Stabilized finite element methods for nonsymmetric, noncoercive and ill-posed problems

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Outline

- The coercive framework for FEM
- Stabilization for positive operators
- FEM, problems without coercivity
- Stabilized FEM, problems without coercivity
- Elliptic problems, analysis - examples
- Hyperbolic pbs, analysis - examples
- Ill-posed pbs, analysis - examples

The classical framework for numerical analysis I

- Variational formulation: find $u \in V$ such that

$$a(u, v) = l(v) \quad \forall v \in V$$

- Wellposedness given by the Lax-Milgram's lemma
  - $a(\cdot, \cdot)$ bilinear; $|a(u, v)| \leq M\|u\|_V\|v\|_V$ for all $u, v \in V$
  - $\alpha\|u\|_V^2 \leq a(u, u)$, for all $u \in V$
  - $l(\cdot)$ linear, $l(v) \leq L\|v\|_V$, $L = \|l\|_{V'}$
- $\rightarrow$ there exists a unique solution
- Continuous dependence on data

$$\|u\|_V \leq M\alpha^{-1}\|l\|_{V'}$$

The classical framework for numerical analysis II

- Galerkin projection: find $u_h \in V_h \subset V$ such that

$$a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h$$

- Best approximation using coercivity, Galerkin orthogonality, continuity, $e = u - u_h \in V$

$$\alpha \|e\|_V^2 \leq a(e, e) = a(e, u - v_h) \leq M \|e\|_V \|u - v_h\|_V$$

as a consequence

$$\|e\|_V \leq M\alpha^{-1} \inf_{v_h \in V_h} \|u - v_h\|_V$$

- Compare with the continuous dependence on data.

$$\|u\|_V \leq M\alpha^{-1} \|l\|_{V'}$$

Stabilization to enhance coercivity I

- Consider the discrete error: $e_h := i_h u - u_h$
- For problems where Lax-Milgram fails the analysis above may lead to

$$\|i_h u - u_h\|_L{}^2 \leq M\alpha^{-1}\|u - i_h u\|_* \|i_h u - u_h\|_V,$$

$\|\cdot\|_*$ with optimal approximation and $\|\cdot\|_V$ a stronger norm than $\|\cdot\|_L$

Example: the transport equation

- find $u_h \in V_h$ such that

$$(\sigma u_h + \beta \cdot \nabla u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h$$

Coercivity in the L^2-norm but continuity on L^2/H^1:

$$\alpha\|i_h u - u_h\|^2_{L^2(\Omega)} \leq \|u - i_h u\|_{L^2(\Omega)}(\|\sigma(i_h u - u_h)\|_{L^2(\Omega)} + \|\beta \cdot \nabla(i_h u - u_h)\|_{L^2(\Omega)})$$

- inverse inequality $\rightarrow$ error estimate for smooth solutions, optimality is lost

Stabilization to enhance coercivity II

- A stabilized formulation may read: find $u_h \in V_h$ such that

$$(\sigma u_h + \beta \cdot \nabla u_h, v_h) + s(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h$$

- $s(u_h, v_h)$: weakly consistent operator, making coercivity and continuity match

$$|||u_h|||^2 := \|u_h\|^2_{L^2(\Omega)} + s(u_h, u_h)$$

- The analysis now becomes with $e_h := i_h u - u_h$,

$$\alpha |||e_h|||^2 = a(e_h, e_h) + s(e_h, e_h) = a(u - i_h u, e_h) + s(i_h u, e_h) \leq M \|u - i_h u\|_* |||e_h|||$$

and hence

$$|||e_h||| \leq M\alpha^{-1} \|u - i_h u\|_*.$$

- $s(\cdot, \cdot)$ chosen to give the best compromise between stability and accuracy.
- $a(\cdot, \cdot)$ must be coercive, at least on some weak norm
- For a complete picture we need an inf-sup condition based analysis

Finite element methods for problems without coercivity I

- Elliptic problems (Schatz, 1974)
  - Well posedness under suitable assumptions on data using Fredholm's alternative
  - The standard Galerkin finite element method produces an invertible linear system and optimally convergent approximations for sufficiently small meshsizes
    - duality (Nitsche):

$$\|u - u_h\|_{L^2(\Omega)} \leq C_a h \|\nabla(u - u_h)\|_{L^2(\Omega)}$$

    - Gårding's inequality

$$C_1\|u - u_h\|^2_{H^1(\Omega)} - C_2\|u - u_h\|^2_{L^2(\Omega)} \leq a(u - u_h, u - u_h)$$

    - therefore, for small enough h the left hand side below is positive

$$(1 - C_a^2 C_2 C_1^{-1} h^2)\|u - u_h\|_{H^1(\Omega)} \leq M C_1^{-1}\|u - i_h u\|_{H^1(\Omega)}$$

- The transport equation (hyperbolic)
  - Well posedness for smooth, non vanishing velocity fields using the method of characteristics
  - No known analysis for the standard Galerkin method
  - Stabilized FEM for non-negative form, exponential weight functions: Johnson-Nävert-Pitkäranta, 1983 ; Sangalli, 2000 ; Guzman 2008; Ayuso-Marini, 2009;

Finite element methods for problems without coercivity II

- To fix the ideas: $\mathcal{L}u := -\mu\Delta u + \beta \cdot \nabla u + \sigma u$
- The Peclet number is low
- Consider the well-posed, but indefinite problem:

$$\mathcal{L}u = f \text{ in } \Omega \quad + \quad \text{BCs on } \partial\Omega$$

with associated weak form: find $u \in V$ such that

$$a(u, v) = (f, v), \quad \forall v \in V.$$

- $a(\cdot, \cdot)$ not coercive $\rightarrow$ the discrete problem, find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h \tag{1}$$

may be ill-posed for fixed h.

