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Building Bridges: Connections and Challenges in Modern Approaches to Numerical PDEs

Approximation by plane and circular waves

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Time-harmonic PDEs, waves and Trefftz methods

Consider time-harmonic PDEs, e.g., Helmholtz and Maxwell eq.s

$$-\Delta u - \omega^2 u = 0, \qquad \nabla \times (\nabla \times \mathbf{E}) - \omega^2 \mathbf{E} = \mathbf{0}, \qquad \omega > 0.$$

Their solutions are "waves", oscillates with wavelength $\lambda = 2\pi/\omega$.

At high frequencies, $\omega \gg 1$, (piecewise) polynomial approximation is very expensive, standard FEMs are not good.

Desired: more accuracy for less DOFs.

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Desired: more accuracy for less DOFs. Possible strategy:

Trefftz methods are finite element schemes such that test and trial functions are solutions of Helmholtz (or Maxwell...) equation in each element K of the mesh $\mathcal{T}_h$, e.g.:

$$V_p \subset T(\mathcal{T}_h) = \Big\{ v \in L^2(\Omega) : -\Delta v - \omega^2 v = 0 \text{ in each } K \in \mathcal{T}_h \Big\}.$$

E.g.: TDG/PWDG, UWVF, VTCR, DEM, (m)DGM, FLAME, WBM, MFS, LS, PUM/PUFEM, GFEM...

Typical Trefftz basis functions for Helmholtz

1 plane waves, $\mathbf{x} \mapsto e^{i\omega \mathbf{x}\cdot\mathbf{d}}$ $\quad \mathbf{d} \in \mathbb{S}^{N-1}$ (PWs)

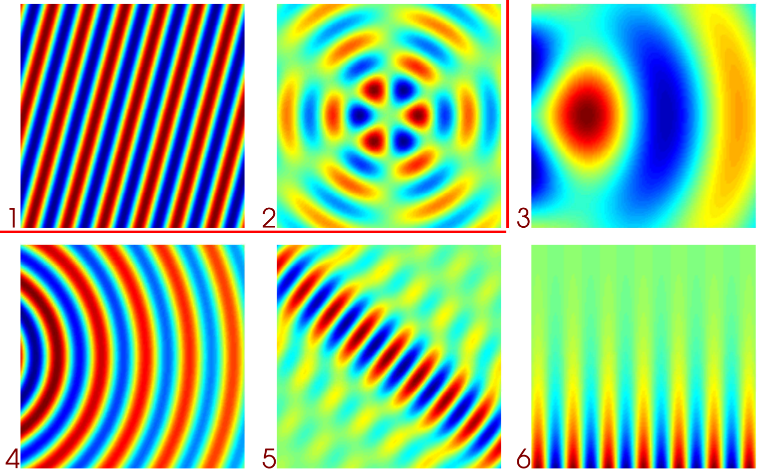
2 circular / spherical waves, $e^{il\psi} J_l(\omega|\mathbf{x}|), \quad Y_l^m(\frac{\mathbf{x}}{|\mathbf{x}|})\, j_l(\omega|\mathbf{x}|)$

3 corner waves,

4 fundamental solutions/multipoles,

5 wavebands,

6 evanescent waves, . . .



Best approximation estimates

The analysis of any plane wave Trefftz method requires best approximation estimates:

$$-\Delta u - \omega^2 u = 0 \qquad \text{in (bdd., Lip.) } D \subset \mathbb{R}^N, \qquad u \in H^{k+1}(D),$$

$$\mathrm{diam}(D) = h, \qquad p \in \mathbb{N}, \qquad \mathbf{d}_1, \dots, \mathbf{d}_p \in \mathbb{S}^{N-1},$$

$$\inf_{\vec{\alpha} \in \mathbb{C}^p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{d}_\ell \cdot \mathbf{x}} \right\|_{H^j(D)} \leq C\, \epsilon(h,p)\ \|u\|_{H^{k+1}(D)},$$

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Goal: precise estimates on $\epsilon(h,p)$

- for plane and circular/spherical waves;
- both in h and p (simultaneously);
- in 2 and 3 dimensions;
- with explicit bounds in the wavenumber ω;
- (suitable for hp-schemes);
- for Helmholtz, Maxwell, elasticity, plates,. . .

