Plane Wave Discontinuous Galerkin Methods

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London Mathematical Society — EPSRC Durham Symposium
Connections and challenges in modern approaches to numerical partial differential equations
July 7-16, 2014

Overwiew

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① Motivation: Classical FEM – Approximation Challenges

② Operator Adapted Trial Spaces

③ Trefftz-Discontinuous Galerkin Discretization

④ *h*-Version of PWDG: Convergence

⑤ *p*-Version of PWDG: Convergence

⑥ *hp*-PWDG: A Priori Error Estimates

⑦ Miscellaneous Issues and Open Problems

Focus on theory!

Model Problem: Acoustic Scattering

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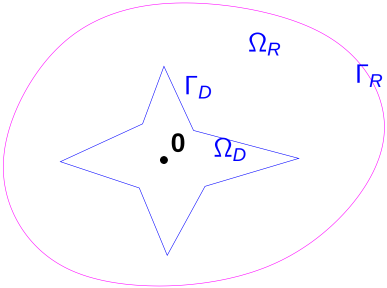

Geometric setting:

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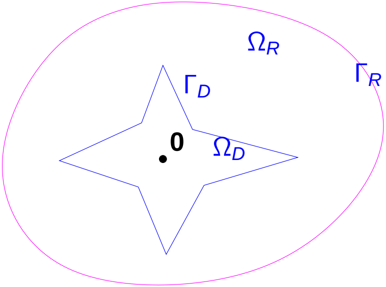

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$$\Omega := \Omega_D \setminus \Omega_R$$

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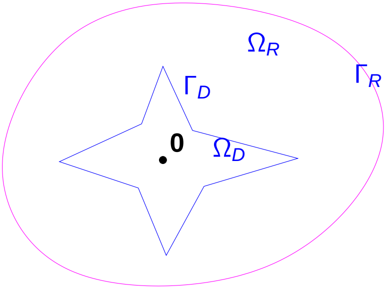

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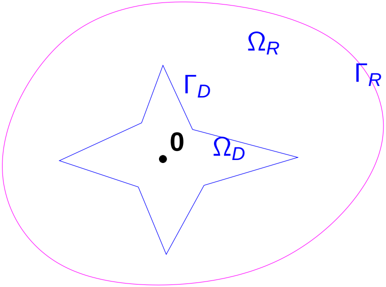

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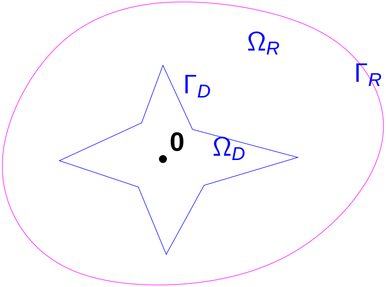

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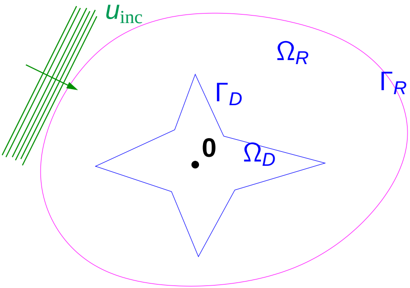

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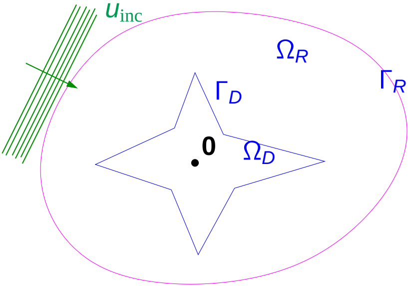

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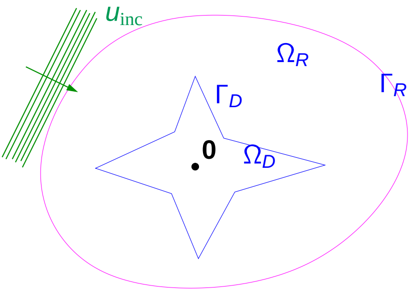

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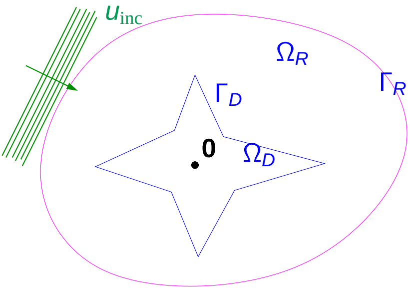

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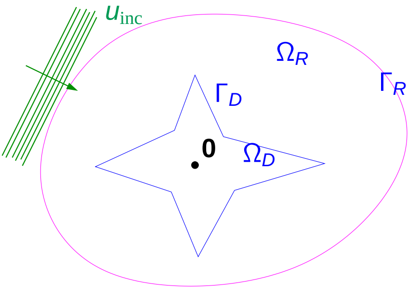

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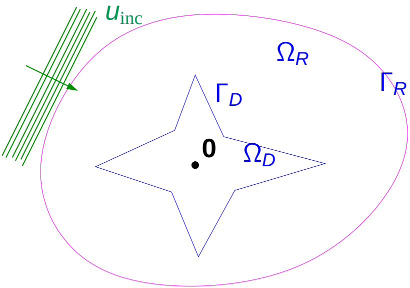

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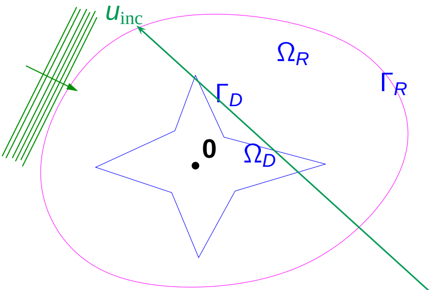

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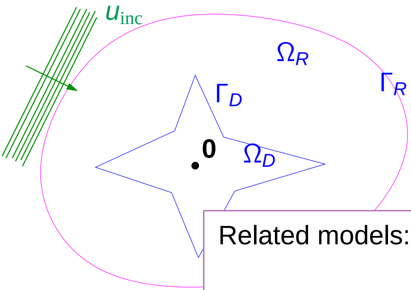

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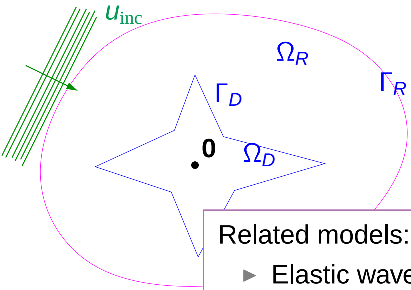

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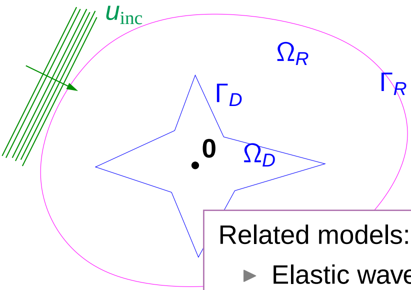

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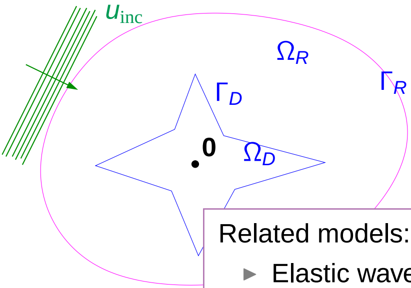

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→ Lecture by I. Perugia

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Oscillatory wave solutions

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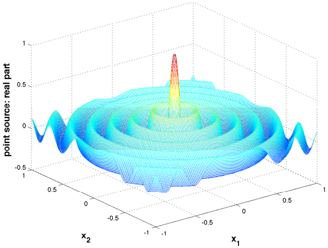

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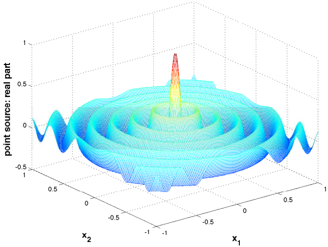

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wavelength $\lambda := \dfrac{2\pi}{k} \rightarrow 0$ for $k \rightarrow \infty$.

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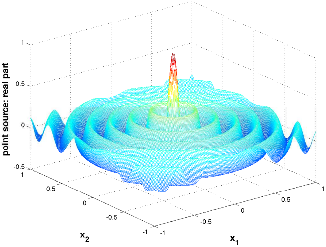

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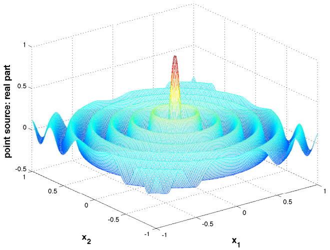

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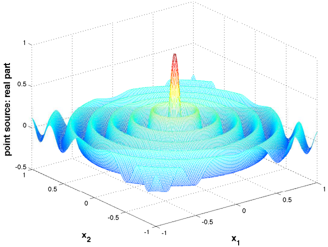

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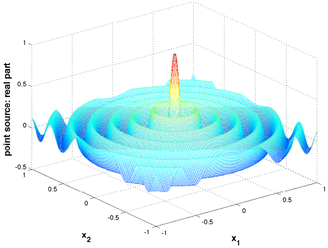

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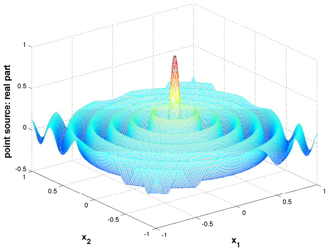

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Numerical dispersion (pollution effect)

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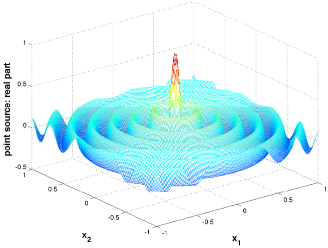

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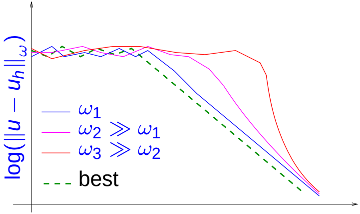

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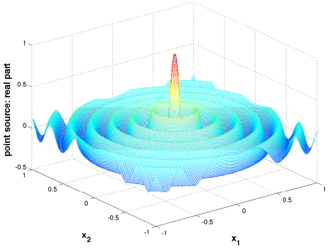

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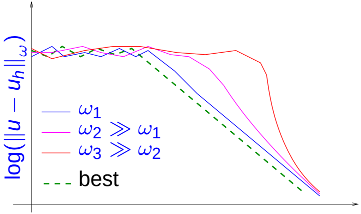
➥ approximation isn't enough !