Failure of coercivity $\rightarrow$ matrix possibly singular

If $A := a(\varphi_j, \varphi_i)$, $F := l(\varphi_i)$, with φ_i nodal basis function,

$$AU = F$$

A may have zero eigenvalues, or be ill-conditioned, even if the continuous problem is well-posed.

1. Non-uniqueness: $\exists \tilde{U} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, $N := \dim(V_h)$ s.t.

$$A\tilde{U} = 0$$

2. Non-existence: $F \notin \mathrm{Image}(A) \rightarrow$ compatibility conditions

Analogy: Stokes' problem,

1. $\sim$ spurious pressure modes
2. $\sim$ locking

A framework for stabilization of noncoercive problems I

Standard stabilization fails

$$a(u_h, v_h) + s(u_h, v_h) \text{ is still typically indefinite.}$$

Inf-sup stability typically either requires some positivity or a mesh condition

Idea

- Consider $a(u_h, v_h) = (f, v_h)$ as the constraint for a minimization problem
- Minimize some weakly consistent stabilization possibly together with penalty for the boundary conditions
- Stabilize the Lagrange multiplier

A framework for stabilization of noncoercive problems II

- Lagrangian:

$$\boxed{L(u_h, z_h) := \frac{1}{2} s_p(u_h - u, u_h - u) - \frac{1}{2} s_a(z_h, z_h) + a_h(u_h, z_h) - (f, z_h)}$$

- "choose" the u_h that minimizes $s(u_h - u, u_h - u)$
- Lack of inf-sup stability handled by stabilizing the Lagrange-multiplier
- Stationary points

$$\boxed{\begin{cases} \dfrac{\partial L}{\partial u_h}(v_h) & = a_h(v_h, z_h) - s_p(u_h - u, v_h) = 0 \\[2ex] \dfrac{\partial L}{\partial z_h}(w_h) & = a_h(u_h, w_h) - s_a(z_h, w_h) - (f, w_h) = 0 \end{cases}}$$

A framework for stabilization of noncoercive problems III

- The resulting Euler-Lagrange equations: find $(u_h, z_h) \in V_h \times V_h$

$$\boxed{\left\{\begin{array}{rcl} a_h(u_h, w_h) - s_a(z_h, w_h) & = & (f, w_h) \\ a_h(v_h, z_h) + s_p(u_h, v_h) & = & s_p(u, v_h) \end{array}\right. \quad \text{for all } (w_h, v_h) \in V_h \times V_h} \qquad (2)$$

- The exact solution is: $u_h = u$ and $z_h = 0$
- The resulting system has twice as many degrees of freedom as FEM
- $s_p(u, v_h)$ must be a known quantity
- imposition of boundary conditions possible in $s_a(\cdot, \cdot)$ and $s_p(\cdot, \cdot)$
- Skew-symmetry gives partial stability: take $w_h = -z_h$, $v_h = u_h$

$$\boxed{|u_h|_{s_p}^2 + |z_h|_{s_a}^2 = -(f, z_h) + s_p(u, u_h)}$$

with $|u_h|_{s_p} := s_p(u_h, u_h)^{\frac{1}{2}}$ and $|z_h|_{s_a} := s_a(z_h, z_h)^{\frac{1}{2}}$

Typically, piecewise affine elements $\rightarrow$ invertibility of the matrix.

Possible stabilization operators: the usual suspects

- Galerkin-Least squares:

$$s_p(u_h - u, w_h) = \gamma \sum_{K \in \mathcal{T}_h} (h^2(\mathcal{L}u_h - f), \mathcal{L}w_h)_K + \gamma \sum_{F \in \mathcal{F}_I} \langle h[\![\partial_n u_h]\!], [\![\partial_n w_h]\!]\rangle_F$$

$$s_a(z_h, v_h) = \gamma \sum_{K \in \mathcal{T}_h} (h^2 \mathcal{L}^* z_h, \mathcal{L}^* v_h)_K + \gamma \sum_{F \in \mathcal{F}_I} \langle h[\![\partial_n z_h]\!], [\![\partial_n v_h]\!]\rangle_F$$

- discontinuous Galerkin (dG): $s_a(\cdot,\cdot) \equiv s_p(\cdot,\cdot)$

$$s_p(u_h, w_h) = \gamma \sum_{F \in \mathcal{F}_I} (\left\langle h^{-1}[\![u_h]\!], [\![w_h]\!]\right\rangle_F + \langle h[\![\partial_n u_h]\!], [\![\partial_n w_h]\!]\rangle_F)$$

- Continuous interior penalty (CIP): $s_a(\cdot,\cdot) \equiv s_p(\cdot,\cdot)$

$$s_p(u_h, w_h) = \gamma \sum_{F \in \mathcal{F}_I} (\left\langle h^3[\![\Delta u_h]\!], [\![\Delta w_h]\!]\right\rangle_F + \langle h[\![\partial_n u_h]\!], [\![\partial_n w_h]\!]\rangle_F)$$

- $\partial_n u_h := n \cdot \nabla u_h$, $[\![u_h]\!]$ is the jump of u_h on internal faces and equal u_h on boundary faces

The elliptic case: analysis by duality (GLS) I

1. Approximability:

$$\|u - i_h u\|_* := \|h^{-\frac{1}{2}}(u - i_h u)\|_{\mathcal{F}} + \|h^{-1}(u - i_h u)\|_\Omega + |u - i_h u|_{s_p} \leq Ch^k |u|_{H^{k+1}(\Omega)}$$

