1,1,2,1,1} - 4\, H_{1,1,1,3} + H_{1,1,1,2,1} - 24\,\zeta_3\, H_{2,1} + 6\,\zeta_3\, H_{1,1,1} \Big) + \mathcal{O}(u)$$

Compressed notation: $\qquad H_{2,1,1,2} \;=\; H(0,1,1,1,0,1;Y)$

$$
\begin{aligned}
x_{13}^2\, x_{24}^2\, E_{14;23} \;=\; & \frac{\log u}{Y}\Big(H_{2,2,1} - H_{2,1,2} + H_{1,3,1} + 2\, H_{1,2,1,1} - H_{1,1,3} - 2\, H_{1,1,1,2} \\
- \;& 6\,\zeta_3\, H_2 - 6\,\zeta_3\, H_{1,1}\Big) - \frac{2}{Y}\Big(2\,\zeta_3\, H_{2,1} - 4\,\zeta_3\, H_{1,2} + 4\,\zeta_3\, H_{1,1,1} + H_{3,2,1} \\
- \;& H_{3,1,2} + H_{2,3,1} - H_{2,1,3} + 2\, H_{1,4,1} + 2\, H_{1,3,1,1} + 2\, H_{1,2,2,1} - 2\, H_{1,1,4} \\
- \;& 2\, H_{1,1,2,2} - 2\, H_{1,1,1,3} - 6\,\zeta_3\, H_3\Big) + \mathcal{O}(u)
\end{aligned}
$$

Rational factors in the limits of E, H:

$$
\frac{1}{Y^2}\,, \qquad \frac{1}{Y}\,, \qquad \frac{1}{Y(1-Y)}
$$

- The **last case** is **not** a limit of $\lambda(\mathbf{u},\mathbf{v})^{-\mathbf{n}}$.
- Calculate **symbol** with the letters $\{u,\, Y,\, 1-Y\}$ **of the limits**.

Rational factors = leading singularities

For **on-shell** amplitudes: [Cazacho (2008)], [Arkani-Hamed, Bourjaily, Cachazo, Trnka (2010)]

Four-mass box function, i.e. $\pi^2\, g(1,2,3,4)$:

$$B \;=\; \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

- **Shift contour to circle the residues when all denominator terms vanish**.
- **Change coordinates** from x_5^μ to $p_i = x_{i5}^2$.

$$B = \int \frac{d^4 p_i}{p_1\, p_2\, p_3\, p_4\, J}\,, \qquad J \;=\; \det\left(\frac{\partial p_i}{\partial x_5^\mu}\right) \;=\; \det\left(-2\, x_{i5}^\mu\right)\,.$$

Using $\det(M) = \sqrt{\det(M M^T)}$ one can see the Jacobian is:

$$J^2 \;=\; \det\left(4\, x_{i5}\cdot x_{j5}\right) \;=\; 16\det\left(x_{ij}^2 - x_{i5}^2 - x_{j5}^2\right)$$

- **Cut $\{\mathbf{p_i}\}$** (i.e. compute the residue at the poles picking up $(2\pi i)^4$ and setting $p_i = 0$ in J).

$$\text{leading singularity of } B \;=\; \frac{4\pi^4}{\lambda_{1234}}\,, \qquad \lambda_{1234} = \sqrt{\det(x_{ij}^2)_{i,j=1..4}} \;=\; x_{13}^2 x_{24}^2 (x-\bar{x})\,.$$

The hard integral

Consider each integral sequentially:

$$H_{12;34} = x_{34}^2 \left\{ \int \frac{d^4x_6}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2} \left[\int \frac{d^4x_5 \; x_{56}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \left(\int \frac{d^4x_7}{x_{37}^2 x_{47}^2 x_{57}^2 x_{67}^2} \right) \right] \right\}$$

First the x_7 integration (massive box)

$$\int \frac{d^4x_7}{x_{37}^2 x_{47}^2 x_{57}^2 x_{67}^2} \to \pm \frac{4\pi^4}{\lambda_{3456}} .$$

