Virtual Element Methods for general elliptic equations

Alessandro Russo

Department of Mathematics and Applications
University of Milano-Bicocca, Milan, Italy
and
IMATI-CNR, Pavia, Italy

Building bridges: connections and challenges in modern approaches to numerical partial differential equations

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Joint work with L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, D. Marini

Outline of the presentation

- The ultimate goal
- Virtual Element spaces in 2D
- Projectors in 2D
- VEM approximation of general elliptic equations
- Extension to 3D
- The VEM paradigm
- Numerical experiments

The ultimate goal

We want to approximate a general second-order elliptic equation in 3D with an arbitrary polyhedral mesh with a conforming finite element method of order k.

$$\begin{cases} -\mathrm{div}\,(\kappa\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \alpha u = f & \text{in } \Omega \subset \mathbb{R}^3 \\ u = g & \text{on } \partial\Omega \end{cases}$$

We start with the two-dimensional case.

The local finite element space $V_k(P)$

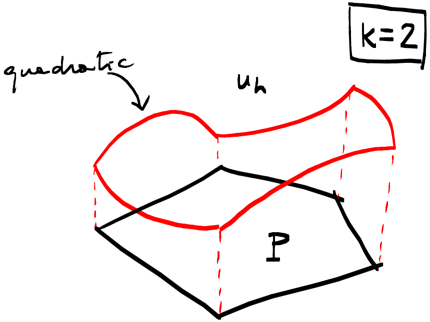
Let P be a polygon. We would like to define a finite element space $V_k(P)$ on P such that:

- $V_k(P)$ contains the space $\mathbb{P}_k(P)$ of polynomials of degree less than or equal to k plus other "bad" functions;
- if two polygons P and P' have an edge in common, the two spaces $V_k(P)$ and $V_k(P')$ must "glue" in $C^0(P \cup P')$;
- I don't want to compute the pointwise value of the "bad" (non-polynomial) functions to approximate my equation.

The local finite element space $V_k(P)$

A function v_h in $V_k(P)$ is defined by the following properties:

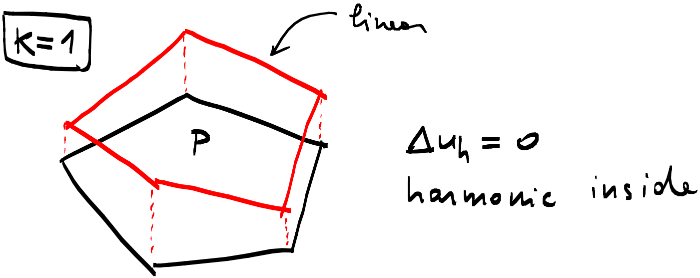
- if e is and edge of P, v_h restricted to e is a polynomial of degree less than or equal to k;
- v_h is continuous on the boundary of P;
- Δv_h is a polynomial of degree less or equal than $k-2$ in P.



$\Delta u_h =$ constant

The local finite element space $V_k(P)$ for $k = 1$

In the case $k = 1$ an element v_h of $V_1(P)$ is linear on each edge e, continuous on the boundary of P and harmonic inside ($\mathbb{P}_{-1}(P) = \{0\}$).



For $k = 1$ this definition corresponds to the well-known notion of Harmonic Barycentric Coordinates on polygons.

The local finite element space $V_k(P)$

It is clear that the condition

$$\boxed{\Delta v_h \in \mathbb{P}_{k-2}(P)}$$

ensures that

$$\boxed{\mathbb{P}_k(P) \subset V_k(P)}.$$

If N^v is the number of vertices (and also the number of edges) of the polygon P, the dimension of $V_k(P)$ is given by

$$\dim V_k(P) = \underbrace{N^v}_{\text{vertices}} + N^v\underbrace{(k-1)}_{\text{edge}} + \underbrace{\frac{k(k-1)}{2}}_{\text{interior}} = N^v k + \frac{k(k-1)}{2}$$

Degrees of freedom in $V_k(P)$

Let (x_P, y_P) be the centroid of P and h_P its diameter. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is a multiindex we define the $\boxed{\text{scaled monomials}}$ of degree $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$:

$$m_{\boldsymbol{\alpha}}(x, y) := \left(\frac{x - x_P}{h_P}\right)^{\alpha_1} \left(\frac{y - y_P}{h_P}\right)^{\alpha_2}.$$

The set $\{m_{\boldsymbol{\alpha}},\ \text{with } |\boldsymbol{\alpha}| \le k\}$ is a basis for $\mathbb{P}_k(P)$.

