The Tutte polynomial: sign and approximability

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The Tutte polynomial (traditional bivariate style)

The *Tutte polynomial* of a graph $G = (V, E)$ is a two-variable polynomial T defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\kappa(A) - \kappa(E)} (y-1)^{|A| + \kappa(A) - n},$$

where $\kappa(A)$ denotes the number of connected components of (V, A), and $n = |V(G)|$.

Evaluations of the Tutte polynomial at various points and along various curves in $\mathbb{R}^2$ yield much interesting information about G.

Evaluations of the Tutte polynomial

- $T(G;1,1)$ counts spanning trees in G.
- $T(G;2,1)$ counts forests in G.
- $T(G;1-q,0)$ counts q-colourings of G.
- More generally, along the hyperbola

$$H_q = \{(x,y) : (x-1)(y-1) = q\},$$

  $T(G;x,y)$ specialises to the partition function of the q-state Potts model.
- $T(G;2,0)$ counts acyclic orientations of G.
- Along the $y > 1$ branch of H_0, $T(G;1,y)$ specialises to the reliability polynomial of G.

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There is a polynomial-time algorithm for evaluating $T(G; 1, 1)$.

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Theorem (Jaeger, Vertigan and Welsh, 1990, rough statement.)

Evaluating $T(G; x, y)$ is #P-complete, except on the hyperbola H_1 (where it is trivial), and at a finite set of "special points".

. . . and what was known (approximate evaluation)

Definition (First attempt)

An *FPRAS* for the Tutte polynomial at (x, y) is a randomised algorithm that estimates $T(G; x, y)$ within relative error $1 \pm \varepsilon$ with high probability. It must run in time $\mathrm{poly}(|G|, \varepsilon^{-1})$.

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Definition (Extended to functions taking negative values)

An *FPRAS* for the Tutte polynomial at (x, y) is a randomised algorithm that decides the sign of $T(G; x, y)$ (one of $+, -, 0$), and estimates $|T(G; x, y)|$ within relative error $1 \pm \varepsilon$ with high probability. It must run in time $\mathrm{poly}(|G|, \varepsilon^{-1})$.

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Theorem (Jerrum and Sinclair, 1990)

There is an FPRAS for $T(G; x, y)$ on the positive branch of the hyperbola H_2.

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Theorem (Jerrum and Sinclair, 1990)

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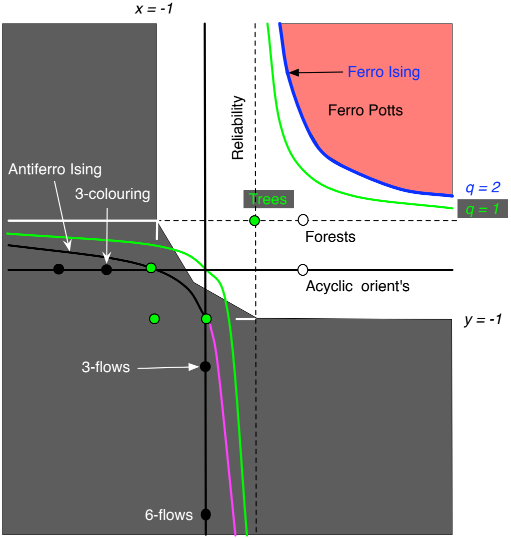
Theorem (Goldberg and Jerrum, 2008, 2012)

Assuming RP $\neq$ NP, *there is no FPRAS for large regions of the Tutte plane. (Classification is far from complete though.)*

The Tutte plane (2010)



The programme for this talk

- Jackson and Sokal have shown that in certain regions of the plane, the sign of the Tutte polynomial is "essentially determined" (i.e., is a simple function of the number of vertices, number of edges, number of connected components, etc).
- What happens when the sign is not essentially determined? We show that computing the sign is often #P-hard. (#P is to counting problems what NP is to decision problems.)
- Where the sign is hard to compute, the Tutte polynomial is a fortiori hard to approximate.

An illustration: the x-axis.

The line $y = 0$ corresponds (up to scaling) to the chromatic polynomial, under the transformation $q = 1 - x$.

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- At integer points $q > 2$ ($x < -1$) the polynomial counts q-colourings and its sign is 0 or $+$. Determining which is NP-hard.

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- At integer points $q > 2$ ($x < -1$) the polynomial counts q-colourings and its sign is 0 or $+$. Determining which is NP-hard.
- At non-integer points $q > 32/27$ ($x < -5/27$) the polynomial can take any sign, and determining which is #P-hard.