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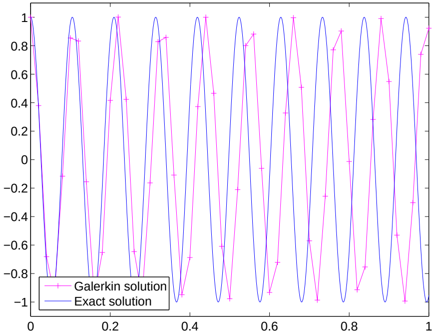

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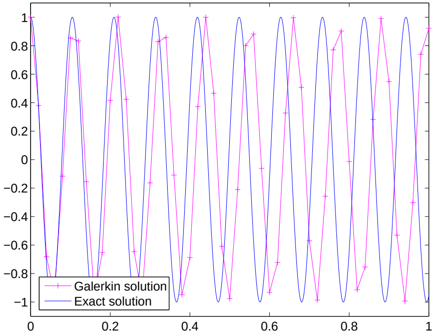

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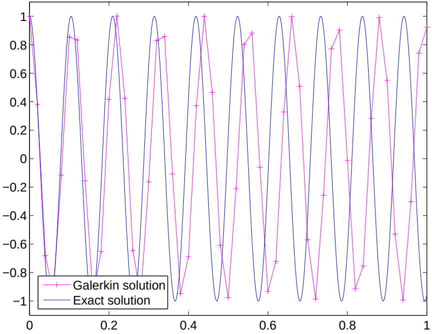

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- ➤ Phase error of Galerkin solution

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- ➤ Principal cause of *h*-FEM discretization error in wave propagation (at medium & high frequencies)

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Best approximation error estimates (p.w. linear FE):

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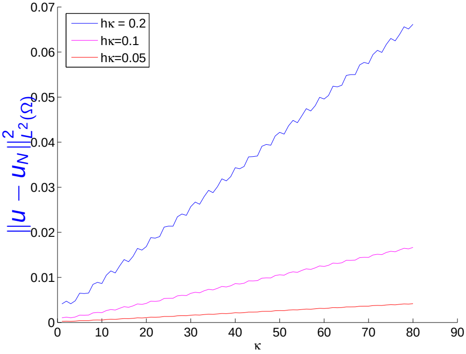

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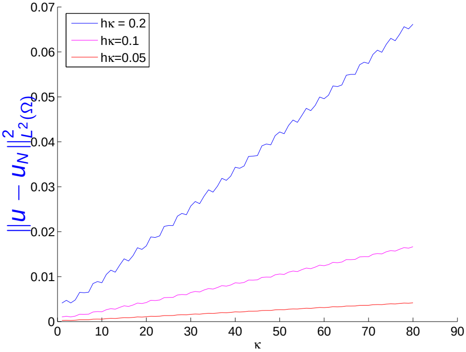

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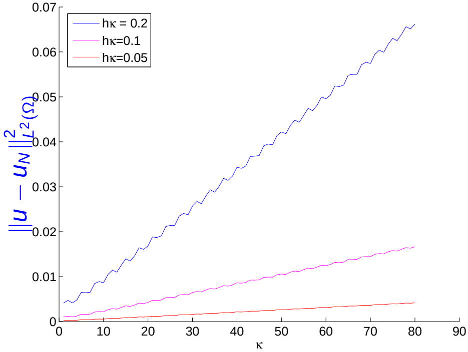

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Approximation error Phase error

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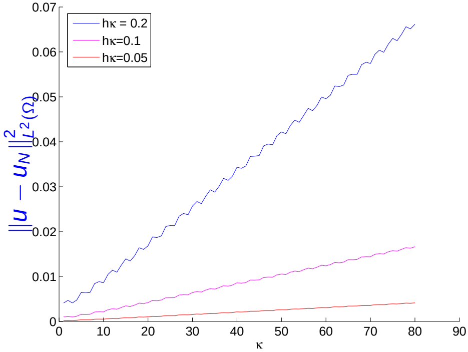

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$\nearrow$ Approximation error $\uparrow$ Phase error

Fixed "no. of points per wavelength" is *not* enough **!**

Wave Propagation: The Pollution Effect (II)

Helmholtz BVP **+** polynomial C^0-FE Galerkin discretization

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typical error behavior for h-refinement:

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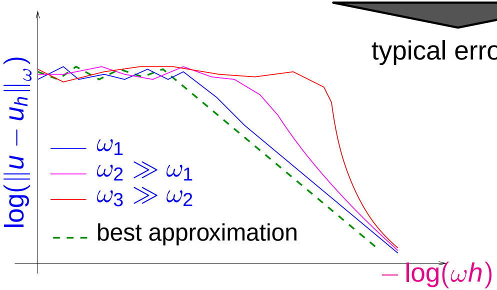

Wave Propagation: The Pollution Effect (II)

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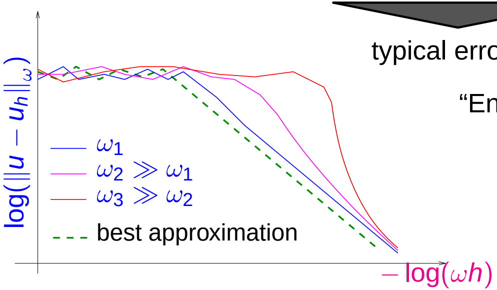

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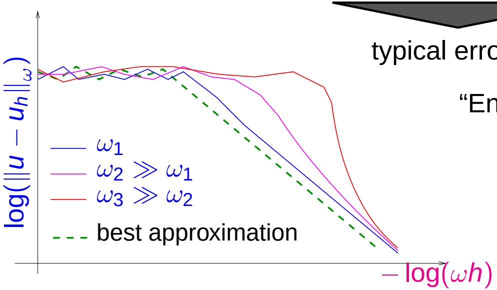

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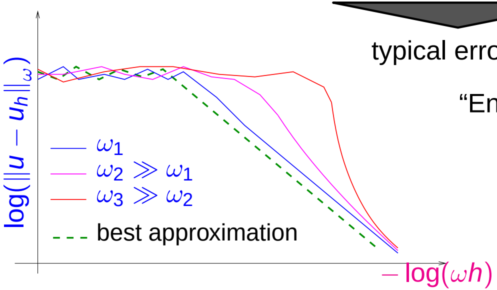

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I. Babuška and S. Sauter, *Is the pollution effect of the FEM avoidable for the Helmholtz equation?*, SIAM Review, 42 (2000), pp. 451–484.

Y. Du and H.-J. Wu, *Preasymptotic error analysis of higher order FEM and CIP-FEM for Helmholtz equation with high wave number*, Tech. Rep. arXiv:1401.4311 [math.NA], 2014

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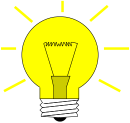

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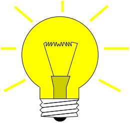

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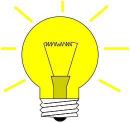

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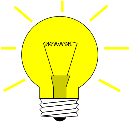

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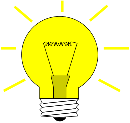

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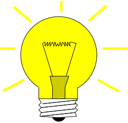

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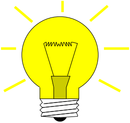

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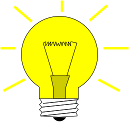

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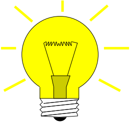

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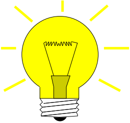

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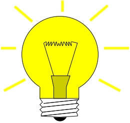

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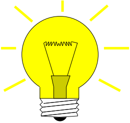

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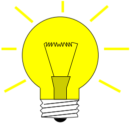

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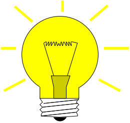

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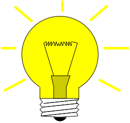

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Convergence theory: Estimates for $\|u - u_{\mathcal{X}}\|_{L^2(\partial\Omega)}$

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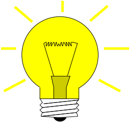

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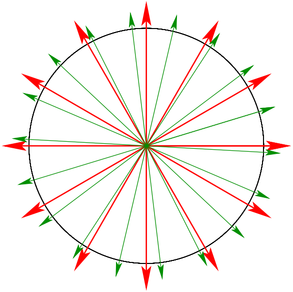

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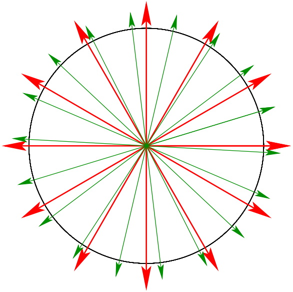

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- T. Huttunen, P. Monk and J. Kaipio (2002), ‘Computational aspects of the ultra-weak variational formulation’, *J. Comp. Phys.* **182**(1), 27–46
- T. Huttunen, J. Kaipio, and P. Monk, *The perfectly matched layer for the ultra weak variational formulation of the 3D Helmholtz equation*, Int. J. Numer. Meth. Eng., 61 (2004), pp. 1072–1092.
- T. Huttunen, J. Kaipio, and P. Monk, *An ultra-weak method for acoustic fluid-solid interaction*, Journal of Computational and Applied Mathematics, 213 (2008), pp. 166–185.
- T. Huttunen, M. Malinen, J. Kaipio, P. White, and K. Hynynen, *A full-wave Helmholtz model for continuous-wave ultrasound transmission*, IEEE Trans. Ultrasonics, Ferroelectrics and Frequency Control, 52 (2005), pp. 397–409.
- T. Huttunen, M. Malinen, and P. Monk, *Solving Maxwell’s equations using the ultra weak variational formulation*, J. Comp. Phys., 223 (2007), pp. 731–758.
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$$\begin{aligned} i\omega\,\boldsymbol{\sigma} &= \nabla u && \text{in } \Omega\,,\\ i\omega\,u - \nabla\cdot\boldsymbol{\sigma} &= \frac{1}{i\omega} f && \text{in } \Omega\,,\\ i\omega\boldsymbol{\sigma}\cdot\boldsymbol{n} - i\omega\,u &= g && \text{on } \partial\Omega\,. \end{aligned}$$

(PW)DG: Derivation (I)

C. GITTELSON, R. HIPTMAIR, AND I. PERUGIA, *Plane wave discontinuous Galerkin methods: Analysis of the h-version*, Math. Model. Numer. Anal., 43 (2009), pp. 297–331.