As degrees of freedom in $V_k(P)$, we choose:

- the value of v_h at the vertices and at $k-1$ equally spaced points on each edge;
- the (scaled) moments $\boxed{\dfrac{1}{|P|}\displaystyle\int_P v_h m_{\boldsymbol{\alpha}}}$ for $|\boldsymbol{\alpha}| \le k-2$.

Degrees of freedom in $V_k(P)$

It can be easily shown that:

- the degrees of freedom above are *unisolvent* in $V_k(P)$.

The choice of the moments $\int_P v_h m_{\boldsymbol{\alpha}}$ for $|\boldsymbol{\alpha}| \leq k-2$ as degrees of freedom implies that, starting from the degrees of freedom of v_h, I can compute

$$\Pi^0_{k-2} v_h := L^2 \text{ projection of } v_h \text{ onto } \mathbb{P}_{k-2}(P).$$

Infact, to compute the L^2 projection of v_h onto $\mathbb{P}_{k-2}(P)$ I need to compute the moments $\int_P v_h\, p$ up to order $k-2$ which are among the degrees of freedom.

Meaning of "I can compute"

In what follows the precise meaning of the statement

$$\boxed{\text{I can compute } \Pi^0_{k-2} v_h}$$

is:

$$\boxed{\text{given the array } \mathrm{dof}_i(v_h)\text{, I can compute } \Pi^0_{k-2} v_h}$$

The same applies for all other quantities which are computable from v_h.

Basis functions in $V_k(P)$

For $i = 1, \dots, N^{\text{dof}}$ we define φ_i as the function in $V_k(P)$ such that

$$\text{dof}_j(\varphi_i) = j\text{-th degree of freedom of } \varphi_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We have the usual Lagrange-type expansion

$$v_h = \sum_{i=1}^{N^{\text{dof}}} \text{dof}_i(v_h)\, \varphi_i.$$

- It is clear that if I could compute directly the bilinear form on the space $V_k(P)$, the resulting finite element method would converge with the right optimal rates.

The projector operator Π_k^∇

Warning: For a while I will restrict to the case $k \geq 2$. The case $k = 1$ is similar but does not fit well in the general case.

We define a projection operator

$$\Pi_k^\nabla : V_k(P) \longrightarrow \mathbb{P}_k(P)$$

which is orthogonal with respect to the H^1 inner product $\int_P \nabla u \cdot \nabla v$, i.e.:

$$(1) \qquad \int_P \nabla p_k \cdot \nabla \Pi_k^\nabla v_h = \int_P \nabla p_k \cdot \nabla v_h \text{ for all } p_k \in \mathbb{P}_k(P)$$

$$(2) \qquad \int_P \Pi_k^\nabla v_h = \int_P v_h$$

Condition (1) defines $\Pi_k^\nabla v_h$ up to a constant function, while condition (2) determines how Π_k^∇ acts on constant functions.

Note that the gradient of $\Pi_k^\nabla v_h$ is completely determined by condition (1).

Computability of Π_k^∇

The operator $\Pi_k^\nabla v_h$ is computable without knowing the values of v_h inside the polygon. In fact:

- to compute $\Pi_k^\nabla \varphi_i$, it is enough to test condition (1) only on $\{m_{\boldsymbol\beta}\}$;
- if we express $\Pi_k^\nabla \varphi_i = \sum_{|\boldsymbol\alpha|\le k} s_{i\boldsymbol\alpha} m_{\boldsymbol\alpha}$, condition (1) becomes:

(1)'
$$\sum_{|\boldsymbol\alpha|\le k} s_{i\boldsymbol\alpha} \int_P \nabla m_{\boldsymbol\beta} \cdot \nabla m_{\boldsymbol\alpha} = \int_P \nabla m_{\boldsymbol\beta} \cdot \nabla \varphi_i, \quad |\boldsymbol\beta| \le k$$

and condition (2) becomes

(2)'
$$\sum_{|\boldsymbol\alpha|\le k} s_{i\boldsymbol\alpha} \int_P m_{\boldsymbol\alpha} = \int_P \varphi_i$$

Computability of Π_k^∇

When $\boldsymbol{\beta} = (0, 0)$, equation (1)' is the identity $0 = 0$ reflecting the fact that condition (1) determines $\Pi_k^\nabla v_h$ only up to a constant function.

Equation (2)' supplies the missing condition.