2. Continuity :

$$\begin{cases} a(u - i_h u, v_h) & \leq & C\|u - i_h u\|_* |v_h|_{s_a} \text{ and} \\ a(u - u_h, w - i_h w) & \leq & Ch|u - u_h|_{s_p} \|w\|_{H^2(\Omega)} \end{cases}$$

Theorem

Assume that $u \in H^{k+1}(\Omega)$ is the unique solution of $a(u, v) = (f, v), \quad \forall v \in V$ and that the adjoint problem $\mathcal{L}^*\varphi = \psi$ is wellposed with $\|\varphi\|_{H^2(\Omega)} \leq C_R \|\psi\|_{L^2(\Omega)}$. Then

$$\boxed{\|u - u_h\|_{L^2(\Omega)} + h\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \underbrace{Ch(|u - u_h|_{s_p} + |z_h|_{s_a})}_{\text{a posteriori quantity}} \leq Ch^{k+1}\|u\|_{H^{k+1}(\Omega)}}$$

GLS: no conditions on the mesh-parameter
dG and CIP: $C_R h^3 |\beta|_{W^{2,\infty}} \lesssim 1$ small if oscillation in data
(c.f. Schatz $C_R^2 h^2 \lesssim 1$)

The elliptic case: analysis by duality (GLS) II

Sketch of proof.

- Step 1: Optimal convergence, stabilization semi-norm by energy arguments, $\xi_h = u_h - i_h u$

$$|\xi_h|_{s_p}^2 + |z_h|_{s_a}^2 = a(\xi_h, z_h) + s_p(\xi_h, \xi_h) - a(\xi_h, z_h) + s_a(z_h, z_h)$$
$$= a(u - i_h u, z_h) - s_p(u - i_h u, \xi_h) \le \|u - i_h u\|_* (|\xi_h|_{s_p}^2 + |z_h|_{s_a}^2)^{\frac{1}{2}}.$$

- Step 2: Prove optimal convergence in the L^2-norm using a duality argument

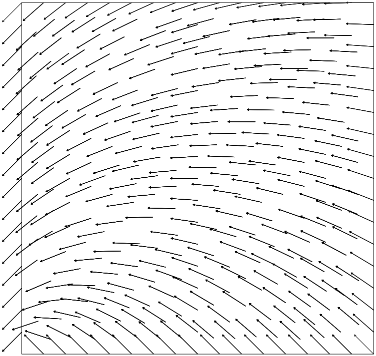
$$\|u - u_h\|_{L^2(\Omega)} + \|z_h\|_{L^2(\Omega)} \le Ch(|\xi_h|_{s_p} + |z_h|_{s_a}) \le Ch^{k+1} |u|_{H^{k+1}(\Omega)}$$

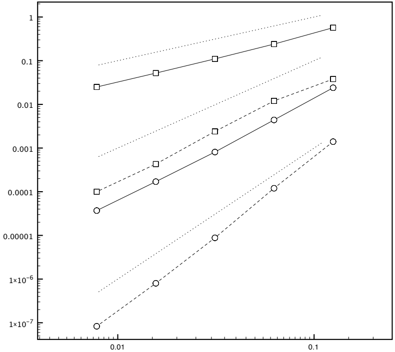
- Step 3: Prove optimal convergence in the H^1-norm using Gårding's inequality, or an inverse inequality.

□

Important observation: no stability of the continuous problem is used in Step 1

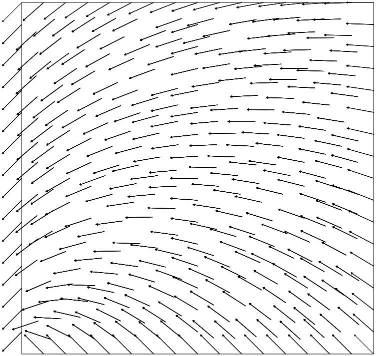
Example within the assumptions: noncoercive convection–diffusion with pure Neumann conditions

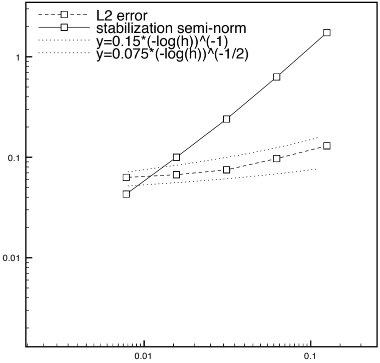




- $\nabla \cdot (\beta u - \nu \nabla u) = f$, Pe= 200, u smooth, $\nabla \cdot \beta = -200$
- Neumann condition on $\partial\Omega$: $(\beta u - \nu \nabla u) \cdot n = g$
- Full lines, $|u - u_h|_{s_p} + |z_h|_{s_a}$, dashed L^2-norm error, dotted $O(h^k)$, $k = 1, 2, 3$
- Squares P_1 approximation, circles P_2 approximation

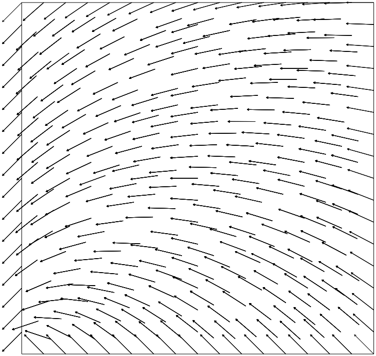
Example beyond the assumptions: the Cauchy problem