How can determining the sign be #P-hard?

Consider a #P-complete counting problem such $\#\mathrm{SAT}$. Let φ be an instance of $\#\mathrm{SAT}$; we want to know how many satisfying assignments φ has. Let this number be $N(\varphi)$.

Suppose we could design a reduction that takes a Boolean formula φ and a number c and produces a graph G_c with the following property:

> *The sign of $N(\varphi) - c$ is the same as the sign of $T(G_c; -\frac{3}{2}, 0)$*

Then an oracle for the sign of $T(G; -\frac{3}{2}, 0)$ could be used to compute $N(\varphi)$ exactly (by binary search on c).

The multivariate Tutte polynomial

As usual [Sokal, 2005], proofs are made easier by the moving to the multivariate Tutte polynomial.

Let G be a graph and γ be a function that assigns a (rational) weight γ_e to every edge $e \in E(G)$.

Definition (The multivariate Tutte polynomial)

$$Z(G; q, \gamma) = \sum_{A \subseteq E(G)} q^{\kappa(V,A)} \prod_{e \in A} \gamma_e.$$

When $\gamma_e = \gamma$ for all e (i.e., the edge weights are constant), we recover the traditional Tutte polynomial via the substitutions $q = (x-1)(y-1)$ and $\gamma = y - 1$.

A key lemma (one of two)

Name $\textsc{SignTutte}(q; \gamma_1, \ldots, \gamma_k)$.

Instance A graph $G = (V, E)$ and a weight function $\gamma : E \to \{\gamma_1, \ldots, \gamma_k\}$.

Output Determine the sign of $Z(G; q, \gamma)$.

Lemma

Suppose $q > 1$ *and that* $\gamma_1 \in (-2, -1)$ *and* $\gamma_2 \notin [-2, 0]$. *Then* $\textsc{SignTutte}(q; \gamma_1, \gamma_2)$ *is* #P*-hard.*

Simulating weights

The problem we actually want to study is:

Name $\textsc{SignTutte}(q, \gamma)$.

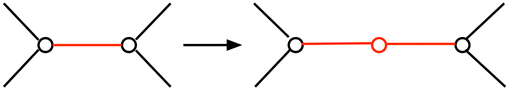
Instance A graph $G = (V, E)$.

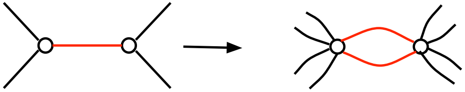
Output Determine the sign of $Z(G; q, \gamma)$.

So the question becomes: can we "simulate" the weights γ_1 and γ_2 required in the key lemma using the single weight γ?

A partial answer is that we can often do this by "stretching" and/or "thickening" [Jaeger et al, 1990].

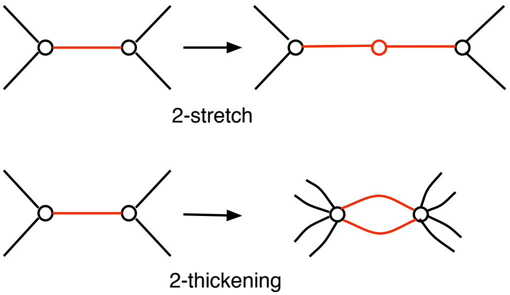
Stretching and thickening





Two γ-edges in *series* “simulate” an edge of weight $\gamma' = \gamma^2/(q + 2\gamma)$. The *2-stretch* of a graph implements $x' = x^2$ and $y' = q/(x' - 1) + 1$.

Stretching and thickening



Two γ-edges in *parallel* simulate an edge of weight $\gamma' = (1+\gamma)^2 - 1$. A *2-thickening* of a graph implements $y' = y^2$ and $x' = q/(y'-1)+1$.

The significance of 32/27

Consider the point $(x, y) = (-0.1, -0.1)$. Note that $q = (x-1)(y-1) = 1.21 > 32/27$.

We already have a point with $y \in (-1, 0)$. To satisfy the lemma we need to simulate a point with $y \notin [-1, 1]$.

Perform alternate 2-stretches and 2-thickenings:

$$
\begin{aligned}
(x_0, y_0) &= (-0.100000000, & -0.100000000) \\
(x_1, y_1) &= (\ \ 0.010000000, & -0.222222222) \\
(x_2, y_2) &= (-0.272857143, & 0.049382715) \\
(x_3, y_3) &= (\ \ 0.074451020, & -0.307332218) \\
& \vdots \\
(x_9, y_9) &= (\ \ 0.240501295, & -0.593156107) \\
(x_{10}, y_{10}) &= (-0.866806208 & 0.351834167) \\
(x_{11}, y_{11}) &= (\ \ 0.751353002, & -3.866336657)
\end{aligned}
$$

The significance of 32/27 (continued)

Consider the point $(x, y) = (0, -0.1)$. Note that $q = (x-1)(y-1) = 1.1 < 32/27$.