Equations (1)' and (2)' form for each i a system of linear equations of dimension

$$\dim \mathbb{P}_k(P) = \frac{(k+1)(k+2)}{2}.$$

Computability of Π_k^∇

- The matrix is always computable (integrals of polynomials on P);
- the right-hand-side $\int_P \varphi_i$ is computable because it is one of the degrees of freedom;
- the right-hand-side $\int_P \nabla m_\beta \cdot \nabla \varphi_i$ can be written as

$$\int_P \nabla m_\beta \cdot \nabla \varphi_i = -\int_P \Delta m_\beta \, \varphi_i + \int_{\partial P} \frac{\partial m_\beta}{\partial n} \, \varphi_i$$

- $\int_P \Delta m_\beta \, \varphi_i$ can be computed because $\Delta m_\beta \in \mathbb{P}_{k-2}(P)$;
- $\int_{\partial P} \frac{\partial m_\beta}{\partial n} \, \varphi_i$ can be computed because on each edge the integrand is a known polynomial of degree $k(k-1)$.

What we have so far

Starting from the degrees of freedom of a function $v_h \in V_k(P)$, we can compute:

- the nabla projector of v_h: $\boxed{\Pi_k^\nabla v_h \in \mathbb{P}_k(P)}$;
- the L^2 projector of v_h on $\mathbb{P}_{k-2}(P)$: $\boxed{\Pi_{k-2}^0 v_h \in \mathbb{P}_{k-2}(P)}$.

We wish to apply the abstract theorem proved by the "volley" team:

Theorem. Suppose that $a_h^P(\cdot,\cdot)$ is a local bilinear form defined on P which approximate the exact bilinear form $a^P(\cdot,\cdot)$ in the following sense:

- it is consistent, i.e. $a_h^P(v_h, p_k) = a^P(v_h, p_k)$ for any $v_h \in V_k(P)$ and any $p_k \in \mathbb{P}_k(P)$;
- it is stable, i.e. $\alpha_* a^P(v_h, v_h) \leq a_h^P(v_h, v_h) \leq \alpha^* a^P(v_h, v_h)$.

Then if we use $a_h(\cdot,\cdot)$ instead of $a(\cdot,\cdot)$ the method converges.

What we have so far

As shown in the previous talk by Lourenco Beirao, the projectors Π_k^∇ and Π_{k-2}^0 allow us to compute an approximate bilinear form $a_h^P(\cdot,\cdot)$ which is consistent and stable in the simple case of the Laplace operator:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = g & \text{on } \partial\Omega \end{cases}$$

In this case we have $a^P(u_h, v_h) = \int_P \nabla u_h \cdot \nabla v_h$ and we can define

$$a_h^P(u_h, v_h) := a^P(\Pi_k^\nabla u_h, \Pi_k^\nabla v_h) + S((I - \Pi_k^\nabla)u_h, (I - \Pi_k^\nabla)v_h)$$

where $S(\varphi_i, \varphi_j) = \delta_{ij}$. The load term $\int_P f\, v_h$ can be approximated by

$$\int_P f\, v_h \approx \int_P f\, \Pi_{k-2}^0 v_h.$$

Computing other projectors: $\Pi^0_{k-1}\nabla v_h$

To deal with more general operator, we need to extract more information out of the space $V_k(P)$.

Up to now we have seen that we are able to compute $\Pi^\nabla_k v_h$ and $\Pi^0_{k-2} v_h$.

We show now that we can easily compute $\boxed{\Pi^0_{k-1}\nabla v_h}$.

To compute $\Pi^0_{k-1}\nabla v_h$ we need to know the moments of ∇v_h up to order $k-1$:

$$\int_P \frac{\partial v_h}{\partial x}\, m_{\boldsymbol{\beta}} = -\int_P v_h\, \frac{\partial m_{\boldsymbol{\beta}}}{\partial x} + \int_{\partial P} v_h\, m_{\boldsymbol{\beta}}\, \boldsymbol{n}_x, \quad |\boldsymbol{\beta}| \le k-1$$

and both terms are computable directly from the degrees of freedom of v_h.

Computing the L^2 projection onto $\mathbb{P}_k(P)$

We go back to the definition of $V_k(P)$:

A function v_h in $V_k(P)$ is defined by the following properties:

- if e is and edge of P, v_h restricted to e is a polynomial of degree less or equal than k;
- v_h is continuous on the boundary of P;
- $\boxed{\Delta v_h \text{ is a polynomial of degree less than or equal to } k-2 \text{ in } P.}$

The boxed condition has been used only to ensure that $\mathbb{P}_k(P) \subset V_k(P)$ and to get the right number of degrees of freedom.

We can change it and slightly modify *(enhance)* the space $V_k(P)$.

Computing the L^2 projection onto $\mathbb{P}_k(P)$

The idea is first to relax the condition $\Delta v_h \in \mathbb{P}_{k-2}(P)$ by asking

$$\Delta v_h \in \mathbb{P}_k(P)$$

and then requiring

$$\int_P v_h m_{\boldsymbol{\alpha}} = \int_P \Pi_k^\nabla v_h \, m_{\boldsymbol{\alpha}} \quad \text{for } |\boldsymbol{\alpha}| = k \text{ and } |\boldsymbol{\alpha}| = k-1$$

We call $W_k(P)$ this new space.