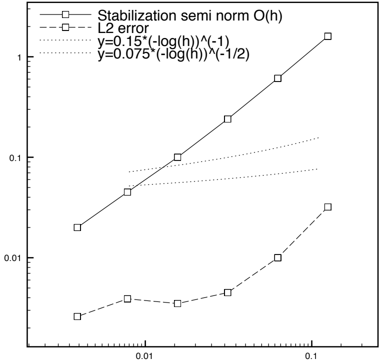




- $\beta \cdot \nabla u - \nu \Delta u = f$, Pe= 200, u smooth
- Dirichlet and Neumann bcs on $\{x \in (0,1),\ y = 0\}$ and $\{x = 1,\ y \in (0,1)\}$
- No boundary data on on $\{x = 0,\ y \in (0,1)\}$ and $\{x \in (0,1),\ y = 1\}$
- $\|\nabla \varphi\| \leq \|u - u_h\|$ can not hold, would give a posteriori upper bound

Example beyond the assumptions: the Cauchy problem





- $\beta \cdot \nabla u - \nu \Delta u = f$, Pe= 200, u smooth
- Dirichlet and Neumann bcs on $\{x = 0,\ y \in (0, 1)\}$ and $\{x \in (0, 1),\ y = 1\}$
- No boundary data on $\{x \in (0, 1),\ y = 0\}$ and $\{x = 1,\ y \in (0, 1)\}$
- $\|\nabla \varphi\| \leq \|u - u_h\|$ can not hold, would give a posteriori upper bound

The hyperbolic case: analysis using inf-sup stability I

- Transport equation:

$$\mathcal{L}u := \nabla \cdot (\beta u) + \sigma u = f, \quad \beta \in W^{2,\infty}(\Omega), \, \sigma \in W^{1,\infty}(\Omega)$$

- For every $x \in \Omega$ $\exists$ streamline leading to boundary data in finite time
- For GLS and dG stabilization the gradient jumps may be dropped.
  For CIP stabilization the jumps in the Laplacian may be dropped.
- Stabilization parameters will scale differently in h

Error estimate for stabilized FEM, hyperbolic case

$$\boxed{\|u - u_h\|_{L^2(\Omega)} + \|h^{\frac{1}{2}} \beta \cdot \nabla(u - u_h)\|_{L^2(\Omega)} \le C h^{k+\frac{1}{2}} |u|_{H^{k+1}(\Omega)}}$$

Mesh conditions:

- standard stabilized FEM: $h^{\frac{1}{2}}$ small
- GLS optimization based: no condition on h under exact quadrature.
- dG and cG optimization based: h^2 small (for nonconstant smooth β and σ).

The hyperbolic case: analysis using inf-sup stability II

Main ideas and tools for proof.

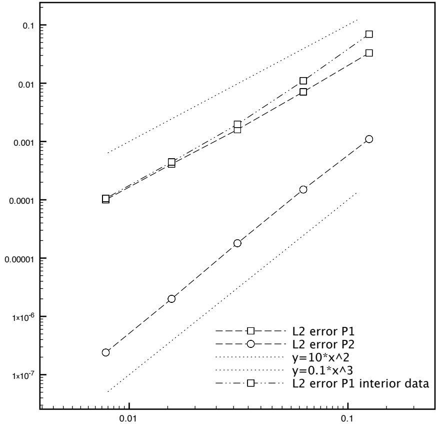
- The stability of the dual problem is replaced by

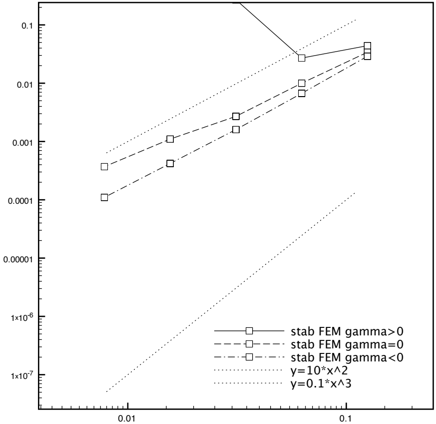
$$\forall v_h \in V_h \exists v_p(v_h) \text{ such that } \|v_h\|^2_{L^2(\Omega)} \leq a(v_h, v_p(v_h))$$

  and similarly for the adjoint problem
- for the transport equation: $v_p(v_h) = (e^{\eta} v_h)$ where $\beta \cdot \nabla \eta \geq c$, with c sufficiently big
- Superapproximation to estimate $\|v_p(v_h) - \pi_h v_p(v_h)\|$
- Steps 1 and 2 of the elliptic case, must be handled together in this case, weighting together the energy stability of $|\cdot|_{s_p}$ and $|\cdot|_{s_a}$ with the inf-sup stability in the L^2-norm

$\square$

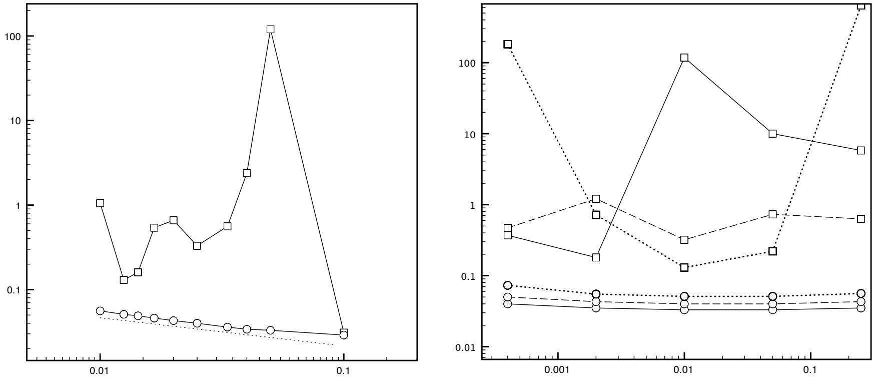
Example within the assumptions: data assimilation





- Problem: $\nabla \cdot (\beta u) = f$, data set on the outflow boundary, smooth solution u
- $\beta = (-(x+1)^4 + y, -8(y-x))^T$
- Left plot: optimization method, L^2-error vs. h, squares P_1, circles P_2
- Right plot: standard stabilized method. Dash-dot: $\gamma < 0$, dashed $\gamma = 0$, full $\gamma > 0$. Observe that for standard stabilization γ must change sign!