Previous results & outline

Only few results available:

- (Cessenat and Després 1998), using Taylor polynomials, h-convergence, 2D, L^2-norm, order is not sharp;
- (Melenk 1995), using Vekua theory, no ω-dependence, p-convergence for plane w., h and p for circular w., 2D.

We follow the general strategy of Melenk.

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Outline:

- algebraic best approximation estimates:
  - Vekua theory;
  - approximation by circular and spherical waves;
  - approximation by plane waves;
- exponential estimates for hp-schemes;
- (extension to Maxwell equations).

Part I

Vekua theory

Vekua theory in N dimensions

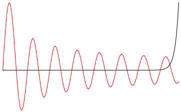
We need an old (1940s) tool from PDE analysis: Vekua theory.

$D \subset \mathbb{R}^N$, open, star-shaped wrt. $\mathbf{0}$, $\omega > 0$.
Define two continuous functions:

$$M_1, M_2 : D \times [0,1] \to \mathbb{R}$$

$$M_1(\mathbf{x}, t) = -\frac{\omega|\mathbf{x}|}{2} \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} J_1\left(\omega|\mathbf{x}|\sqrt{1-t}\right),$$

$$M_2(\mathbf{x}, t) = -\frac{i\omega|\mathbf{x}|}{2} \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} J_1\left(i\omega|\mathbf{x}|\sqrt{t(1-t)}\right).$$



$J_1(t), J_1(it)$

The Vekua operators

$$V_1, V_2 : C^0(D) \to C^0(D),$$

$$V_j[\phi](\mathbf{x}) := \phi(\mathbf{x}) + \int_0^1 M_j(\mathbf{x}, t)\phi(t\mathbf{x})\,\mathrm{d}t \qquad \forall\, \mathbf{x} \in D,\ j = 1, 2.$$

4 properties of Vekua operators

1. $$V_2 = (V_1)^{-1}$$

2. $$\Delta\phi = 0 \qquad \Longleftrightarrow \qquad (-\Delta - \omega^2)\ V_1[\phi] = 0$$

Main idea of Vekua theory:

$$\text{Harmonic functions} \quad \underset{V_1}{\overset{V_2}{\rightleftarrows}} \quad \text{Helmholtz solutions}$$

3. Continuity in (ω-weighted) Sobolev norms, explicit in ω

$(H^j(D), W^{j,\infty}(D),\ j \in \mathbb{N})$

4. $P = $ Harmonic polynomial $\quad \Longleftrightarrow \quad V_1[P] = $ circular/spherical wave

$$\Big[\underbrace{e^{il\psi}\ J_l(\omega r)}_{2D}, \quad \underbrace{Y_l^m\big(\tfrac{\mathbf{x}}{|\mathbf{x}|}\big)\ j_l(\omega|\mathbf{x}|)}_{3D}\Big]$$

Part II

Approximation by circular waves

Vekua operators & approximation by GHPs

$$-\Delta u - \omega^2 u = 0, \qquad\qquad u \in H^{k+1}(D),$$

$$\downarrow V_2$$

$V_2[u]$ is harmonic $\Longrightarrow$ can be approximated by harmonic polynomials

(harmonic Bramble–Hilbert in h,
$\mathbb{C}$omplex analysis in p-2D (Melenk), new result in p-3D),

$$\downarrow V_1$$

u can be approximated by GHPs:

generalized harmonic polynomials $:= V_1 \left[\begin{array}{c} \text{harmonic} \\ \text{polynomials} \end{array} \right] =$ circular/spherical waves.

The approximation by GHPs: h-convergence

$$\inf_{P\in\left\{\substack{\text{harmonic}\\ \text{polynomials}\\ \text{of degree } \le L}\right\}} \big\| \underbrace{u - V_1[P]}_{=V_1[V_2[u]-P]} \big\|_{j,\omega,D} \le C \inf_P \|V_2[u] - P\|_{j,\omega,D} \qquad \text{contin. of } V_1,$$

$$\le C\, h^{k+1-j}\, \epsilon(L)\ \|V_2[u]\|_{k+1,\omega,D} \qquad \text{harmonic approx. results,}$$

$$\le C\, h^{k+1-j}\, \epsilon(L)\ \|u\|_{k+1,\omega,D} \qquad \text{contin. of } V_2.$$

For the h-convergence, Bramble–Hilbert theorem is enough: it provides a harmonic polynomial!