Helmholtz BVP:
$$\begin{aligned} -\Delta u - \omega^2 u &= f && \text{in } \Omega\,, \\ \nabla u \cdot \boldsymbol{n} - i\omega u &= g && \text{on } \partial\Omega\,. \end{aligned}$$

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+ partitioning $\mathcal{T}_h = \{T\}$ of Ω into cells (→ generalized FE mesh)

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first order system:

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$$\int_T i\omega\,\boldsymbol{\sigma}\cdot\overline{\boldsymbol{\tau}}\,\mathrm{d}V + \int_T u\,\overline{\nabla\cdot\boldsymbol{\tau}}\,\mathrm{d}V - \int_{\partial T} u\,\overline{\boldsymbol{\tau}\cdot\boldsymbol{n}}\,\mathrm{d}S = 0\,,$$
$$\int_T i\omega\,u\,\overline{v}\,\mathrm{d}V + \int_T \boldsymbol{\sigma}\cdot\overline{\nabla v}\,\mathrm{d}V - \int_{\partial T} \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\overline{v}\,\mathrm{d}S = \frac{1}{i\omega}\int_T f\,\overline{v}\,\mathrm{d}V\,,$$

for all $\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}; T)$, $v \in H^1(T)$.

(PW)DG: Derivation (II)

$$\begin{aligned}
&\int_T i\omega\,\boldsymbol{\sigma}\cdot\overline{\boldsymbol{\tau}}\,\mathrm{d}V + \int_T u\,\overline{\nabla\cdot\boldsymbol{\tau}}\,\mathrm{d}V - \int_{\partial T} u\,\overline{\boldsymbol{\tau}\cdot\boldsymbol{n}}\,\mathrm{d}S = 0\;,\\
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Replace:	$\mathbf{H}(\mathrm{div};T)$	$\rightarrow$	$\boldsymbol{\Sigma}_h(T)$	$\widehat{=}$	trial/test space for cell fluxes
	$H^1(T)$	$\rightarrow$	$V_h(T)$	$\widehat{=}$	local trial/test space for u

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$$\int\limits_T i\omega\,\boldsymbol{\sigma}_h\cdot\overline{\boldsymbol{\tau}}_h\,\mathrm{d}V + \int\limits_T u_h\,\overline{\nabla\cdot\boldsymbol{\tau}_h}\,\mathrm{d}V - \int\limits_{\partial T} \widehat{u}_h\,\overline{\boldsymbol{\tau}_h\cdot\boldsymbol{n}}\,\mathrm{d}S = 0\,,$$
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\end{aligned}$$

for all $\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h(T)$, $v_h\in V_h(T)$.

+ numerical fluxes $\widehat{u}_h$, $\widehat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}$ on cell interfaces

Trefftz DG

$$\int_T i\omega\,\boldsymbol{\sigma}_h\cdot\overline{\boldsymbol{\tau}_h}\,\mathrm{d}V+\int_T u_h\,\overline{\nabla\cdot\boldsymbol{\tau}_h}\,\mathrm{d}V-\int_{\partial T}\widehat{u}_h\,\overline{\boldsymbol{\tau}_h\cdot\boldsymbol{n}}\,\mathrm{d}S=0\,,$$

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Use local Trefftz-type trial spaces: $\boxed{(-\Delta-\omega^2)V_h(T)=0\quad\forall T\in\mathcal{T}_h}$.

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Concrete choice: $V_h(T) = PW_p$

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← assume $\boxed{\nabla_h V_h(T) \subset \boldsymbol{\Sigma}_h(T)}$

Trefftz DG

$$
\begin{aligned}
&\int\limits_T i\omega\,\boldsymbol{\sigma}_h\cdot\overline{\nabla v_h}\,\mathrm{d}V + \int\limits_T u_h\,\overline{\nabla\cdot\nabla v_h}\,\mathrm{d}V - \int\limits_{\partial T}\widehat{u}_h\,\overline{\nabla v_h\cdot\boldsymbol{n}}\,\mathrm{d}S = 0\;,\\
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▼ ← assume $\boxed{\nabla_h V_h(T)\subset\boldsymbol{\Sigma}_h(T)}$

$$
\boxed{\int_{\partial T}\widehat{u}_h\,\overline{\nabla v_h\cdot\boldsymbol{n}}\,\mathrm{d}S - \int_{\partial T} i\omega\widehat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}\,\overline{v}_h\,\mathrm{d}S = \int_T f\,\overline{v}_h\,\mathrm{d}\boldsymbol{x} \quad \forall v_h\in V_h(T)}\;.
$$

Trefftz DG

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A skeleton variational formulation!

$T) \subset \boldsymbol{\Sigma}_h(T)$

$$\boxed{\int_{\partial T}\widehat{u}_h\,\overline{\nabla v_h\cdot\boldsymbol{n}}\,\mathrm{d}S - \int_{\partial T} i\omega\widehat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}\,\overline{v}_h\,\mathrm{d}S = \int_T f\,\overline{v}_h\,\mathrm{d}\boldsymbol{x} \quad \forall v_h \in V_h(T)}\ .$$

DG: Numerical fluxes

DG: Numerical fluxes

D. Arnold, F. Brezzi, B. Cockburn, and L. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2002), pp. 1749–1779.

Conservative & consistent fluxes (commonly used in *polynomial* DG):

- Interior penalty (IP) DG:

penalty parameter

$$\widehat{u}_h = \{u_h\} \quad , \quad \widehat{\boldsymbol{\sigma}}_h = \{\nabla u_h\} - \overset{\downarrow}{\alpha}\,[u_h] \;.$$

- Mixed DG:

$$\widehat{u}_h = \{u_h\} + \boldsymbol{\gamma}\cdot[u_h] - \beta\,[\boldsymbol{\sigma}_h] \quad , \quad \widehat{\boldsymbol{\sigma}}_h = \{\boldsymbol{\sigma}_h\} - \alpha\,[u_h] - \boldsymbol{\gamma}\,[\boldsymbol{\sigma}_h] \;.$$

- Local DG (LDG):

$$\widehat{u}_h = \{u_h\} - \beta\,[u_h] \quad , \quad \widehat{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}_h\} - \beta\,[u_h] - \alpha\,[\boldsymbol{\sigma}_h] \;.$$

DG notations: $[\cdot] \mathrel{\widehat{=}}$ jump·normal, $\{\cdot\} \mathrel{\widehat{=}}$ average

DG: Numerical fluxes

Our favorite choice: primal DG numerical fluxes (on faces):

$$i\omega\widehat{\boldsymbol{\sigma}}_h = \begin{cases} \{\nabla_h u_h\} - \alpha\, i\omega\, [u_h] & \text{on } \mathcal{F}_h^I, \\ \nabla_h u_h - (1-\delta)\left(\nabla_h u_h + i\omega u_h \mathbf{n} - g_R \mathbf{n}\right) & \text{on } \mathcal{F}_h^R, \\ [\nabla_h u_h - \alpha\, i\omega\, u_h \mathbf{n} & \text{on } \mathcal{F}_h^D] \,. \end{cases}$$

$$\widehat{u}_h = \begin{cases} \{u_h\} - \beta\,(i\omega)^{-1}\,[\nabla_h u_h] & \text{on } \mathcal{F}_h^I, \\ u_h - \delta\left((i\omega)^{-1}\nabla_h u_h \cdot \mathbf{n} + u_h - (i\omega)^{-1} g_R\right) & \text{on } \mathcal{F}_h^R, \\ [0 & \text{on } \mathcal{F}_h^D] \,. \end{cases}$$

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DG: Numerical fluxes

Alternative choice: mixed DG numerical fluxes (on interior faces)

$$\widehat{\boldsymbol{\sigma}}_h = \{\boldsymbol{\sigma}_h\} - \alpha\,[u_h] - \boldsymbol{\gamma}\,[\boldsymbol{\sigma}_h]\ ,$$
$$\widehat{u}_h = \{u_h\} + \boldsymbol{\gamma}\,[u_h] - \beta\,[\boldsymbol{\sigma}_h]\ .$$

R. HIPTMAIR AND I. PERUGIA, *Mixed plane wave discontinuous Galerkin methods*, Springer LNCSE 70, 2009, pp. 51–62.

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$$\widehat{u}_h = \begin{cases} \{u_h\} - \beta\,(i\omega)^{-1}\,[\nabla_h u_h] & \text{on } \mathcal{F}_h^I, \\ u_h - \delta\left((i\omega)^{-1}\nabla_h u_h \cdot \mathbf{n} + u_h - (i\omega)^{-1} g_R\right) & \text{on } \mathcal{F}_h^R, \\ [0 & \text{on } \mathcal{F}_h^D]\ . \end{cases}$$

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Flux parameters: $\alpha, \beta > 0$, $0 < \delta < 1$

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Remark. $\alpha = 1/2\,,\quad \beta = 1/2\,,\quad \delta = 1/2$ ➢ UWVF **!**

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Remark. $\alpha = 1/2\,,\quad \beta = 1/2\,,\quad \delta = 1/2$ ➢ UWVF **!**

$$\alpha = \mathrm{a}\frac{p}{\omega h \log p},\quad \beta = \mathrm{b}\frac{\omega h \log p}{p},\quad \delta = \mathrm{d}\frac{\omega h \log p}{p}$$ “classical”

DG: Variational Problem (I)

Local variational problem ($T \mathrel{\widehat{=}}$ cell of the mesh): $u_h \in PW_p$

$$\int_T \nabla u_h \cdot \nabla \overline{v}_h - \omega^2 u_h \overline{v}_h \, \mathrm{d}V - \int_{\partial T} (u_h - \widehat{u}_h) \, \overline{\nabla v_h \cdot \boldsymbol{n}} \, \mathrm{d}S \\ - \int_{\partial T} i\omega \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} \, \overline{v}_h \, \mathrm{d}S = \int_T f \, \overline{v}_h \, \mathrm{d}V \quad \forall v_h \in PW_p \; .$$