Example beyond the assumptions: strong oscillation



- Problem: $\nabla \cdot (\beta u) = f$
- data set on the inflow, smooth solution u, 64×64 unstructured mesh.
- $\beta = (10 \arctan(\frac{y-\frac{1}{2}}{\varepsilon}) - \frac{x^2}{\varepsilon}, \sin(x/\varepsilon) + \sin(y/\varepsilon)\frac{x^2}{\varepsilon})^T$
- circles: optimization method; squares: standard stabilized method
- Left plot: SD-error vs ε with $\gamma_{CIP} = 0.01$, dotted line $O(\epsilon^{-\frac{1}{3}})$
- Right plot: SD-error vs γ_{CIP} for $\epsilon = \{0.05 \text{ (full)}, 0.025 \text{ (dash)}, 0.0125 \text{ (dot)}\}$

Ill-posed problems. Example: the Cauchy problem

Let Ω be a convex polygonal (polyhedral) domain in $\mathbb{R}^d$, $d = 2, 3$

$$\left\{ \begin{array}{rcl} -\Delta u & = & f, \text{ in } \Omega \\ u = 0 \text{ and } \nabla u \cdot n & = & \psi \text{ on } \Gamma \end{array} \right. \tag{3}$$

- $\Gamma \subset \partial\Omega$, Γ simply connected, $\Gamma' := \partial\Omega \setminus \Gamma$
- $f \in L^2(\Omega)$, $\psi \in H^{\frac{1}{2}}(\Gamma)$
- $V := \{v \in H^1(\Omega) : v|_\Gamma = 0\}$ and $W := \{v \in H^1(\Omega) : v|_{\Gamma'} = 0\}$
- $a(u, w) = \int_\Omega \nabla u \cdot \nabla w \, \mathrm{dx}$, and $l(w) := \int_\Omega f w \, \mathrm{dx} + \int_\Gamma \psi w \, \mathrm{ds}$
- abstract weak formulation,

$$\boxed{\text{find } u \in V \text{ such that } a(u, w) = l(w) \quad \forall w \in W} \tag{4}$$

The ill-posed case: analysis by continuous dependence I

Consider the abstract problem: find $u \in V$ such that

$$a(u, w) = l(w) \quad \forall w \in W. \tag{5}$$

- Assumption: $l(w)$ is such that the problem (5) admits a unique solution $u \in V$.
- *Observe that we do not assume that* (5) *admits a unique solution for all* $l(w)$ *such that* $\|l\|_{W'} < \infty$

Assumption: continuous dependence on data

Consider the functional $j : V \mapsto \mathbb{R}$. Let $\Xi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous, monotone increasing function with $\lim_{x \to 0^+} \Xi(x) = 0$.

$$\boxed{\text{If } \|l\|_{W'} \leq \epsilon \text{ in (5) then } |j(u)| \leq \Xi(\epsilon). \text{ if } \epsilon > 0 \text{ sufficiently small}} \tag{6}$$

Finite element formulation of the abstract problem I

- Assume that $V_h \subset V$ and $W_h \subset W$
- Finite element formulation: find $(u_h, z_h) \in V_h \times W_h$ such that,

$$\left.\begin{array}{rcl} a(u_h, w_h) - s_W(z_h, w_h) & = & l(w_h) \\ a(v_h, z_h) + s_V(u_h, v_h) & = & s_V(u, v_h) \end{array}\right\} \quad \text{for all } (v_h, w_h) \in V_h \times W_h. \tag{7}$$

- Stabilization operators may be chosen as before

Finite element formulation of the abstract problem II

Main assumptions on $a(\cdot,\cdot)$, $s_W(\cdot,\cdot)$ and $s_V(\cdot,\cdot)$

Assume that the form $a(u,v)$ satisfies the continuities

$$a(v - i_V v, w_h) \leq \|v - i_V v\|_{*,V} |w_h|_{s_W}, \; \forall v \in V, \; w_h \in W_h \tag{8}$$

and for u solution of (5),

$$a(u - u_h, w - i_W w) \leq \delta_I(h)\|w\|_W + \|w - i_W w\|_{*,W} |u - u_h|_{s_V}, \; \forall w \in W. \tag{9}$$

Assume approximation estimates for $v - i_V v$ and $w - i_W w$

$$|v - i_V v|_{s_V} + \|v - i_V v\|_{*,V} \leq C_V(v) h^t \tag{10}$$

$$\|w - i_W w\|_{*,W} + |i_W w|_{s_W} \leq C_W \|w\|_W, \quad \forall w \in W. \tag{11}$$

Finite element formulation of the abstract problem III

Lemma (Convergence of stabilizing terms)

Let u be the solution of (5) *and* (u_h, z_h) *the solution of the formulation* (14) *for which* (8) *and* (10) *hold. Then*

$$|u - u_h|_{s_V} + |z_h|_{s_W} \leq (1 + \sqrt{2}) C_V(u) h^t.$$

Theorem (Convergence using continuous dependence)

Let u be the solution of (5) *(which has the stability property* (6)*) and* (u_h, z_h) *the solution of the formulation* (14) *(for which* (8)-(10) *hold). Then*

$$|j(u - u_h)| \leq \Xi(\eta(u_h, z_h)) \tag{12}$$

With the a posteriori quantity $\eta(u_h, z_h) := \delta_I(h) + C_W(|u - u_h|_{s_V} + |z_h|_{s_W})$.
For sufficiently smooth u there holds

$$\eta(u_h, z_h) \leq \delta_I(h) + (1 + \sqrt{2}) C_W C_V(u) h^t. \tag{13}$$

The approximation will be optimal with respect to continuous dependence!