This may seem weird because we defined Π_k^∇ only on $V_k(P)$, but if we go back to the definition we see that actually Π_k^∇ is defined on the whole space $H^1(P)$ (but of course it's not computable in general!).

Computing the L^2 projection onto $\mathbb{P}_k(P)$

It can be shown that $W_k(P)$ has the same dimension of $V_k(P)$ and can be described by the same degrees of freedom of $V_k(P)$.

The projection operators $\Pi_k^{\nabla} w_h$ and $\Pi_{k-1}^0 \nabla w_h$ can still be computed.

The additional property that the k and $k-1$ moments are computable (through the projector Π_k^{∇}) implies that

in $W_k(P)$ we can compute the full L^2 projection onto $\mathbb{P}_k(P)$.

The spaces $V_k(P)$ and $W_k(P)$

- $V_k(P)$ and $W_k(P)$ have the same dimension;
- $V_k(P)$ and $W_k(P)$ can be described with the same set dof_i of degrees of freedom;
- both $V_k(P)$ and $W_k(P)$ contain the polynomials of degree k;
- given $v_h \in V_k(P)$ and $w_h \in W_k(P)$,

$$\text{if } \mathrm{dof}_i(v_h) = \mathrm{dof}_i(w_h) \quad \text{then} \quad \Pi_k^{\nabla} v_h = \Pi_k^{\nabla} w_h$$

  (obviously, also $\Pi_{k-2}^0 v_h = \Pi_{k-2}^0 w_h$);
- the basis functions φ_i are different in $V_k(P)$ and in $W_k(P)$ but their Π_k^{∇} and Π_{k-2}^0 projections are equal;
- in $W_k(P)$ we can also compute $\Pi_k^0 w_h$;
- for $k = 1$ and $k = 2$ we have $\Pi_k^0 w_h = \Pi_k^{\nabla} w_h$.

VEM approximation of general elliptic equations

We consider now a general second order elliptic operator with variable coefficients:

$$-\mathrm{div}\,(\kappa\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \alpha u = f$$

and we approximate the various local consistency terms as:

- $\displaystyle\int_P \kappa\,\nabla u_h\cdot\nabla v_h \quad\rightsquigarrow\quad \int_P \kappa\left[\Pi^0_{k-1}\nabla u_h\right]\cdot\left[\Pi^0_{k-1}\nabla v_h\right]$
- $\displaystyle\int_P (\boldsymbol{\beta}\cdot\nabla u_h)\,v_h \quad\rightsquigarrow\quad \int_P \left(\boldsymbol{\beta}\cdot\left[\Pi^0_{k-1}\nabla u_h\right]\right)\Pi^0_k v_h$
- $\displaystyle\int_P \alpha\,u_h\,v_h \quad\rightsquigarrow\quad \int_P \alpha\left[\Pi^0_k u_h\right]\left[\Pi^0_k v_h\right]$

and for the right-hand-side:

- $\displaystyle\int_P f\,v_h \quad\rightsquigarrow\quad \int_P f\,\Pi^0_k v_h \quad \left(\Pi^0_{k-2}v_h \text{ is enough for } k\geq 2\right)$

VEM approximation of general elliptic equations

The approximations above produce, as usual, rank-deficient matrices that must be stabilized.

For the stabilization we can take the same term we had for the Laplace operator, i.e.

$$S((I-\Pi_k^\nabla)u_h,(I-\Pi_k^\nabla)v_h)$$

with $S(\varphi_i,\varphi_j)=\delta_{ij}$.

Summarizing, the approximate local stiffness matrix provided by the Virtual Element Method is

$$\begin{aligned}(\mathbf{K}^P_{\text{VEM}})_{ij} &:= a_h^P(\varphi_i,\varphi_j) := \\ \int_P \kappa\left[\Pi^0_{k-1}\nabla\varphi_j\right]\cdot\left[\Pi^0_{k-1}\nabla\varphi_i\right] &+ \int_P\left[\boldsymbol{\beta}\cdot\Pi^0_{k-1}\nabla\varphi_j\right]\Pi^0_k\varphi_i + \int_P \alpha\left[\Pi^0_k\varphi_j\right]\left[\Pi^0_k\varphi_i\right] \\ &+ S((I-\Pi_k^\nabla)\varphi_i,(I-\Pi_k^\nabla)\varphi_j)\end{aligned}$$