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Local variational problem ($T \;\widehat{=}$ cell of the mesh): $u_h \in PW_p$

$$\int_T \nabla u_h \cdot \nabla \overline{v}_h - \omega^2 u_h \overline{v}_h \,\mathrm{d}V - \int_{\partial T} (u_h - \widehat{u}_h)\, \overline{\nabla v_h \cdot \boldsymbol{n}} \,\mathrm{d}S$$
$$- \int_{\partial T} i\omega \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\, \overline{v}_h \,\mathrm{d}S = \int_T f\, \overline{v}_h \,\mathrm{d}V \qquad \forall v_h \in PW_p \;.$$

+ generalized “UW fluxes”:
(boundary fluxes ignored)

$$\widehat{\boldsymbol{\sigma}}_h = \frac{1}{i\omega} \{\nabla_h u_h\} - \alpha\, [u_h] \;, \qquad \alpha, \beta > 0 \;.$$
$$\widehat{u}_h = \{u_h\} - \frac{\beta}{i\omega}\, [\nabla_h u_h] \;.$$

DG: Variational Problem (I)

Local variational problem ($T \;\widehat{=}$ cell of the mesh): $u_h \in PW_p$

$$\begin{aligned}\int_T \nabla u_h \cdot \nabla \overline{v}_h - \omega^2 u_h \overline{v}_h \,\mathrm{d}V - \int_{\partial T} (u_h - \widehat{u}_h)\,\overline{\nabla v_h \cdot \boldsymbol{n}}\,\mathrm{d}S \\ - \int_{\partial T} i\omega \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\,\overline{v}_h\,\mathrm{d}S = \int_T f\,\overline{v}_h\,\mathrm{d}V \quad \forall v_h \in PW_p \;.\end{aligned}$$

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$$\begin{aligned}\widehat{\boldsymbol{\sigma}}_h &= \frac{1}{i\omega}\{\nabla_h u_h\} - \alpha\,[u_h] \;, \\ \widehat{u}_h &= \{u_h\} - \frac{\beta}{i\omega}\,[\nabla_h u_h] \;.\end{aligned} \qquad \alpha, \beta > 0 \;.$$

+ “DG magic formula”

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial \boldsymbol{n}_T}\, v\,\mathrm{d}S = \int_{\mathcal{F}_h} \{\nabla u\}\,[v]\;\mathrm{d}S + \int_{\mathcal{F}_h^I} [\nabla u]\,\{v\}\;\mathrm{d}S \;.$$

✎ notation: $\mathcal{F}_h/\mathcal{F}_h^I \;\widehat{=}$ edges/interior edges of $\mathcal{T}_h$

PWDG: Variational Problem (II)

- Trefftz DG: global variational problem

$$u_h \in V_h: \quad a_h(u_h, v_h) - \omega^2(u_h, v_h) = (f, v_h) + \{\text{boundary terms}\} \quad \forall v_h \in V_h \ .$$

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- existence/uniqueness of solutions of discretized problem

What Next ?

1. Motivation: Classical FEM – Approximation Challenges
2. Operator Adapted Trial Spaces
3. Trefftz-Discontinuous Galerkin Discretization
4. *h*-Version of PWDG: Convergence
5. *p*-Version of PWDG: Convergence
6. *hp*-PWDG: A Priori Error Estimates
7. Miscellaneous Issues and Open Problems

PWDG *h*-Version: 2D Numerical Experiments (I)

h-version: increase resolution by (uniform) mesh refinement

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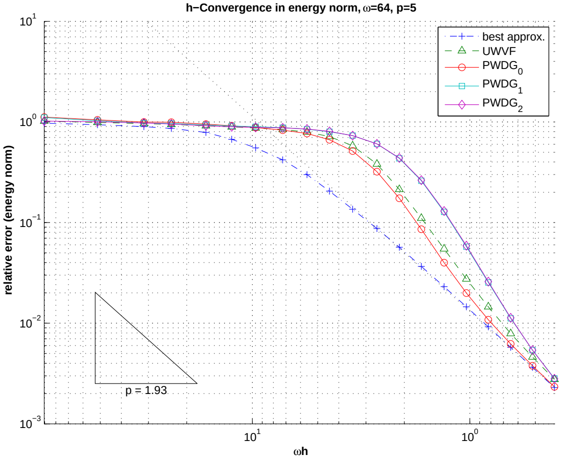

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- cylindrical wave solution
- $\omega = 64$
- quasi-uniform meshes
- $p = 5$

method	α	β	γ
UWVF	$\frac{1}{2}$	$\frac{1}{2}$	0
PWDG0	$\frac{2}{\omega h}$	0	0
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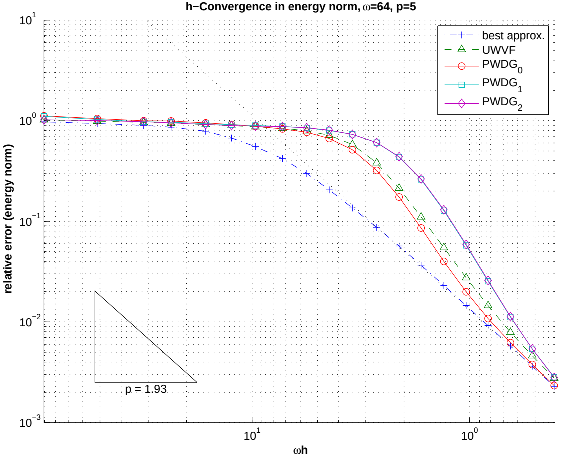

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Observation: algebraic convergence for $h \to 0$, rate = $\begin{cases} (p-1)/2 & \text{, if } f = 0\,, \\ 1 & \text{, if } f \neq 0\,. \end{cases}$

h-PWDG a Priori Estimates: Tools

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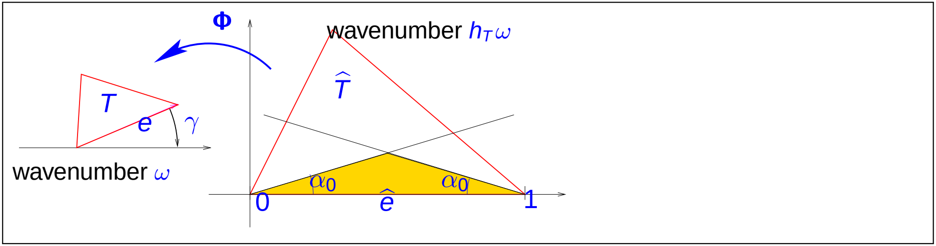

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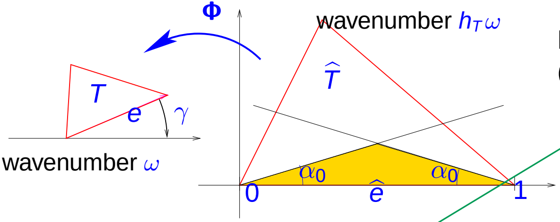
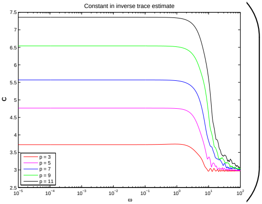


Needed: ω-uniform (inverse/trace) estimates

ω-uniform inverse trace estimate:

$$\|w\|_{0,\partial T} \leq Ch_T^{-\frac{1}{2}} \|w\|_{0,T} \quad \forall w \in PW_p, \forall \omega \ ,$$

$C = C(p) > 0$ depending on shape-regularity.



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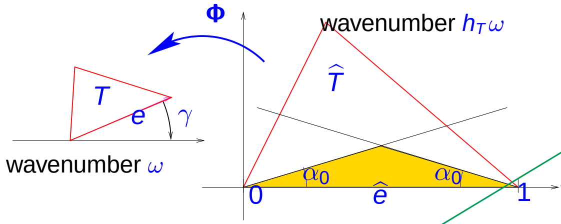

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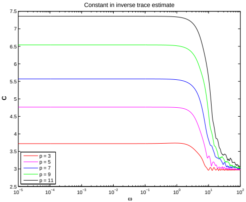
Constant in inverse trace estimate

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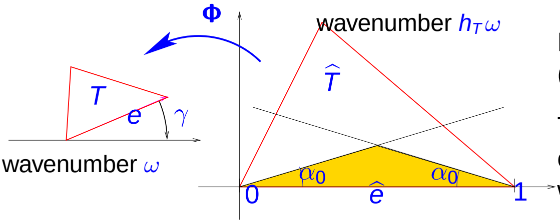

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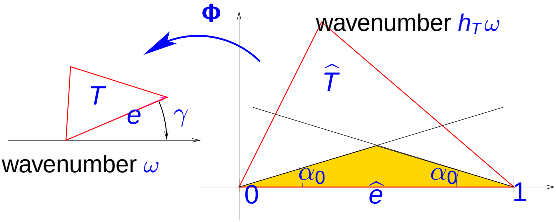

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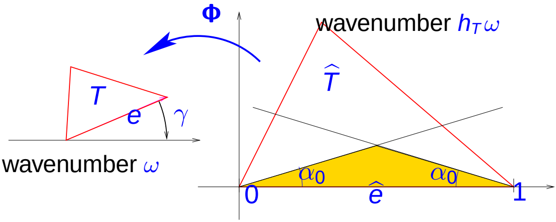

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C. GITTELSON, R. HIPTMAIR, AND I. PERUGIA, *Plane wave discontinuous Galerkin methods: Analysis of the h-version*, Math. Model. Numer. Anal., 43 (2009), pp. 297–331.

h-PWDG a Priori Estimates: Duality Technique

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attack with duality estimates

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adjoint problem ($\Gamma_D = \emptyset$) : $w_h \in V_h$:

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$\mathsf{Q}_\omega \mathrel{\widehat{=}} L^2(\Omega)$-orthogonal projection onto V_h (piecewise definition)

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A. Schatz, *An observation concerning Ritz-Galerkin methods with indefinite bilinear forms*, Math. Comp., 28 (1974), pp. 959–962.