Continuous dependence. Example: the Cauchy problem

- The Cauchy problem is not wellposed in the sense of Hadamard
- However if (3) admits a solution $u \in H^1(\Omega)$, a (conditional) continuous dependence of the form (6), with $0 < \epsilon < 1$, holds for: (interior estimate)

$$\boxed{\begin{array}{l} j(u) := \|u\|_{L^2(\omega)},\ \omega \subset \Omega :\ \mathrm{dist}(\omega, \partial\Omega) =: d_{\omega,\partial\Omega} > 0 \text{ with } \Xi(x) = C_{u\varsigma} x^{\varsigma}, \\ C_{u\varsigma} > 0,\ \varsigma := \varsigma(d_{\omega,\partial\Omega}) \in (0,1) \end{array}}$$

and for: (global estimate)

$$\boxed{j(u) := \|u\|_{L^2(\Omega)} \text{ with } \Xi(x) = C_u(|\log(x)| + C)^{-\varsigma} \text{ with } C_u, C > 0,\ \varsigma \in (0,1)}$$

The constant $C_{u\varsigma}$ grows monotonically in $\|u\|_{L^2(\Omega)}$ and C_u grows monotonically in $\|u\|_{H^1(\Omega)}$

- For details see:
G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella.
The stability for the Cauchy problem for elliptic equations.
Inverse Problems, 25(12):123004, 47, 2009.

Stabilized FEM for the Cauchy problem

Stabilized FEM for the Cauchy problem

- Let $V_h \in V$, $W_h \in W$, with piecewise affine functions
- CIP-stabilization for u_h and z_h (+ boundary penalty for Neumann condition)
- Find $(u_h, z_h) \in V_h \times W_h$ such that

$$\left\{ \begin{array}{rcl} a(u_h, w_h) - s_a(z_h, w_h) & = & (f, w_h) + \langle \psi, w_h \rangle_\Gamma \\ a(v_h, z_h) + s_p(u_h, v_h) & = & s_p(u, v_h) \end{array} \right. \quad \text{for all } (v_h, w_h) \in V_h \times W_h$$

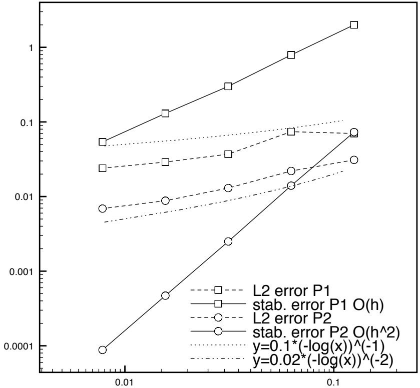
where a possible choice of stabilization operators is

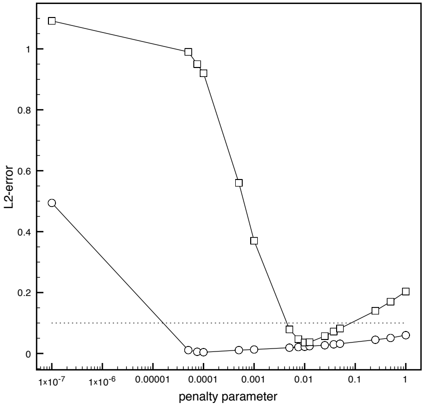
$$s_V(u_h, v_h) := \sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} \int_F h_F [\![\partial_n u_h]\!] [\![\partial_n v_h]\!] \, ds, \quad \text{with } h_F := \mathrm{diam}(F)$$

$$s_W(z_h, w_h) := a(z_h, w_h) \quad \text{or} \quad s_W(z_h, w_h) := \sum_{F \in \mathcal{F}_I \cup \mathcal{F}_{\Gamma'}} \int_F h_F [\![\partial_n z_h]\!] [\![\partial_n w_h]\!] \, ds$$

This formulation satisfies the assumptions of the convergence theorem

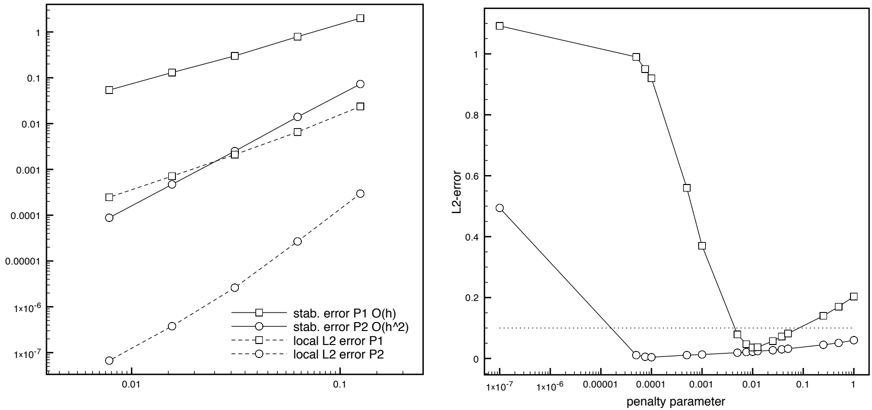
Numerical results for the Cauchy problem