The stabilization term $S((I-\Pi_k^\nabla)\varphi_i,(I-\Pi_k^\nabla)\varphi_j)$

If we expand $\Pi_k^\nabla \varphi_i$ in the basis $\{\varphi_\ell\}$ itself, we have

$$\Pi_k^\nabla \varphi_j = \sum_{\ell=1}^{N^{\text{dof}}} \pi_{j\ell}\varphi_\ell \quad \text{and} \quad (I-\Pi_k^\nabla)\varphi_j = \sum_{\ell=1}^{N^{\text{dof}}} (\delta_{j\ell}-\pi_{j\ell})\varphi_\ell$$

so that

$$S((I-\Pi_k^\nabla)\varphi_j,(I-\Pi_k^\nabla)\varphi_i) = \sum_{\ell,m=1}^{N^{\text{dof}}} (\delta_{j\ell}-\pi_{j\ell})(\delta_{im}-\pi_{im})\, S(\varphi_\ell,\varphi_m)$$

and we can use (see [Beirao, Brezzi, Cangiani, Manzini, Marini, R. 2013])

$\boxed{S(\varphi_\ell,\varphi_m)=\delta_{\ell m}}$ (because we are in 2D).

Any symmetric and positive definite matrix which scale like 1 with respect to h will work.

The matrices $\mathbf{K}^P$ and $\mathbf{K}^P_{\text{VEM}}$

If we compare the exact local stiffness matrix

$$(\mathbf{K}^P)_{ij} := a^P(\varphi_i, \varphi_j)$$

with the VEM local stiffness matrix $\mathbf{K}^P_{\text{VEM}}$ defined above,

it is *NOT* true that $(\mathbf{K}^P)_{ij} \approx (\mathbf{K}^P_{\text{VEM}})_{ij}$

as we would have if we had approximated $\mathbf{K}^P$ by numerical integration.

The local VEM spaces in three dimensions

We begin by considering the simple Laplace equation in three dimensions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^3 \\ u = g & \text{on } \partial\Omega \end{cases}$$

We assume that the domain Ω is partitioned in a family of polyhedra $\{P\}$ and we want to define a Virtual Element Method of order k for this problem.

We define the scaled moments $m_{\boldsymbol{\alpha}}$ for $|\boldsymbol{\alpha}| \leq k$ as in the two dimensional case.

The local VEM spaces in three dimensions

We know from the previous analysis that we need to include polynomials of degree k and we should compute the projectors Π_k^∇ and Π_{k-2}^0.

Hence we start to define the local virtual space in 3D by requiring

$$v_{h|e} \in \mathbb{P}_k(e) \text{ for each edge } e \quad \text{and} \quad \Delta u_h \in \mathbb{P}_{k-2}(P)$$

We need to understand what to do for the faces $\{f\}$ of P.

The projection Π_{k-2}^0 can be computed directly by the internal degrees of freedom.

For the projection Π_k^∇ we need to compute the integrals

$$\int_P \nabla m_{\boldsymbol{\beta}} \cdot \nabla v_h \quad \text{for } |\boldsymbol{\beta}| \leq k$$

The local VEM spaces in three dimensions

Integrating by parts:

$$\int_P \nabla m_\beta \cdot \nabla v_h = -\int_P \Delta m_\beta \, v_h + \sum_f \int_f \frac{\partial m_\beta}{\partial n_f} \, v_h$$

The first term is an internal moment of order $k-2$ and can be computed directly from the internal degrees of freedom.

The second term is a moment of order $k-1$ on the face f.

This means that we can choose

$$v_{h|f} \in W_k(f) \quad \text{for each face } f$$

because in $W_k(f)$ (the enhanced $V_k(f)$) we can compute the L^2 projector $\Pi^0_{f,k}$, and hence all moments up to order k.

The local VEM spaces in three dimensions

The VEM approximate bilinear form for the Laplace equation in 3D is:

$$a_h^P(u_h, v_h) = \int_P \nabla \Pi_k^\nabla u_h \cdot \nabla \Pi_k^\nabla v_h + S((I - \Pi_k^\nabla)u_h, (I - \Pi_k^\nabla)v_h)$$

where this time

$$S(\varphi_i, \varphi_j) = h_P \, \delta_{ij}.$$

If we want to approximate a more general operator, we have to compute $\Pi_{k-1}^0 \nabla v_h$ and the L^2 projector Π_k^0.

$\Pi_{k-1}^0 \nabla v_h$ can be computed directly from the degrees of freedom.

For the L^2 projector Π_k^0 we can enhance the space as shown in two dimensions.