Lemma. With $C > 0$ depending only on shape regularity and p

$$\|u - u_h\|_{DG} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{DG^+} + C\omega \sup_{0 \neq w_h \in V_h} \frac{|(u - u_h, w_h)|}{\|w_h\|_{0,\Omega}} .$$

attack with duality estimates

adjoint problem ($\Gamma_D = \emptyset$) : $w_h \in V_h$:

$$\begin{aligned} -\Delta\varphi - \omega^2\varphi &= w_h && \text{in } \Omega, \\ -\nabla\varphi \cdot \boldsymbol{n} + i\omega\,\varphi &= 0 && \text{on } \partial\Omega \end{aligned}$$

[+ consistency **&** adjoint consistency]

$$\Downarrow$$

$$|(u - u_h, w_h)| \leq C\ \|u - u_h\|_{DG}\, \|\varphi - \mathsf{Q}_\omega\varphi\|_{DG^+} + |(f - \mathsf{Q}_\omega f, \varphi - \mathsf{Q}_\omega\varphi)|\ .$$

$\mathsf{Q}_\omega \mathrel{\widehat{=}} L^2(\Omega)$-orthogonal projection onto V_h (piecewise definition)

+ ω-explicit elliptic lifting estimates (M. Melenk 1995)

h-PWDG a Priori Estimates: Results

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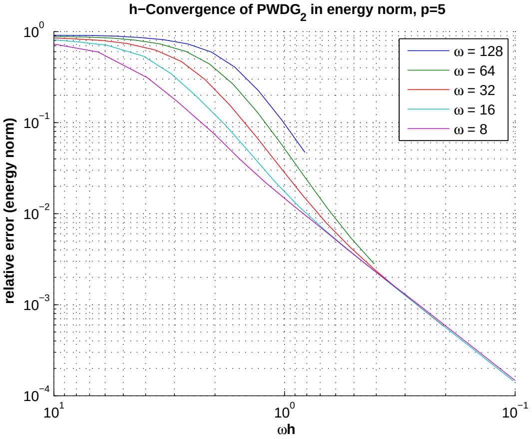
Pollution effect **?**

Remark. $f \equiv 0 \Rightarrow$ asymptotic convergence $O(h^q)$

h-PWDG: Numerical Experiments (II)



- $\Omega =]0,1[^2$
- cylindrical wave solution
- quasi-uniform meshes
- PWDG variant PWDG2
- $p = 5$

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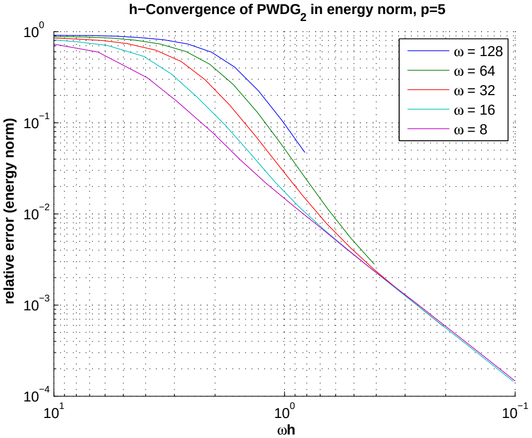

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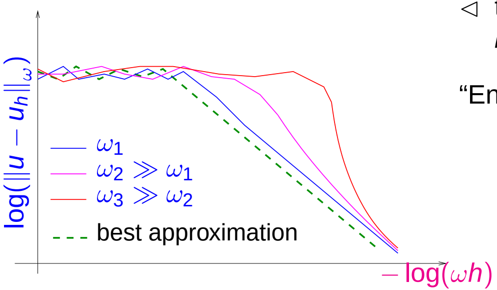
Oberservation ($f \equiv 0$): $\|u - u_h\| \sim (\omega h)^{\frac{p-1}{2}} + \omega(\omega h)^{p-1}$ for $h \to 0$

$\nearrow$

pollution error !

Recall: The Pollution Effect

Local (low order FD, FEM, FV, DG) discretization of Helmholtz BVP:



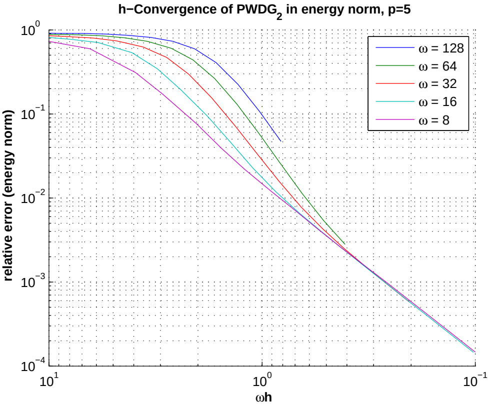
◁ typical error behavior for *h*-refinement:

"Energy" norm:

$$\|u\|_\omega^2 = |u|_1^2 + \omega^2 \|u\|_0^2 \ .$$

delayed onset of
asymptotic convergence **!**

h-PWDG: Numerical Experiments (II)



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Observation ($f \equiv 0$):

$$\|u - u_h\| \sim (\omega h)^{\frac{p-1}{2}} + \omega(\omega h)^{p-1}$$

$\nearrow$ pollution error !

(Numerical) dispersion analysis in:

C. GITTELSON AND R. HIPTMAIR, *Dispersion analysis of plane wave discontinuous Galerkin methods*, Tech. Rep. 2012-42, SAM, ETHZ. To appear in IJNME.

h-PWDG: Numerical Dispersion (I)

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Discrete Bloch-wave analysis

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Discrete Bloch-wave analysis
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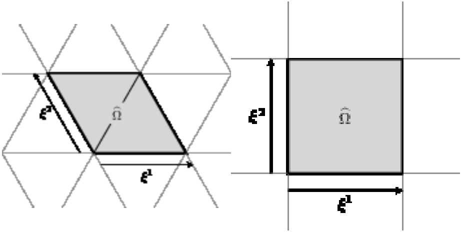
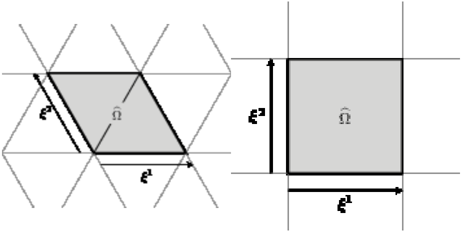


h-PWDG: Numerical Dispersion (I)

Discrete Bloch-wave analysis
on infinite lattice meshes

→ $\omega_h(\theta)$ $\widehat{=}$ numerical wave number
(for propagation in direction θ)



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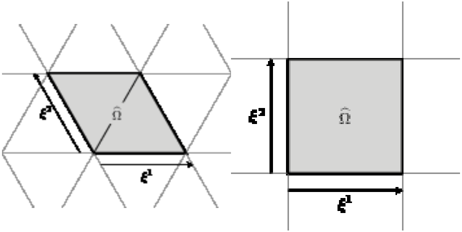

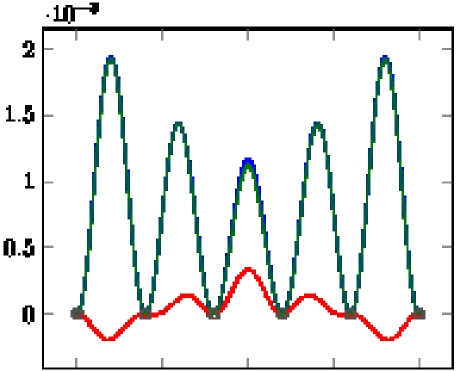

$\triangleleft$ $\omega - \omega_h$ (imaginary and real part, triangular lattice mesh)

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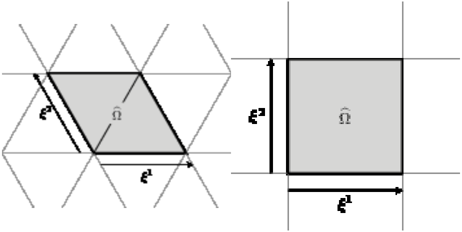
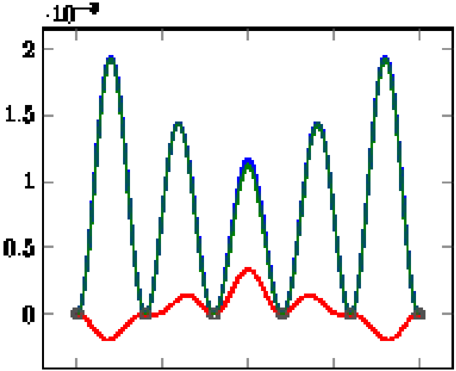




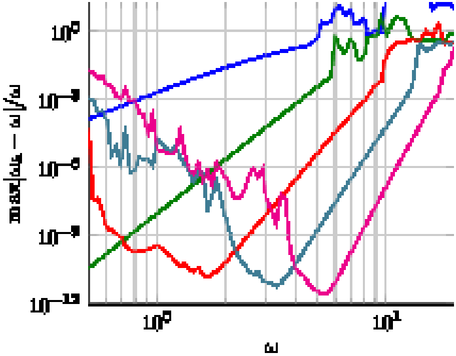
◁ $\omega - \omega_h$ (imaginary and real part, triangular lattice mesh)

Observed:

Numerical dispersion
+
numerical dissipation

h-PWDG: Numerical Dispersion (II)

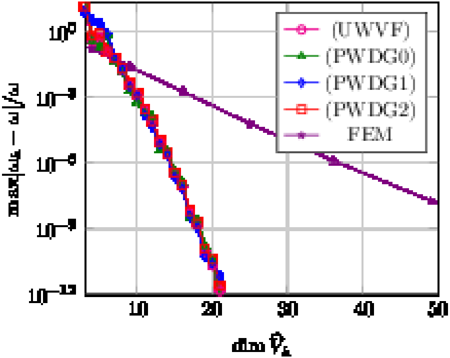
triangular mesh ($h = 1$):



$p \in \{3, 5, 7, 9, 11\}$:

$$\frac{|\omega_h - \omega|}{|\omega|} \sim \omega^{p-1}$$

square mesh ($h = 1$):

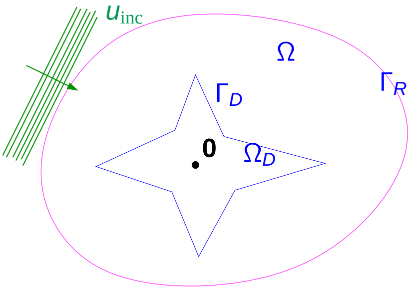


$$\frac{|\omega_h - \omega|}{|\omega|} \sim q^p,\ 0 < q < 1$$

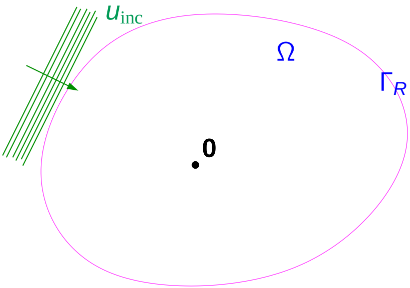
What Next ?