The constant C depends on ωh, not on ω alone:

$$C = C \cdot (1 + \omega h)^{j+6} e^{\frac{3}{4}\omega h}.$$

Harmonic approximation: p-convergence

Assume D is star-shaped wrt B_{ρ_0}.

In 2 dimensions,
sharp p-estimate! (MELENK):

$$\epsilon(L) = \left(\frac{\log(L+2)}{L+2}\right)^{\lambda(k+1-j)} .$$

If D convex, $\lambda = 1$. Otherwise $\lambda = \min(\text{re-entrant corner of } D)/\pi$.

In 2D, use complex analysis: $\mathbb{R}^2 \leftrightarrow \mathbb{C}$, harmonic $\leftrightarrow$ holomorphic.

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We can prove an analogous result in N dimensions:

$$\epsilon(L) = L^{-\lambda(k+1-j)},$$

where $\lambda > 0$ is a geometric unknown parameter.

If u is the restriction of a solution in a larger domain (2 or 3D),
the convergence in L is exponential.

Part III

Approximation by plane waves

The approximation of GHPs by plane waves

Link between plane waves and circular/spherical waves:
Jacobi–Anger expansion

2D $$e^{iz\cos\theta} = \sum_{l\in\mathbb{Z}} i^l J_l(z)\, e^{il\theta} \qquad z\in\mathbb{C},\ \theta\in\mathbb{R},$$

3D $$\underbrace{e^{ir\boldsymbol{\xi}\cdot\boldsymbol{\eta}}}_{\text{plane wave}} = 4\pi \sum_{l\geq 0}\sum_{m=-l}^{l} i^l \underbrace{j_l(r)\ Y_l^m(\boldsymbol{\xi})}_{\text{GHP}}\ \overline{Y_l^m(\boldsymbol{\eta})} \qquad \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{S}^2,\ r\geq 0.$$

We need the other way round:

GHP $\approx$ linear combination of plane waves

- truncation of J–A expansion,
- careful choice of directions (in 3D),
- solution of a linear system,
- residual estimates,

$\rightarrow$ explicit error bound,
$\sim h^k q^{-\frac{q}{2}}$.

The choice of the PW directions in 3D

(In 2D any choice of PW directions is allowed,
estimate depends on minimal angular distance.)

3D Jacobi–Anger gives the matrix $\{\mathbf{M}\}_{l,m;k} = Y_l^m(\mathbf{d}_k)$
that depends on the choice of the directions $\mathbf{d}_k$.

Problem: an upper bound on $\left\|\mathbf{M}^{-1}\right\|$ is needed
but $\mathbf{M}$ is not even always invertible!

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but $\mathbf{M}$ is not even always invertible!

Solution:

- there exists an optimal choice of $\mathbf{d}_k$ s.t. $\|\mathbf{M}^{-1}\|_1 \le 2\sqrt{\pi}\, p$;
- it corresponds to the extremal systems of Sloan–Womersley for quadrature on $\mathbb{S}^2$, computable/downloadable;
- some simple choices of points give good result, heuristic: $\mathbf{d}_k$ have to be as "equispaced" as possible.

With these choices $\rightarrow$ analogous results as in 2D.

The final approximation by plane waves

$$-\Delta u - \omega^2 u = 0 \quad \xrightarrow{V_2} \quad -\Delta V_2[u] = 0$$

harmonic approx. $\downarrow$

$$\text{Circular waves} \quad \xleftarrow{V_1} \quad \text{Harmonic polyn.}$$

$\downarrow$ (Jacobi–Anger)$^{-1}$

Plane waves

Final estimate (algebraic convergence)

$$\inf_{\alpha \in \mathbb{C}^p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{x}\cdot\mathbf{d}_\ell} \right\|_{j,\omega,D} \leq C(\omega h)\ h^{k+1-j} q^{-\lambda(k+1-j)}\ \|u\|_{k+1,\omega,D}$$

In 2D: $p = 2q+1$, $\lambda(D)$ explicit, $\forall\ \mathbf{d}_\ell$.