- $\Omega := [0,1] \times [0,1]$, smooth exact solution u
- Dirichlet and Neumann bcs on $\{x = 0,\, y \in (0,1)\}$ and $\{x \in (0,1),\, y = 1\}$
- Left: convergence plots global errors
- Right: L^2-error against stabilization parameter (squares P_1, circles P_2)

Numerical results for the Cauchy problem



- $\Omega := [0,1] \times [0,1]$, smooth exact solution u
- Dirichlet and Neumann bcs on $\{x = 0,\ y \in (0,1)\}$ and $\{x \in (0,1),\ y = 1\}$
- Left: convergence plots local errors, $\{x > 0.5,\ y < 0.5\}$
- Right: L^2-error against stabilization parameter (squares P_1, circles P_2)

Variations on the theme: discrete inf-sup condition

Instead of using positivity in the derivation of the first estimate

$$|u - u_h|_{s_p} + |z_h|_{s_a} \leq Ch^k |u|_{H^{k+1}(\Omega)}$$

we can in some cases stabilize less and derive a discrete inf-sup condition:

$\exists c_s > 0$ such that $\forall x_h \in V_h,\ y_h \in W_h$ there holds

$$\boxed{c_s \|\!| x_h, y_h \|\!| \leq \sup_{v_h, w_h \in V_h \times W_h} \frac{A_h[(x_h, y_h), (v_h, w_h)]}{\|\!| v_h, w_h \|\!|}}$$

where

$$A_h[(x_h, y_h), (v_h, w_h)] := a_h(x_h, w_h) - s_a(y_h, w_h) + a_h(v_h, y_h) + s_p(x_h, v_h)$$

and ideally (so far only for piecewise affine elements)

$$\|\!| x_h, y_h \|\!| := \|h \nabla x_h\|_{L^2(\Omega)} + \|\nabla y_h\|_{L^2(\Omega)} + \|h^{\frac{1}{2}} [\![\partial_n x_h]\!]\|_{\mathcal{F}_I \cup \mathcal{F}_\Gamma} + |x_h|_{s_p} + |y_h|_{s_a}$$

Then we may prove:

$$\boxed{\|\!| u - u_h, z_h \|\!| \leq Ch |u|_{H^2(\Omega)}}$$

Example: the Cauchy problem, Crouzeix-Raviart element I

- the Crouzeix-Raviart space

$$X_h^\Gamma := \{v_h \in L^2(\Omega) : \int_F [v_h] \, \mathrm{d}s = 0, \forall F \in \mathcal{F}_i \cup \mathcal{F}_\Gamma \text{ and } v_h|_\kappa \in \mathbb{P}_1(\kappa), \forall \kappa \in \mathcal{K}_h\}$$

- $V_h := X_h^\Gamma$ and $W_h := X_h^{\Gamma'}$
- broken norms

$$\|x\|_h^2 := \sum_{\kappa \in \mathcal{T}_h} \|x\|_\kappa^2 \text{ and } \|x\|_{1,h}^2 := \|x\|_h^2 + \|\nabla x\|_h^2$$

- Finite element formulation: find $(u_h, z_h) \in V_h \times W_h$ such that,

$$\begin{aligned} a_h(u_h, w_h) - s_W(z_h, w_h) &= l(w_h) \\ a_h(v_h, z_h) + s_V(u_h, v_h) &= 0 \end{aligned} \tag{14}$$

for all $(v_h, w_h) \in V_h \times W_h$

Example: the Cauchy problem, Crouzeix-Raviart element II

- Here the bilinear forms are defined by

$$a_h(u_h, w_h) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u_h \cdot \nabla w_h \, \mathrm{d}x,$$

$$s_W(z_h, w_h) := \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \gamma_W \nabla z_h \cdot \nabla w_h \, \mathrm{d}x \tag{15}$$

or

$$s_W(z_h, w_h) := \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma'}} \int_F \gamma_W h_F^{-1} [z_h][w_h] \, \mathrm{d}s \tag{16}$$

and finally

$$s_V(u_h, v_h) := \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma}} \int_F \gamma_V h_F^{-1} [u_h][v_h] \, \mathrm{d}s \tag{17}$$

Example: the Cauchy problem, Crouzeix-Raviart element III

- Compact form: find $(u_h, z_h) \in \mathcal{V}_h := V_h \times W_h$ such that,

$$A_h[(u_h, z_h), (v_h, w_h)] = l(w_h) \text{ for all } (v_h, w_h) \in \mathcal{V}_h$$

- The bilinear form is then given by

$$A_h[(u_h, z_h), (v_h, w_h)] := a_h(u_h, w_h) - s_W(z_h, w_h) + a_h(v_h, z_h) + s_V(u_h, v_h)$$

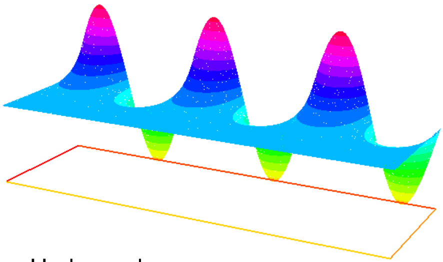
Theorem (Inf-sup stability for the Crouzeix-Raviart based method)

Assume that $(\gamma_V \gamma_W) \le (C_i c_{\mathcal{T}})^{-2}$. Then there exists a positive constant c_s independent of γ_V, γ_W such that there holds

$$c_s |||x_h, y_h||| \le \sup_{(v_h, w_h) \in \mathcal{V}_h} \frac{A_h[(x_h, y_h), (v_h, w_h)]}{|||v_h, w_h|||}$$