1. Motivation: Classical FEM – Approximation Challenges
2. Operator Adapted Trial Spaces
3. Trefftz-Discontinuous Galerkin Discretization
4. *h*-Version of PWDG: Convergence
5. *p*-Version of PWDG: Convergence
6. *hp*-PWDG: A Priori Error Estimates
7. Miscellaneous Issues and Open Problems

Model Problems



full model problem



simplified model problem

Frequency domain models for acoustic wave propagation

Helmholtz equation:
$$-\Delta u - \omega^2 u = 0 \quad \text{in } \Omega ,$$
$$u = 0 \quad \text{on } \Gamma_D , \quad \nabla u \cdot \boldsymbol{n} - i\omega u = g \quad \text{on } \Gamma_R ,$$

with wave number $\omega > 0$.

PWDG p-Version: Numerical Experiments

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Square, $\omega = 10$, $h = 1/\sqrt{2}$, L^2-norm of errors:

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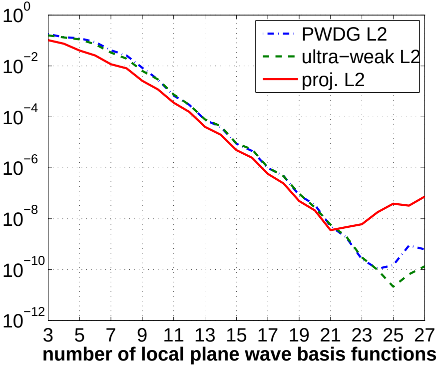

Smooth solution in $C^\infty(\mathbb{R}^2)$

$$u = J_1(\omega|x|)\cos\theta$$

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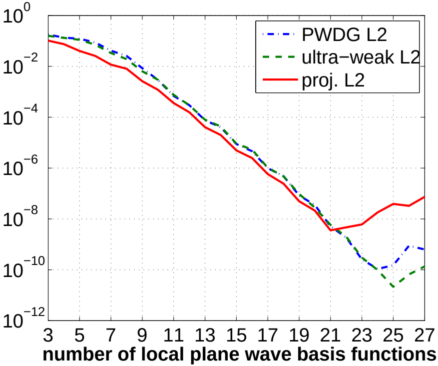

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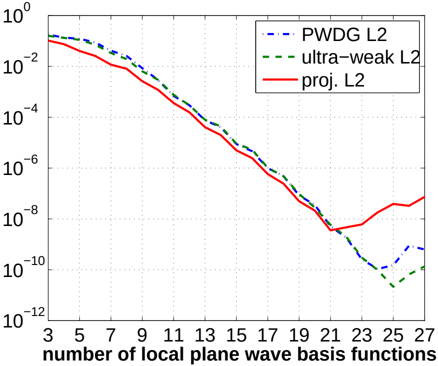

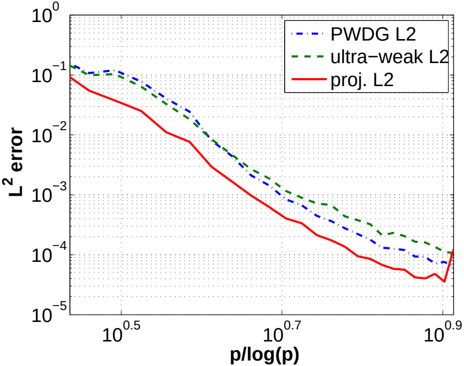

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$u = J_{\frac{3}{2}}(\omega|x|)\cos(\frac{3}{2}\theta)$

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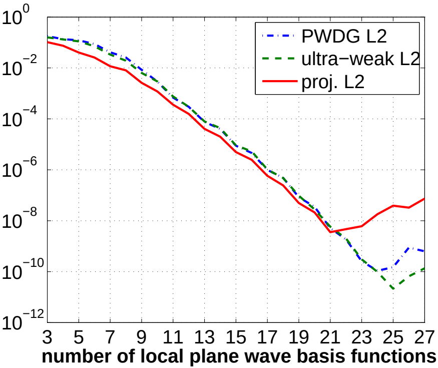

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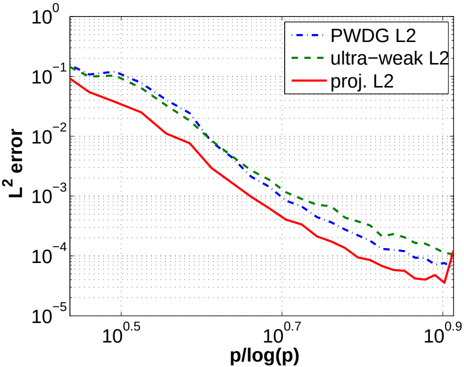

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algebraic convergence.

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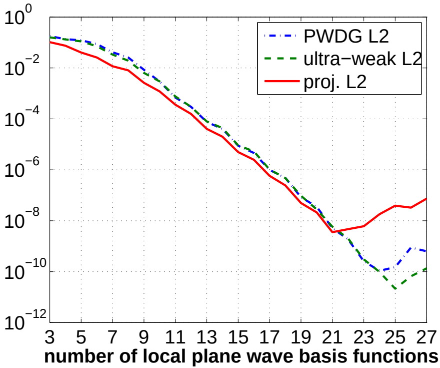

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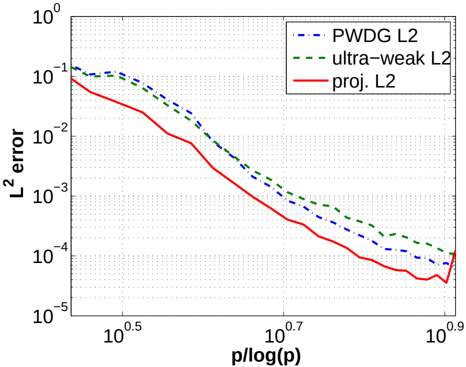

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Numerical instability for high p!

Quasi-Optimality

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Recall PWDG linear variational problem:

$$u_h \in V_h: \quad a_h(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h \ .$$

with $\quad (\Gamma_D = \emptyset)$

$$\begin{aligned} a_h(u,v) = & \int_{\mathcal{F}_h^{\mathcal{I}}} [\nabla u] \{\overline{v}\} - \int_{\mathcal{F}_h^{\mathcal{I}}} [u] \cdot \{\overline{\nabla_h v}\} \, \mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{I}}} \{\nabla_h u\} \cdot [\overline{v}] \, \mathrm{d}S \\ & - \int_{\mathcal{F}_h^{\mathcal{B}}} \delta \, u \, \overline{\nabla_h v} \, \mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta) \, \nabla_h u \cdot \boldsymbol{n} \, \overline{v} \, \mathrm{d}S \\ & + \frac{i}{\omega} \int_{\mathcal{F}_h^{\mathcal{I}}} \beta \, [\nabla_h u] \left[\overline{\nabla_h v}\right] \, \mathrm{d}S + \frac{i}{\omega} \int_{\mathcal{F}_h^{\mathcal{B}}} \delta \, \nabla_h u \cdot \boldsymbol{n} \, \overline{\nabla_h v \cdot \boldsymbol{n}} \, \mathrm{d}S \\ & + i\omega \int_{\mathcal{F}_h^{\mathcal{I}}} \alpha \, [u] \cdot [\overline{v}] \, \mathrm{d}S + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta) \, u \, \overline{v} \, \mathrm{d}S \ , \quad \alpha, \beta, \delta > 0 \ . \end{aligned}$$

[$\mathcal{F}_h^I \mathrel{\widehat{=}}$ interior faces , $\mathcal{F}_h^I \mathrel{\widehat{=}}$ boundary faces]

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local Trefftz space

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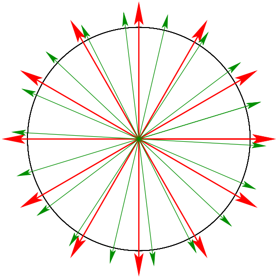
with $(\Gamma_D = \emptyset)$

$$V_h := \{v \in L^2(\Omega) : \; v|_T \in PW_{p_T} \, \forall T \in \mathcal{T}_h\}$$

Plane wave space:

$$PW_p := \text{Span}\,\{\boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d}_j \cdot \boldsymbol{x})\}_{j=1}^{p},$$

$$\boldsymbol{d}_j = \begin{pmatrix} \cos(\frac{2\pi}{p}(j-1)) \\ \sin(\frac{2\pi}{p}(j-1)) \end{pmatrix}, j = 1, \dots, p .$$



$p \in \mathbb{N} \mathrel{\widehat{=}}$ no. of plane wave directions

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$$u_h \in V_h\colon \quad a_h(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h\ .$$

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$$\begin{aligned} a_h(u,v) = &\int_{\mathcal{F}_h^{\mathcal{I}}} [\![\nabla u]\!]\,\{\overline{v}\} - \int_{\mathcal{F}_h^{\mathcal{I}}} [\![u]\!] \cdot \{\overline{\nabla_h v}\}\ \mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{I}}} \{\nabla_h u\} \cdot [\![\overline{v}]\!]\ \mathrm{d}S \\ &- \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, u\, \overline{\nabla_h v}\,\mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, \nabla_h u \cdot \boldsymbol{n}\, \overline{v}\,\mathrm{d}S \\ &+ \frac{i}{\omega} \int_{\mathcal{F}_h^{\mathcal{I}}} \beta\, [\![\nabla_h u]\!]\, [\![\overline{\nabla_h v}]\!]\ \mathrm{d}S + \frac{i}{\omega} \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, \nabla_h u \cdot \boldsymbol{n}\, \overline{\nabla_h v \cdot \boldsymbol{n}}\,\mathrm{d}S \\ &+ i\omega \int_{\mathcal{F}_h^{\mathcal{I}}} \alpha\ [\![u]\!] \cdot [\![\overline{v}]\!]\ \mathrm{d}S + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, u\, \overline{v}\,\mathrm{d}S\ , \quad \alpha, \beta, \delta > 0\ . \end{aligned}$$