Now turn to the x_5 integration

$$\int \frac{d^4x_5 \; x_{56}^2 \; 4\pi^4}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 \, \lambda_{3456}} .$$

- Cut x_{i5}^2. This implies $\lambda_{3456} = \pm x_{34}^2 x_{56}^2$. The numerator is cancelled and we eventually find

$$H_{12;34} \to \pm \frac{(4\pi^4)^3}{\lambda_{1234}^2}$$

- Try to **cut three propagators** and $\boldsymbol{\lambda_{3456}}$.
- If we cut x_{35}^2, x_{45}^2 the root λ_{3456} collapses, we get back to the last case.
- Cut e.g. $x_{15}^2, x_{25}^2, x_{35}^2, \lambda_{3456} = \pm(x_{34}^2 x_{56}^2 - x_{36}^2 x_{45}^2)$

The new Jacobian is

$$J = \det\left(\frac{\partial(x_{15}^2, x_{25}^2, x_{35}^2, \lambda_{3456})}{\partial x_5^\mu}\right)$$

Result of x_5 integral on the cut:

$$\frac{(4\pi^4)^2\ x_{56}^2}{J\ x_{45}^2} = \frac{(4\pi^4)^2\ x_{36}^2}{J\ x_{34}^2}$$

so to freeze the final x_6 integral we must cut $x_{16}^2, x_{26}^2, x_{46}^2, J$.

At $x_{15}^2 = x_{25}^2 = x_{35}^2 = x_{16}^2 = x_{26}^2 = x_{46}^2 = 0$ we find

$$J\ \longrightarrow\ (x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2)\, x_{36}^2$$

and finally

$$H_{12;34}\ \longrightarrow = \pm\frac{(4\pi^4)^3}{(x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2)\,\lambda_{1234}}\ .$$

Form of H:

$$H_{12;34} = \left(\frac{(4\pi^4)^3}{x_{13}^4 x_{24}^4}\right)\left(\frac{H^{(a)}(x,\bar{x})}{(x-\bar{x})^2} + \frac{H^{(b)}(x,\bar{x})}{(v-1)(x-\bar{x})}\right)$$

$H^{(a)}, H^{(b)}$ are symmetric under $x_1 \leftrightarrow x_2$ while $H^{(a)}$ $\left(H^{(b)}\right)$ is symmetric(antisymmetric) under $x \leftrightarrow \bar{x}$.

The easy integral

$$E_{12;34} = x_{23}^2 x_{24}^2 \left[\int \frac{d^4x_6 \; x_{16}^2}{x_{26}^2 x_{36}^2 x_{46}^2} \left(\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{56}^2} \right) \left(\int \frac{d^4x_7}{x_{17}^2 x_{27}^2 x_{47}^2 x_{67}^2} \right) \right]$$

Freeze the outer integrations first (massive boxes). At the central point:

$$(4\pi^4)^2 \int \frac{d^4x_6 \; x_{16}^2}{x_{26}^2 x_{36}^2 x_{46}^2 \lambda_{1236} \lambda_{1246}}$$

- Cut all three propagators. The roots collapse and we obtain $\pm(4\pi^4)^3/\lambda_{1234}$.
- Cut x_{26}^2, x_{36}^2 or x_{26}^2, x_{46}^2. One root collapses. Numerator cancelled, we come back to the last case.
- Cut x_{36}^2, x_{46}^2, $\lambda_{1236} = \pm(x_{13}^2 x_{26}^2 - x_{23}^2 x_{16}^2)$, $\lambda_{1246} = \pm(x_{14}^2 x_{26}^2 - x_{24}^2 x_{16}^2)$. This implies $x_{16}^2, x_{26}^2 = 0$.

  Depending on which cut condition is used to replace x_{16}^2/x_{26}^2 we obtain the residues

$$\left\{ \pm(4\pi^4)^3 \frac{x_{13}^2 x_{24}^2}{\lambda_{1234}(x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2)}, \quad \pm(4\pi^4)^3 \frac{x_{14}^2 x_{23}^2}{\lambda_{1234}(x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2)} \right\}$$

- The first residue is the difference of these.