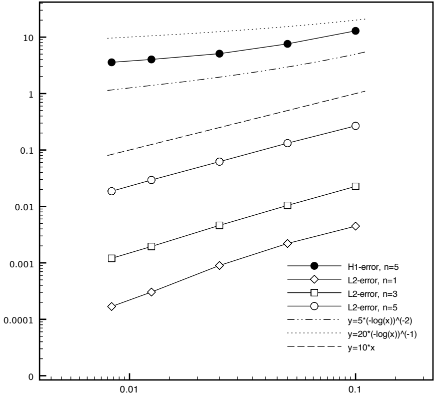
where $|||x_h, y_h||| := \gamma_V^{\frac{1}{2}} \|h \nabla x_h\|_h + \gamma_V^{\frac{1}{2}} \|h [\partial_n x_h]\|_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_C}} + |x_h|_{s_V} + |y_h|_{s_W}$

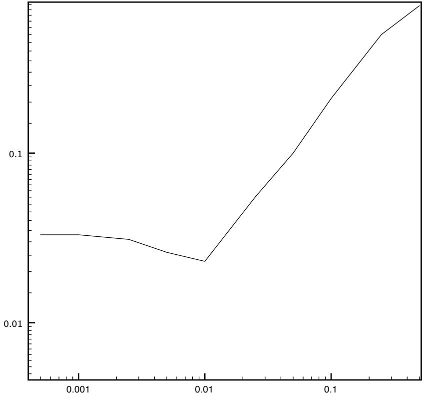
Numerical results for the Cauchy problem (CR-element) I



- Original problem by Hadamard
- $\Omega := [0, \pi] \times [0, 1]$
- $u(x, y) = (1/n) \sin(nx) \sinh(ny)$, n parameter
- Dirichlet and Neumann bcs on $\{x \in (0, \pi),\ y = 0\}$
- Dirichlet on $\{x = 0,\ y \in (0, 1)\}$ and $\{x = \pi,\ y \in (0, 1)\}$
- increasing n increases the rate of exponential growth and size of Sobolev norms

Numerical results for the Cauchy problem (CR-element) II





- Left: global L^2-error for $n = 1$, $n = 3$, $n = 5$, $\gamma_V = \gamma_W = 0.01$
- Right: stabilization parameter $\gamma_V = \gamma_W$ against L^2-error on a 10×10 mesh
- Higher values of n does not yield converging solution on these meshes.
  $\|u\|_{H^2(\Omega)}$-norm too large

Conclusions and outlook

1. Stabilized finite element methods in an optimization framework
2. Error estimates for non-coercive problems
3. A posteriori and a priori error estimates are obtained similarly, constants unknown
4. Ill-posed problems: error analysis using continuous dependence
5. New ideas on data assimilation and inverse problems using stabilized FEM
6. New ideas on the design and analysis of Tikhonov regularization methods

Numerical example: source identification I

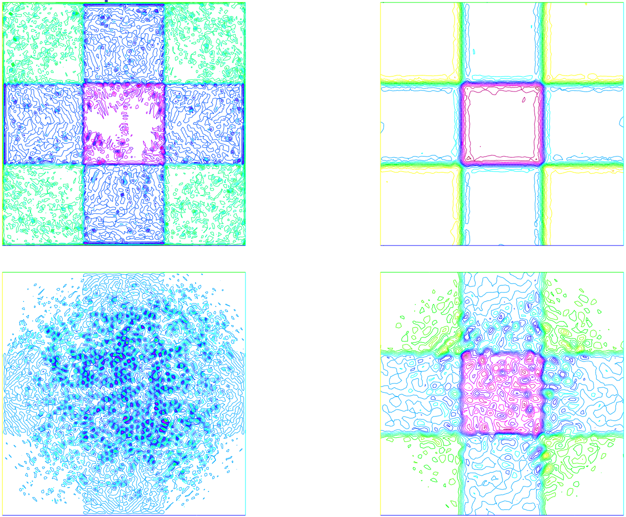
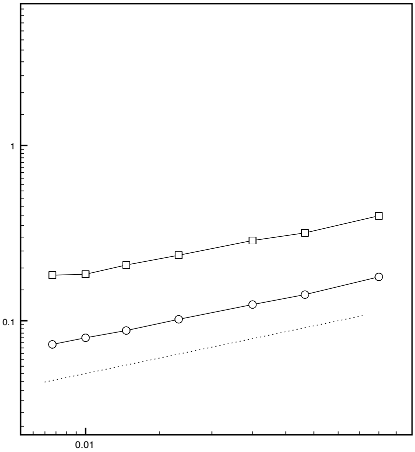


Figure : Left: naive application of the stiffness matrix, Right: stabilized reconstruction, top unpertubed data, bottom perturbed data

Numerical example: source identification II



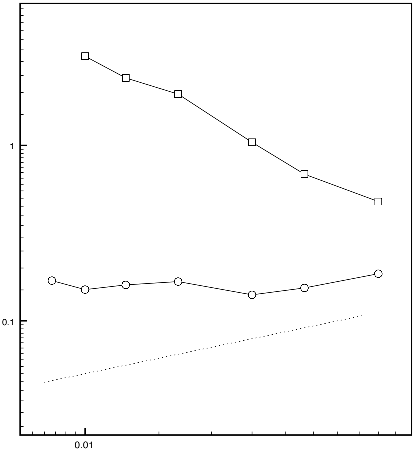


Figure : Convergence plots in the L^2-norm, Left: unperturbed data; Right: perturbed data