[$\mathcal{F}_h^I \mathrel{\widehat{=}}$ interior faces , $\mathcal{F}_h^I \mathrel{\widehat{=}}$ boundary faces]

Quasi-Optimality

Recall PWDG linear variational problem: local Trefftz space

$$u_h \in V_h : \quad a_h(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h \, .$$

Coercivity:

$$\mathrm{Im}\{a_h(u_h, u_h)\} = \|u_h\|_{\mathcal{F}_h}^2$$

&

Continuity:

$$|a_h(u_h, v_h)| \leq 2 \, \|v_h\|_{\mathcal{F}_h} \, \|u_h\|_{\mathcal{F}_h^+}$$

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$$\|w\|^2_{\mathcal{F}_h} := \omega^{-1} \left\| \beta^{1/2} \, [\nabla_h w]_N \right\|^2_{0,\mathcal{F}_h^I} + \omega \left\| \alpha^{1/2} \, [w]_N \right\|^2_{0,\mathcal{F}_h^I} + \text{b.t.}$$

$$\|w\|^2_{\mathcal{F}_h^+} := \|w\|^2_{\mathcal{F}_h} + \omega \left\| \beta^{-1/2} \, \{w\} \right\|^2_{0,\mathcal{F}_h^I} + \omega^{-1} \left\| \alpha^{-1/2} \, \{\nabla_h w\} \right\|^2_{0,\mathcal{F}_h^I} + \omega \left\| \delta^{-1/2} w \right\|^2_{0,\partial\Omega} \; .$$

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“skeleton norms”

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$$\|u - u_h\|_{\mathcal{F}_h} \leq 3 \inf_{v_h \in V_h} \|u - v_h\|_{\mathcal{F}_h^+} \ .$$

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p-PWDG: "Simple" Duality Estimates

Goal: can estimate $\left\|u - u_h\right\|_{\mathcal{F}_h}$ ➢ want estimate $\left\|u - u_h\right\|_{L^2(\Omega)}$

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- R. us *Ga* *p-v*

"DG magic formula":

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial u}{\partial \mathbf{n}_T}\,v\,\mathrm{d}S = \int_{\mathcal{F}_h}\{\nabla u\}\,[v]\,\mathrm{d}S + \int_{\mathcal{F}_h^I}[\nabla u]\,\{v\}\,\mathrm{d}S\,.$$

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► $$\|w\|_{0,\Omega}^2 \leq C(\alpha^{-1},\beta^{-1})\,\|w\|_{\mathcal{F}_h}\cdot\Bigg(\sum_{T\in\mathcal{T}_h}\{\omega h_T^{-1}\|\varphi\|_{0,T}^2+\omega\|\varphi\|_{0,T}|\varphi|_{1,T} + \omega^{-1}h_T^{-1}|\varphi|_{1,T}^2+\omega^{-1}|\varphi|_{1,T}|\varphi|_{2,T}\}\Bigg)^{\frac12}.$$

(Uniform) “Star-Shapedness”

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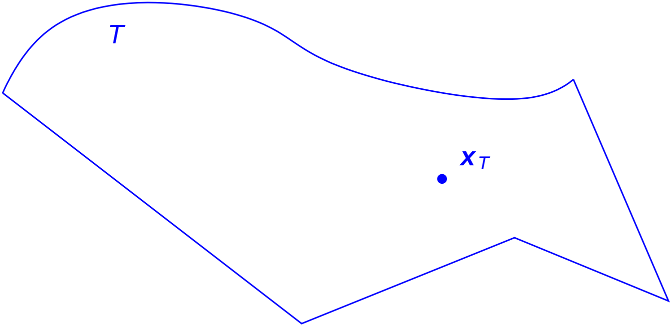

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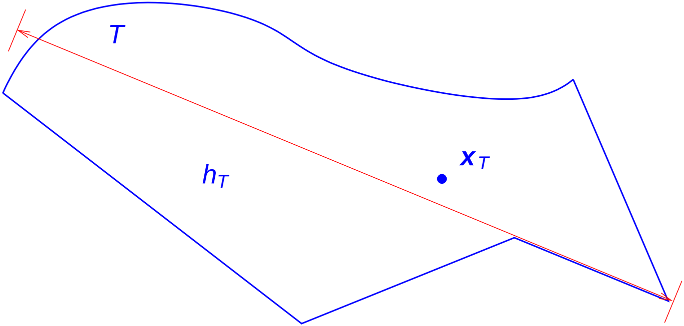

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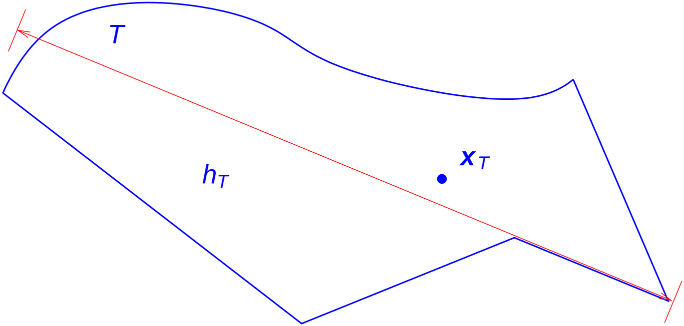

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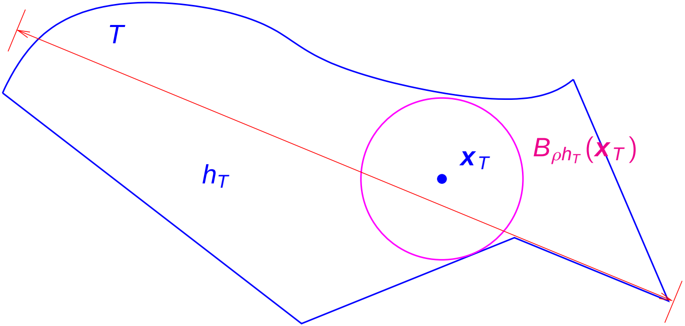

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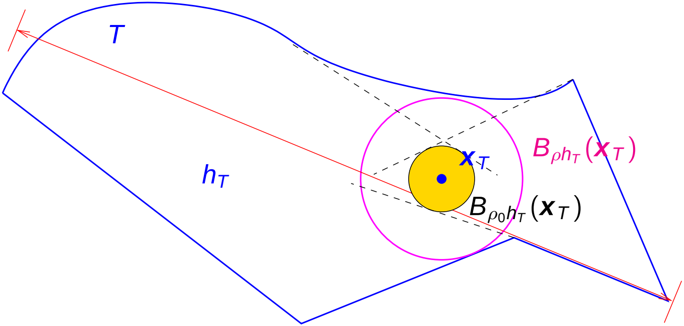

$\exists\ 0 < \rho_0 < \rho \leq \frac{1}{2}$: for all $T \in \mathcal{T}_h$ there is a “center” $\boldsymbol{x}_T$ such that

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- T is star-shaped w.r.t. $B_{\rho_0}(\boldsymbol{x}_T)$

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Setting:
- No scatterer $\Gamma_D = \emptyset$, Ω_R convex
- flux coefficients α, β, δ independent of p, h

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Theorem. stability estimates [MELENK 2D, CUMMINGS&FENG 3D]:

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$$\exists C_1(\Omega), C_2(\Omega) > 0\colon \quad \begin{cases} |\varphi|_{1,\Omega} + \omega\,\|\varphi\|_{0,\Omega} \leq C_1\,\|w\|_{0,\Omega} \ , \\ |\varphi|_{2,\Omega} \leq C_2(1+\omega)\,\|w\|_{0,\Omega} \ . \end{cases}$$

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▶ $$\boxed{\|w\|^2_{0,\Omega} \leq C \left(C' \omega^{-1} h^{-1} + \omega h \right)^{\frac{1}{2}} \cdot \|w\|_{\mathcal{F}_h}} .$$

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$$\exists C_1(\Omega), C_2(\Omega) > 0: \quad \begin{cases} |\varphi|_{1,\Omega} + \omega \|\varphi\|_{0,\Omega} \leq C_1 \|w\|_{0,\Omega} , \\ |\varphi|_{2,\Omega} \leq C_2(1+\omega) \|w\|_{0,\Omega} . \end{cases}$$

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depends on *star-shapedness*

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Estimates explicit in ω!

(Constants must not depend on ω)

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▼ ← [by duality estimate]

With $C > 0$ depending only on Ω, shape-regularity, and ωh

$$\|\nabla\mathcal{P}(u-u_p)\|^2_{0,\Omega} \leq C\left(\omega^2\,\|u-u_p\|^2_{0,\Omega} + (\omega h)^{-1}\,\|u-u_p\|^2_{\mathcal{F}_h}\right)\;.$$

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▼

$$\|\nabla(u-\mathcal{P}(u_p))\|_{0,\Omega} \leq C\,(\mathrm{diam}(\Omega)+\omega^{-1})\,h^{k-1}\left(\frac{\log(p)}{p}\right)^{k-1/2}\|u\|_{k+1,\omega,\Omega} .$$

p-PWDG: Non-Uniform Meshes

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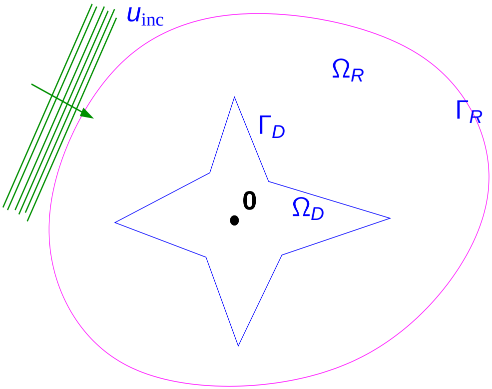