Form of E:

$$E_{12;34} = \frac{(4\pi^4)^3}{x_{13}^2 x_{24}^2} \left(\frac{E(x,\bar{x}) + v\, E(x/(x-1), \bar{x}/(\bar{x}-1))}{(x-\bar{x})(v-1)} \right)$$

with a single pure function $E(x,\bar{x})$ antisymmetric under $x \leftrightarrow \bar{x}$ but without any other symmetry.

- The **sequence** of freezing the integrations **does not matter**.
- **General method**

Single variable limit: $x_{12},\, x_{34} \to 0$ or $\bar{x} \to 0$

- $E_{12;34},\ E_{13;24},\ E_{14;23},\ H_{12;34},\ H_{13;24},\ H_{14;23}$ at $O(u^0)$ but to all orders in $Y = x$ [Eden (2012)]
- All limits take the form $R_i * F_i,\ R_i \in \{1/x^2, 1/x, 1/(x\,(1-x))\},\ F_i = \sum \log^n(u)\,\mathrm{HPL}(x)$.
- Associate symbols with the letters $\{u,\, x,\, 1-x\}$ with the "pure functions" F_i.
- Symbols for E, H: Put $\bar{x} = u/x$ and let $\bar{x} \to 0$. **Equate**!

The symbol

A **Goncharov polylog** is defined as

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_2}{t_2 - a_2} \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n - a_n}$$

Not all independent; they satisfy **functional identities** which can be found using the **symbol**:

$$dF = \sum_i F_i \, d\log R_i \,, \qquad \mathcal{S}(F) = \sum_i \mathcal{S}(F_i) \otimes R_i$$

HPL's: $a_i \in \{0, 1\}$

$$\mathcal{S}\left(G(1,0,1,0;x)\right) = x \otimes (1-x) \otimes x \otimes (1-x)$$

Properties:

- Log-like **functional identities**:

$$\begin{aligned} \ldots \otimes x\, y \otimes \ldots &= \ldots \otimes x \otimes \ldots + \ldots \otimes y \otimes \ldots \,, \\ \ldots \otimes 1/x \otimes \ldots &= - \ldots \otimes x \otimes \ldots \end{aligned}$$

- **Integrability**: A symbol corresponds to a function, if

$$\sum c_{\omega_1, \ldots, \omega_n} \; d\log \omega_i \wedge d\log \omega_{i+1} \;\; \omega_1 \otimes \ldots \otimes \omega_{i-1} \otimes \omega_{i+2} \otimes \ldots \otimes \omega_n = 0 \,, \qquad i \in \{1, \ldots, n-1\} \,.$$

Easy integral

- General ansatz with **letters** $\{x, 1-x, \bar{x}, 1-\bar{x}\}$
- Impose $\mathbf{x}, \bar{\mathbf{x}}$ **antisymmetry, integrability, single-valuedness** ($x, \bar{x}$ recombine to u).
- Any one of the three **single variable limits** fixes the symbol.

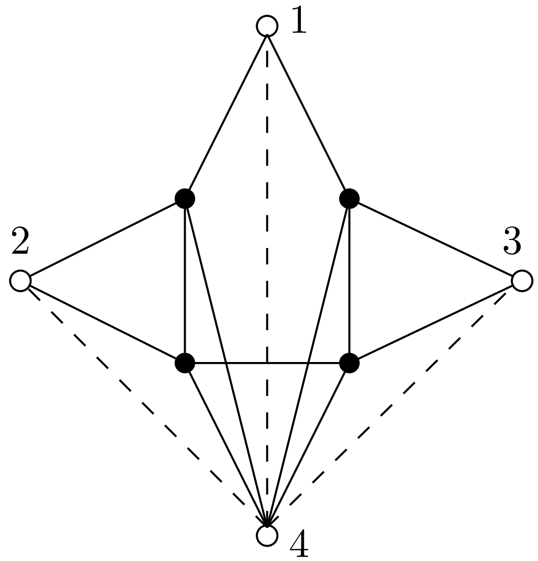
  Find 1024 terms, all with coefficients $\pm 1, \pm 2$.
- **Function** short, given in terms of generalised polylogs. Passes numerical tests.