In 3D: $\underbrace{p = (q+1)^2}_{\text{better than poly.!}}$, $\lambda(D)$ unknown, special $\mathbf{d}_\ell$.

(p = dimension, q = "degree" of approximating space.)

Part IV

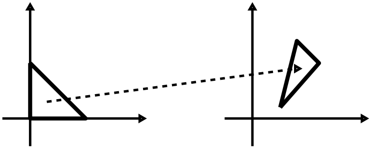
Exponential bounds for hp-schemes

What do we need?

Assume u can be extended outside D (true for most elements).

Bounds with exponential dependence on "plane wave degree" q are easy.

But it is harder to have explicit dependence on the size of the extension and on the element shape (needed because Trefftz methods do not allow mappings to reference elements).



Even for affine scaling:
$\mathbb{P}^q(\hat{K}) \longrightarrow \mathbb{P}^q(K)$
$PW^q(\hat{K}) \longrightarrow ???$

Only step to be improved is harmonic approximation.
Only 2D considered.

Assumption and tools

Assumption on element D: (Very weak!)

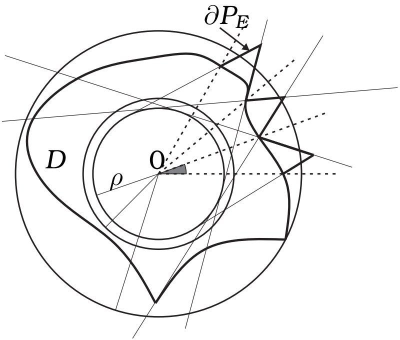
- $D \subset \mathbb{R}^2$ s.t. $\operatorname{diam}(D) = 1$, star-shaped wrt B_ρ, $0 < \rho < 1/2$.

Define:

- $D_\delta := \{z \in \mathbb{R}^2, d(z, D) < \delta\}$, $\quad \xi := \begin{cases} 1 & D \text{ convex,} \\ \frac{2}{\pi} \arcsin \frac{\rho}{1-\rho} < 1. \end{cases}$

Use:

- M. Melenk's ideas;
- complex variable, identification $\mathbb{R}^2 \leftrightarrow \mathbb{C}$, harmonic $\leftrightarrow$ holomorphic;
- conformal map level sets, Schwarz–Christoffel;
- Hermite interpolant q_n;
- lot of "basic" geometry and trigonometry, nested polygons, plenty of pictures...



Explicit approximation estimate

Approximation result

Let $n \in \mathbb{N}$, f holomorphic in $D_\delta := \{z \in \mathbb{R}^2, d(z, D) < \delta\}$, $\delta \leq 1/2$,
$H := \min\left\{(\xi\delta/27)^{1/\xi}/3,\ \rho/4\right\}$, $\Rightarrow$ $\exists q_n$ of degree $\leq n$ s.t.

$$\|f - q_n\|_{L^\infty(D)} \leq 7\rho^{-2}\, H^{-\frac{72}{\rho^4}} (1+H)^{-n} \|f\|_{L^\infty(D_\delta)}.$$

- Fully explicit bound;
- exponential in degree n;
- $H \geq$ "conformal dist." $(D, \partial D_\delta)$, related to physical dist. δ;
- in convex case $H = \min\{\delta/27,\ \rho/4\}$;
- extends to harmonic f/q_n and derivatives ($W^{j,\infty}$-norm);
- extended to Helmholtz solutions and circular/plane waves (fully explicit $W^{j,\infty}(D)$-continuity of Vekua operators).

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$$\Rightarrow \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega \mathbf{x}\cdot\mathbf{d}_\ell} \right\|_{W^{j,\infty}(D)} \leq C_{(\rho,\delta,j,\omega h)} h^{-j}\, e^{-bp}\, \|u\|_{W^{1,\infty}(D_\delta)}\,.$$

Part V

The electromagnetic case

Maxwell plane waves

The vector field $\mathbf{E}$ is solution of Maxwell's equations if

$$\nabla \times (\nabla \times \mathbf{E}) - \omega^2 \mathbf{E} = \mathbf{0} \qquad \Longleftrightarrow \qquad \begin{cases} -\boldsymbol{\Delta} \mathbf{E}_j - \omega^2 \mathbf{E}_j = \mathbf{0} & j = 1, 2, 3, \\ \operatorname{div} \mathbf{E} = 0. \end{cases}$$