The VEM paradigm

We believe that the Virtual Element Method has a wide range of applicability. The keypoints of VEM are:

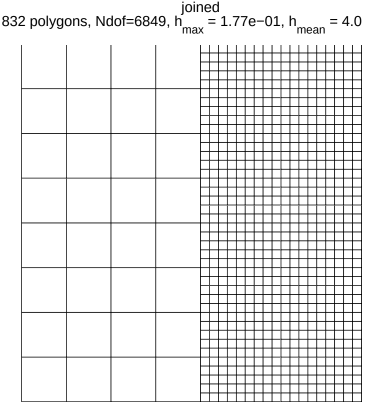
- The definition of the local finite element spaces and of the associated degrees of freedom. These spaces contain polynomials plus other functions which are not computable.
- The costruction of variuos projectors onto polynomial spaces.
- The definition of a consistent and stable bilinear form using these projectors which is the VEM approximation of the exact bilinear form.
- A general theorem that guarantees convergence for a consistent and stable approximate bilinear form.

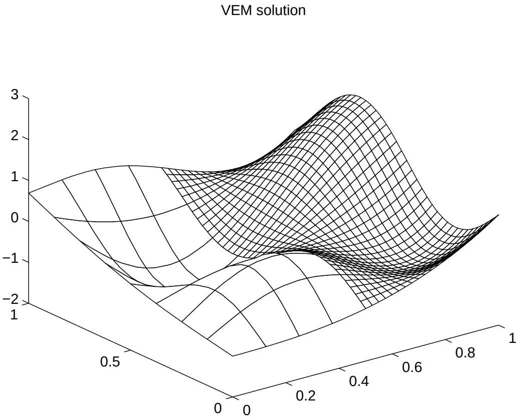
The VEM paradigm

The VEM paradigm is currently being applied for various problem, including:

- H(div) and H(curl) VEM [Beirao da Veiga, Brezzi, Marini, R.]
- Non conforming VEM [Ayuso, Lipnikov, Manzini] *talk this afternoon*
- SUPG stabilization of convection-dominated equations [Cangiani, Manzini, R., Sutton]
- C^m finite elements [Beirao da Veiga, Manzini]
- Eigenvalue problems [Beirao da Veiga, Mora] *next talk*
- Elasticity Problems [Beirao da Veiga, Brezzi, Marini, Paulino]
- Plates and Shells [Brezzi, Marini]
- Helmholtz equations [Perugia, Pietra, R.]

Joining meshes 1





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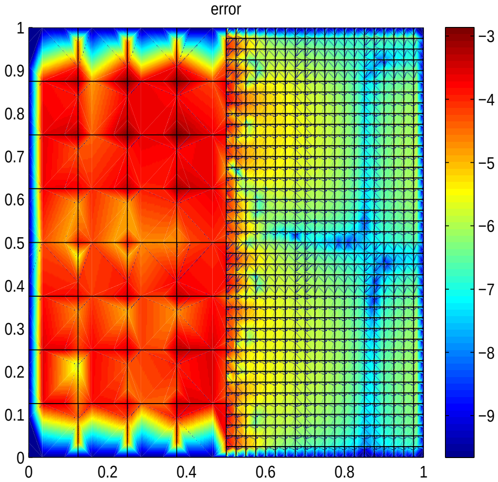

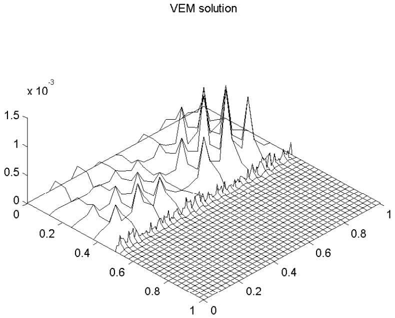

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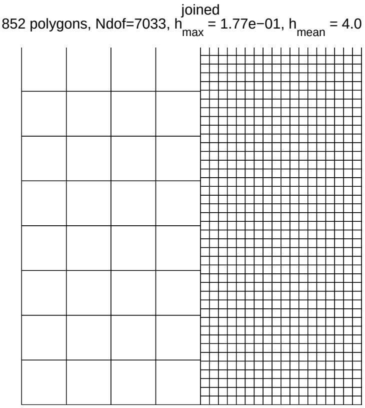