Hard integral

- **Same strategy**
- Ansatz with $\{x, 1-x, \bar{x}, 1-\bar{x}\}$ **fails**.
- **Include** $x - \bar{x}$. [Chavez, Duhr (2012)]
- $\mathbf{x}, \bar{\mathbf{x}}$ **symmetry/antisymmetry, flip symmetry, vanishing of** $\mathcal{S}\left(H^{(a)}\right)$ **at** $x = \bar{x}$**, integrability, single-valuedness, any single variable limit** fix symbols.
- $\mathcal{S}\left(H^{(b)}\right)$ has 1536 terms, each with coefficient ± 2. No $x - \bar{x}$.

  $\mathcal{S}\left(H^{(a)}\right)$ has 3456 terms, varied coefficients. $x - \bar{x}$ present!
- Can obtain $H^{(a)}$ symbol from $G(\ldots, 1/x, \ldots, 1/\bar{x}, \ldots; 1)$. Passes numerical tests.

A four-loop example



$$I^{(4)}_{14;23} = \frac{1}{\pi^8}\int \frac{d^4x_5\, d^4x_6\, d^4x_7\, d^4x_8\ x^2_{14}x^2_{24}x^2_{34}}{x^2_{15}x^2_{18}x^2_{25}x^2_{26}x^2_{37}x^2_{38}x^2_{45}x^2_{46}x^2_{47}x^2_{48}x^2_{56}x^2_{67}x^2_{78}}$$
$$= \frac{1}{x^2_{13}x^2_{24}}\,\frac{1}{x-\bar{x}}\,\hat{f}(x,\bar{x})$$

Unique leading singularity!

Magic identity on two-loop ladder subintegral:

$$h_{17;24} = \frac{1}{\pi^4}\int \frac{d^4x_5\, d^4x_6\ x^2_{24}}{x^2_{15}x^2_{25}x^2_{26}x^2_{45}x^2_{46}x^2_{67}x^2_{56}}\,, \qquad h_{17;24} = h_{24;17}$$

The four-loop integral becomes

$$I^{(4)}_{14;23} = \frac{1}{\pi^4}\int \frac{d^4x_7\, d^4x_8\ x^2_{14}x^2_{34}}{x^2_{18}x^2_{37}x^2_{38}x^2_{47}x^2_{48}x^2_{78}}\ h_{17;24} = \frac{1}{\pi^4}\int \frac{d^4x_7\, d^4x_8\ x^2_{14}x^2_{34}}{x^2_{18}x^2_{37}x^2_{38}x^2_{47}x^2_{48}x^2_{78}}\ h_{24;17}$$
$$= \frac{1}{\pi^8}\int \frac{d^4x_5\, d^4x_6\, d^4x_7\, d^4x_8\ x^2_{17}x^2_{14}x^2_{34}}{x^2_{18}x^2_{37}x^2_{38}x^2_{47}x^2_{48}x^2_{78}\,\mathbf{x^2_{25}}\,x^2_{15}x^2_{16}x^2_{75}x^2_{76}x^2_{64}x^2_{56}}\,.$$

A differential equation

Point 2 occurs only **once**. Employ **Laplace** operator:

$$\Box_2 \frac{1}{x_{25}^2} = -4\pi^2 \delta^4(x_{25})$$

On $I^{(4)}$ we have

$$\Box_2\, I^{(4)}(x_1,x_2,x_3,x_4) = -\frac{4}{\pi^6} \int \frac{d^4x_6\, d^4x_7\, d^4x_8\; x_{17}^2 x_{14}^2 x_{34}^2}{x_{18}^2 x_{37}^2 x_{38}^2 x_{47}^2 x_{48}^2 x_{78}^2 x_{12}^2 x_{16}^2 x_{72}^2 x_{76}^2 x_{64}^2 x_{26}^2} = -\,4\, \frac{x_{14}^2}{x_{12}^2 x_{24}^2}\; E_{14;23}\,.$$