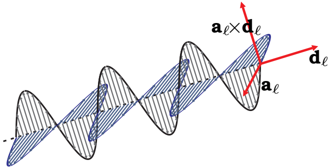
A vector plane wave $\mathbf{a} e^{i\omega \mathbf{x}\cdot\mathbf{d}}$ is a Maxwell solution iff

$$\operatorname{div}(\mathbf{a} e^{i\omega \mathbf{x}\cdot\mathbf{d}}) = i\omega(\mathbf{d}\cdot\mathbf{a}) e^{i\omega \mathbf{x}\cdot\mathbf{d}} = 0, \quad \text{i.e., } \mathbf{d}\cdot\mathbf{a} = 0.$$

Basis of Maxwell plane waves:

$$\left\{ \mathbf{a}_\ell e^{i\omega \mathbf{x}\cdot\mathbf{d}_\ell}, \quad \mathbf{a}_\ell \times \mathbf{d}_\ell e^{i\omega \mathbf{x}\cdot\mathbf{d}_\ell} \right\}_{\ell = 1, \dots, (q+1)^2}$$

$$|\mathbf{a}_\ell| = |\mathbf{d}_\ell| = 1, \; \mathbf{d}_\ell \cdot \mathbf{a}_\ell = 0.$$



Goal: prove convergence using $2(q+1)^2$ plane waves and preserving the Trefftz property.

Maxwell plane wave approximation

1. $\mathbf{E}$ Maxwell $\Rightarrow$ $\nabla \times \mathbf{E}$ Maxwell $\Rightarrow$ $(\nabla \times \mathbf{E})_{1,2,3}$ Helmholtz

$$\left\| \nabla \times \mathbf{E} - \begin{matrix}\text{Helmholtz}\\ \text{vector p.w.}\end{matrix} \right\|_{j,\omega,D} \le C(hq^{-\lambda})^{k+1-j} \left\| \nabla \times \mathbf{E} \right\|_{k+1,\omega,D}.$$

2. With $j \geq 1$, apply $\nabla \times$ and reduce j (bad!):

$$\left\| \nabla \times \nabla \times \mathbf{E} - \nabla \times \begin{bmatrix}\text{Helmholtz}\\ \text{vector p.w.}\end{bmatrix} \right\|_{j-1,\omega,D} \le C(hq^{-\lambda})^{k+1-j} \left\| \nabla \times \mathbf{E} \right\|_{k+1,\omega,D}.$$

$$\Downarrow$$

3. $$\left\| \omega^2 \mathbf{E} - \text{Maxwell p.w.} \right\|_{j-1,\omega,D} \le C(hq^{-\lambda})^{k+1-j} \left\| \nabla \times \mathbf{E} \right\|_{k+1,\omega,D}.$$

Mismatch between Sobolev indices and convergence order:
not sharp!

Improvements and extensions

1. In the previous bound, we need only:

$$\nabla \times \left\{ \begin{array}{c} \text{vector Helmholtz} \\ \text{trial space} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Maxwell} \\ \text{trial space} \end{array} \right\},$$

$\Rightarrow$ same result for Maxwell spherical waves!
The space is defined via vector spherical harmonics.

2. How to get better orders?

- h-conv., spherical w.: ✓ with Vekua theory,
- h-conv., plane w.: $\approx$ probably with vector Jacobi–Anger,
- p-conv.: **!?** no clue!

3. Same technique (+ special potential representation) used for elastic wave equation and Kirchhoff–Love plates (Chardon).

Conclusions

We have estimates for

- the approximation of Helmholtz and Maxwell solutions,
- by circular, spherical and plane waves,
- in 2D and 3D,
- with orders in $h\&p$,
- explicit constants in ω, and
- exponential bounds, explicit in the geometry (in 2D).

Open problems:

- explicit convergence order (λ) in p in 3D (simple) domains,
- sharp bounds for vector equations,
- improved bounds for PWs with "optimal" directions,
- smooth coefficients (see IMBERT-GÉRARD),
- ...

Thank you!