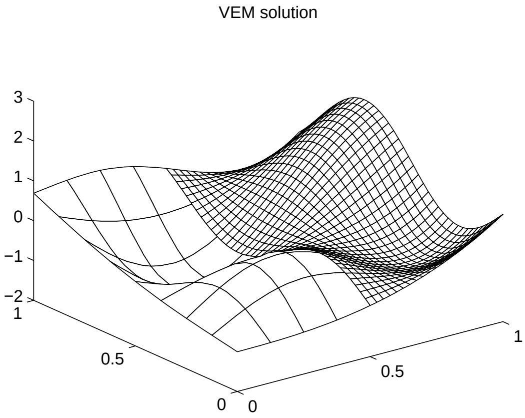

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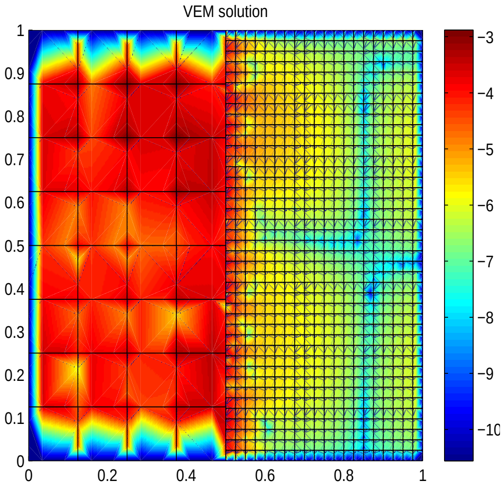

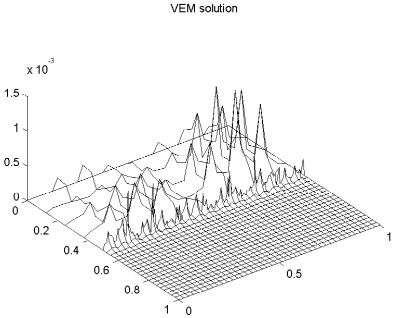

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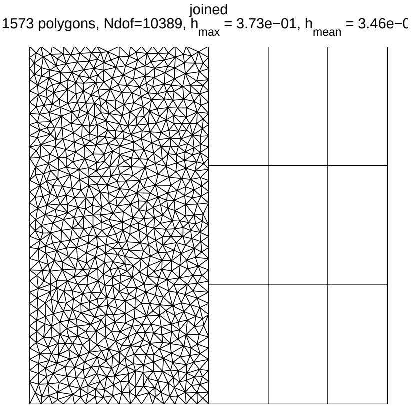