With

$$E_{14;23} = \frac{1}{x_{13}^2 x_{24}^2}\, \frac{1}{x-\bar{x}}\, \hat{f}_E(x,\bar{x})\,, \qquad \hat{f}_E(x,\bar{x}) = \frac{1}{1-x\bar{x}} \left[E(1-x, 1-\bar{x}) + x\bar{x}\; E\!\left(1-\frac{1}{x}, 1-\frac{1}{\bar{x}}\right) \right]$$

we find

$$\boxed{x\bar{x}\, \partial_x \partial_{\bar{x}}\, \hat{f}(x,\bar{x}) = -f_E(x,\bar{x})}$$

- $\hat{f}$ has **weight 8**
- final **entries of symbol** are functions only of x or $\bar{x}$, but $1-x\bar{x}$ in seventh slot
- homogeneous solution $h(x) - h(\bar{x})$ **fixed by single-valuedness**

According to symmetry under $x \leftrightarrow 1/x$ define

$$E_\pm(x,\bar{x}) \;=\; \frac{1}{2}\left[E(1-x,1-\bar{x}) \pm E\Big(1-\frac{1}{x},1-\frac{1}{\bar{x}}\Big)\right]\,, \qquad E_\pm \;=\; \sum_i c^i_\pm\, G_{\omega_i}(x)\, G_{\omega'_i}(\bar{x})\,.$$

The differential equation

$$(1-x\bar{x})x\bar{x}\;\partial_x\partial_{\bar{x}}\;\hat{f}(x,\bar{x}) = -(1-x\bar{x})E_-(x,\bar{x}) - (1+x\bar{x})E_+(x,\bar{x})$$

may now be split into

$$x\bar{x}\;\partial_x\partial_{\bar{x}}\;f_a(x,\bar{x}) = -E_-(x,\bar{x})\,,$$
$$(1-x\bar{x})x\bar{x}\;\partial_x\partial_{\bar{x}}\;f_b(x,\bar{x}) = -(1+x\bar{x})\;E_+(x,\bar{x})\,.$$

Split further

$$f_b \;=\; f_1 + f_2, \quad f_1(x,\bar{x}) \;=\; -f_2(x,\bar{x})$$

$$(1-x\bar{x})x\bar{x}\;\partial_x\partial_{\bar{x}}\;f_1(x,\bar{x}) \;=\; -E_+(x,\bar{x})\,,$$
$$(1-x\bar{x})\;\partial_x\partial_{\bar{x}}\;f_2(x,\bar{x}) \;=\; -E_+(x,\bar{x})\,.$$

and solve:

$$f_2(x,\bar{x}) \;=\; -\sum_i c^i_+ \int_1^x \frac{dt}{t}\,G_{\omega_i}(t)\;\int_1^{\bar{x}} \frac{d\bar{t}}{\bar{t}-\frac{1}{t}}\,G_{\omega'_i}(\bar{t})$$

- Functions like $G_{\hat{3},2,1} \;=\; G\left(0,0,\frac{1}{x\bar{x}},0,\frac{1}{x},\frac{1}{x};1\right)$

Conclusions ...

- **General technique** for calculating **rational factors**, twistor variables inessential
- **Asymptotic expansions** suffice to fix symbols/functions.
- Use of the **symbol at three and four loops**
- Four-loop example: A differential equation moves the letter $1 - u$ from the residues of E into the symbol. Symbol with 11136 terms, function passes numerical checks.

... and outlook

- **Automat** calculation of **residues**
- **Triangle** and **bubble** subgraphs, relation to master contours, c.f. K. Larsen's talk
- Symbols at four loops in C^{++}
- **Planar four-loop** correction to the **correlator** of four stress tensor multiplets in $\mathcal{N} = 4$ SYM. **Integrability** properties (at least for **structure constants**)?
- Three-point limit of **E, H** — **master integrals** also for **standard field theory**?
- **Generalisation of SVHPLs** to the new classes of functions → current work of O. Schnetz?