Perform alternate 2-thickenings and 2-stretches:

$$\begin{array}{lll}
(x_0, y_0) & = (0.000000000, & -0.100000000) \\
(x_1, y_1) & = (-0.111111111, & 0.010000000) \\
(x_2, y_2) & = (0.012345678, & -0.113750000) \\
 & \vdots & \\
(x_{10}, y_{10}) & = (0.013145124, & -0.114652243) \\
(x_{11}, y_{11}) & = (-0.114652256, & 0.013145136) \\
(x_{12}, y_{12}) & = (0.013145139, & -0.114652260) \\
(x_{13}, y_{13}) & = (-0.114652261, & 0.013145140) \\
(x_{14}, y_{14}) & = (0.013145140, & -0.114652261) \\
(x_{15}, y_{15}) & = (-0.114652261, & 0.013145140)
\end{array}$$

A further illustration: the y-axis.

The line $x = 0$ corresponds (up to scaling) to the flow polynomial, under the transformation $q = 1 - x$.

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- At $q = 2$ (i.e., $y = -1$), the Tutte/flow polynomial counts nowhere-zero 2-flows in a graph. Although not essentially determined, the sign (and indeed the polynomial itself) is easy to compute.
- At integer points $q = 3$ ($y = -2$) and $q = 4$ ($y = -3$) the polynomial counts, respectively, 3-colourings of a planar graph and 3-edge-colourings of a cubic graph. The sign is NP-hard to determine.

The y-axis (continued)

- At integer points $q \geq 6$ ($y \leq -5$), the sign is essentially determined (Seymour's 6-flow Theorem).

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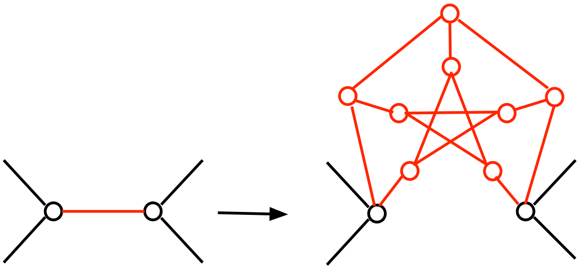
- At non-integer points $32/27 < q < 4$ ($-3 < y < -5/32$) the polynomial can take any sign, and determining which is $\#$P-hard.
- Other points are unresolved.

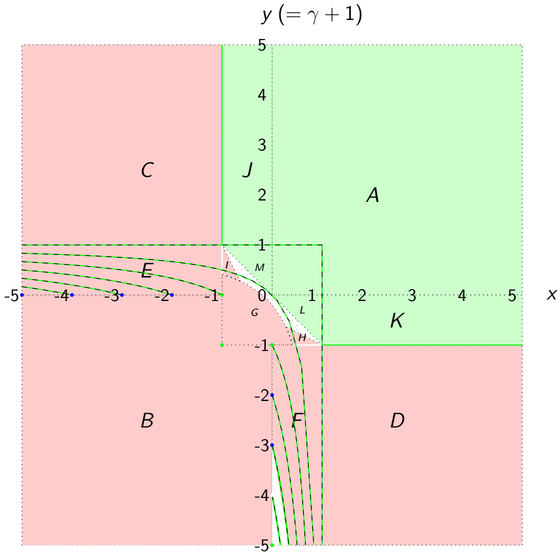
More exotic "shifts"

To approach $y = -3$ close to the y axis, the usual stretchings and thickenings are not enough. Instead we use a graph transformation based on taking a "2-sum" with a Petersen graph along each edge.



2-sum with Petersen graph

The Tutte plane more generally



Relation to approximate counting.

Fix an evaluation point (x, y). There are three possibilities.

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- The sign is NP-hard to determine. This tends to occur when the Tutte polynomial has a combinatorial interpretation, e.g., the number of 3-colourings of a graph. The number of structures may be estimated by iterated random bisection [Valiant and Vazirani], using an NP-oracle. Approximation of the Tutte polynomial is "essentially NP-complete".

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- The sign is easily determined. In this case we have only incomplete information about the complexity of approximating the Tutte polynomial.

The Tutte plane (2010, reprise)