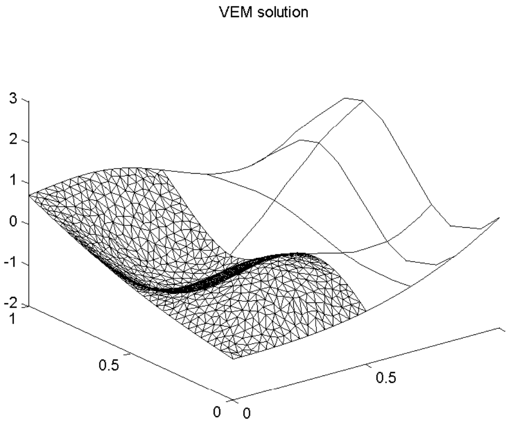

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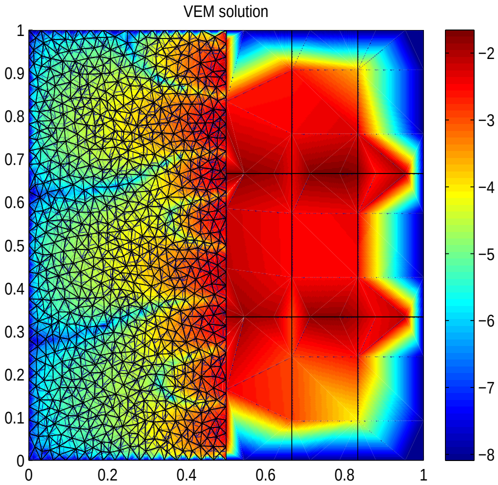
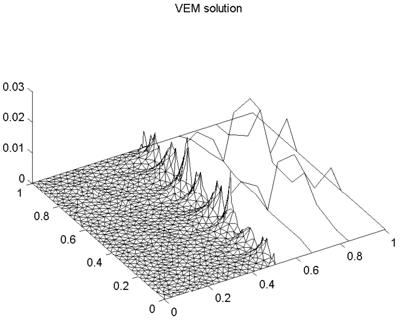
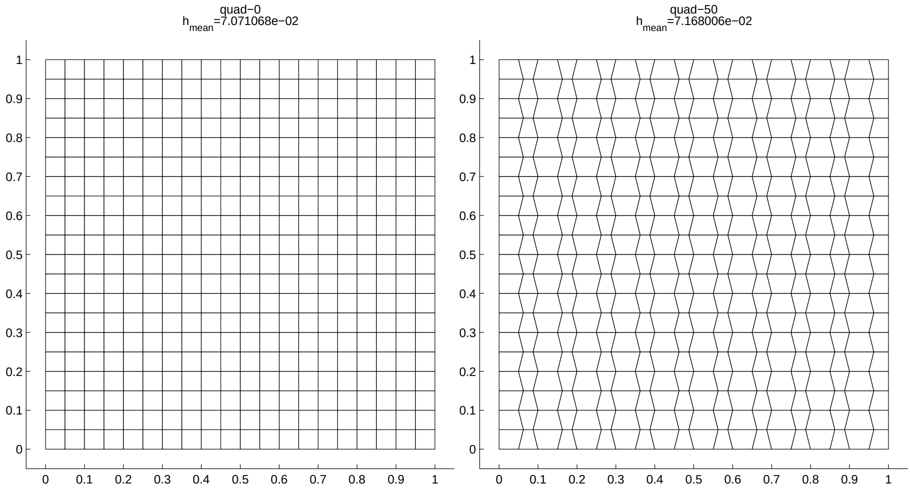
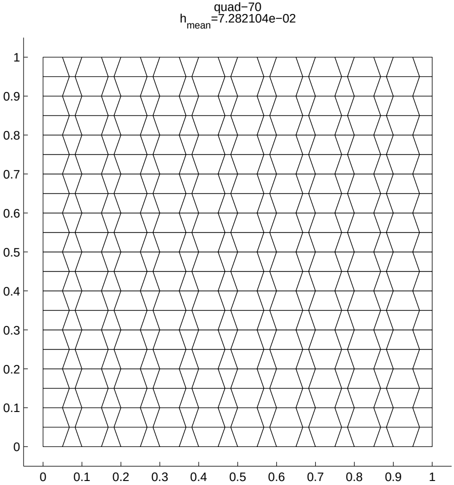


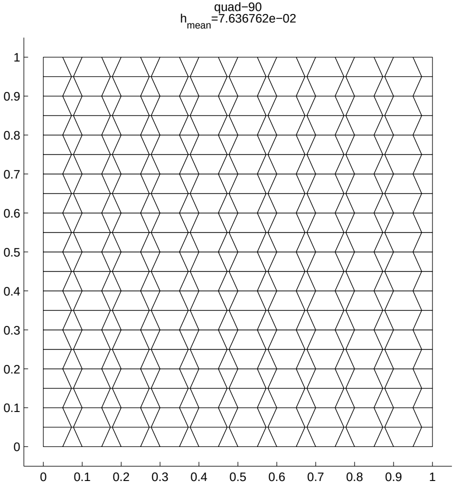


Robustness


quad−0
h_mean=7.071068e−02
quad−50
h_mean=7.168006e−02
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Robustness


quad−70
h$_{mean}$=7.282104e−02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


quad−90
h$_{mean}$=7.636762e−02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

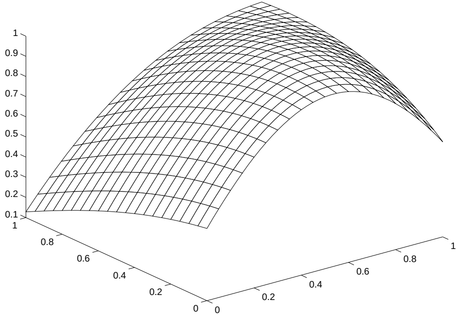
Robustness

We solved the equation

$$-\Delta u + u = f$$

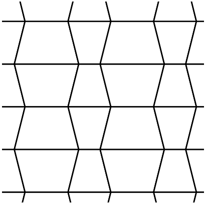
on the unit square with $k = 2$ where f and the Dirichlet boundary condition are taken in such a way that the exact solution is

$$u(x, y) = \sin(2x + 0.5)\cos(y + 0.3) + \log(1 + xy)$$

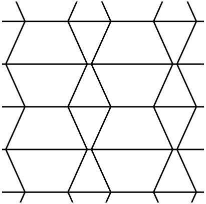


Robustness

degeneracy	h	error L^2
0%	7.0711e-02	1.0437e-07
50%	7.1680e-02	1.6338e-06
60%	7.2003e-02	2.0469e-06
70%	7.2821e-02	2.5287e-06
80%	7.4576e-02	3.1152e-06
90%	7.6368e-02	3.8700e-06
99%	7.8013e-02	4.7784e-06
99.99%	7.8198e-02	4.8968e-06



50%



90%

Basic References

Basic Principles of Virtual Element Methods, *L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini, A. R.*, Math. Models and Methods in Applied Sciences Vol. 23, No. 01 (2013), pp. 199-214;

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Thanks for your attention!

alessandro.russo@unimib.it