Estimate of $\|u - u_h\|_{0,\Omega}$ for model problem

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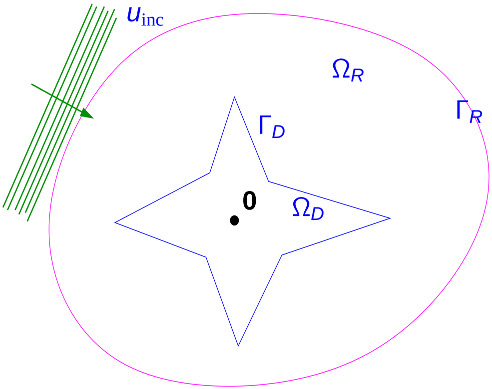

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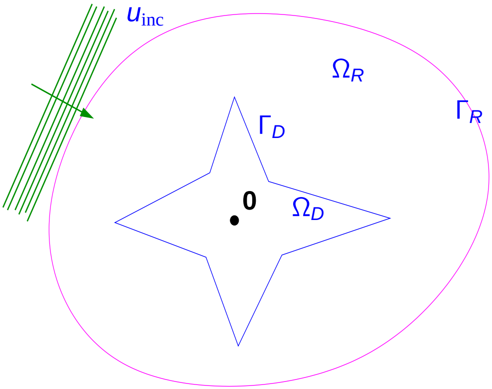

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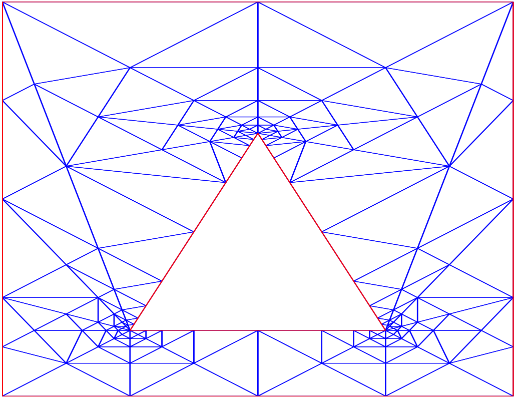

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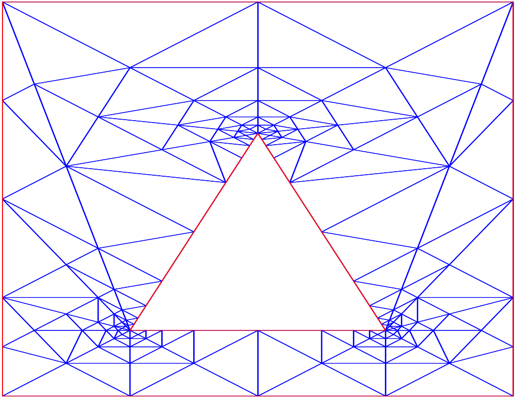

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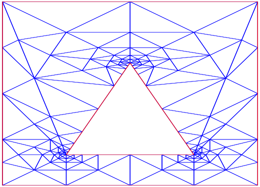
R. Hiptmair, A. Moiola, and I. Perugia, *Trefftz discontinuous Galerkin methods for acoustic scattering on locally refined meshes*, Appl. Num. Math., 79 (2013), pp. 79–91.

p-PWDG: L^2-Estimates on Non-uniform Meshes (I)



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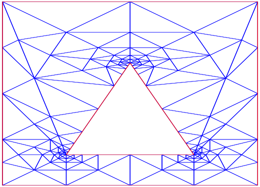

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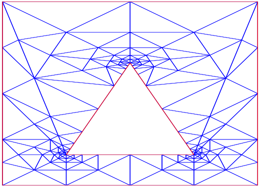

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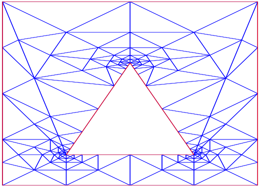

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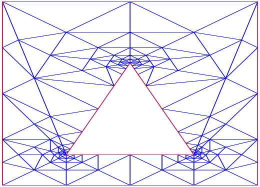

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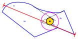

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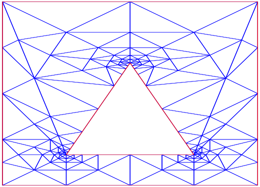

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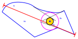

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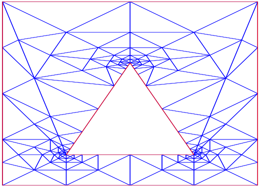

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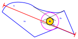
- uniform star-shapedness 
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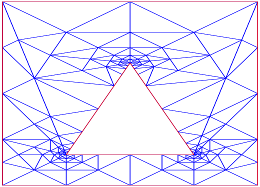

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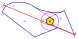
- uniform star-shapedness 
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Duality technique ➥ for all $\mathcal{T}_h$-piecewise Trefftz functions:

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p-PWDG: L^2-Estimates on Non-uniform Meshes (II)

Theorem. Refined ω-explicit lifting theorem:

$$-\Delta\varphi - \omega^2\varphi = f \text{ in } \Omega, \quad \nabla\varphi\cdot\boldsymbol{n} \pm i\omega\varphi = g \text{ on } \Gamma_R, \quad \varphi = 0 \text{ on } \Gamma_D$$

$$\Rightarrow \quad |\nabla\varphi|_{\frac{1}{2}+s,\Omega} \lesssim (1 + d_\Omega\omega)(d_\Omega^{\frac{1}{2}-s}\,\|f\|_{0,\Omega} + d_\Omega^{-s}\,\|g\|_{0,\Gamma_R}) + \|g\|_{s,\Gamma_R} \ .$$

for some $s = s(\Omega) > 0$. $\quad (d_\Omega = \mathrm{diam}(\Omega))$

Lemma. Refined scale invariant trace estimate:

$$\|v\|_{0,\partial T}^2 \lesssim h_T^{-1}\,\|v\|_{0,T}^2 + h_T^{2s}\,|v|_{\frac{1}{2}+s,T} \quad \forall v \in H^{\frac{1}{2}+s}(T)\ .$$

Duality technique ➥ for all $\mathcal{T}_h$-piecewise Trefftz functions:

$$\|w\|_{0,\Omega} \le C d_\Omega(\omega^{-\frac{1}{2}}h^{-\frac{1}{2}} + d_\Omega^{\frac{1}{2}-s}\omega^{\frac{1}{2}}h^s)\,\|w\|_{\mathcal{F}_h} \ .$$

independent of *global* quasi-uniformity

What Next ?

1. Motivation: Classical FEM – Approximation Challenges
2. Operator Adapted Trial Spaces
3. Trefftz-Discontinuous Galerkin Discretization
4. *h*-Version of PWDG: Convergence
5. *p*-Version of PWDG: Convergence
6. *hp*-PWDG: A Priori Error Estimates
7. Miscellaneous Issues and Open Problems

hp-PWDG: 2D Model Problem

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R. HIPTMAIR, A. MOIOLA, AND I. PERUGIA, *Plane wave discontinuous Galerkin methods: Exponential convergence of the hp-version*, Report 2013-31, SAM, ETH Zürich, 2013.

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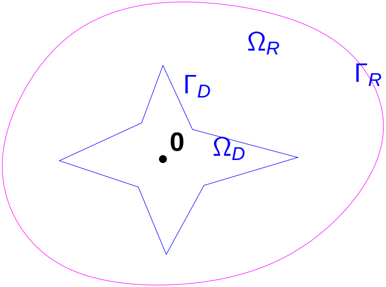

2D geometric setting: $\Omega := \Omega_R \setminus \Omega_D$

Γ_R: (analytic) artificial truncation boundary

Ω_D: sound-soft scatterer star-shaped w.r.t. 0

$$\begin{aligned} -\Delta u - \omega^2 u &= 0 && \text{in } \Omega\ , \\ \nabla u \cdot \boldsymbol{n} - i\omega u &= g && \text{on } \partial\Omega_D\ , \\ u &= 0 && \text{on } \partial\Omega_R\ . \end{aligned}$$

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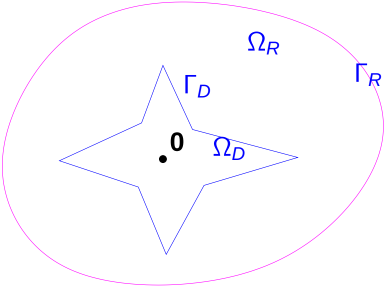

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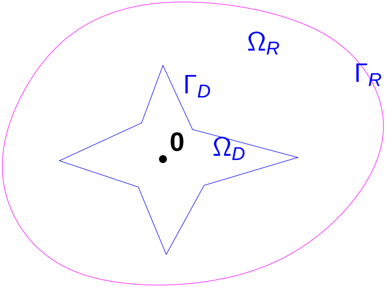

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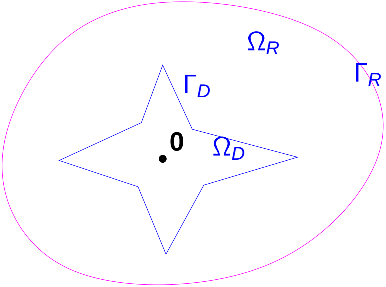

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Analytic Regularity

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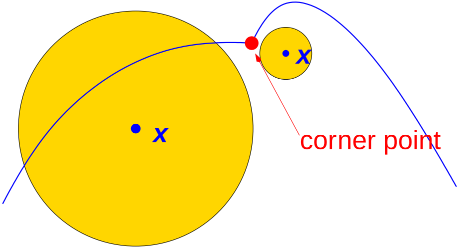

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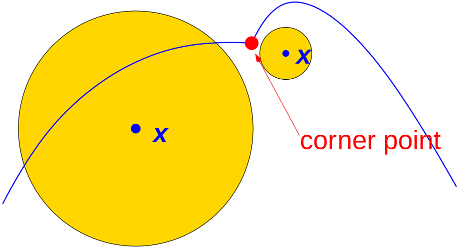

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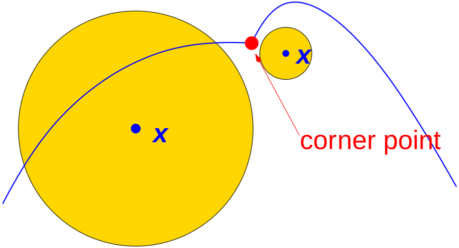

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► PW approximation converges exponentially in no. of directions!

hp-PWDG: Approximation Policy

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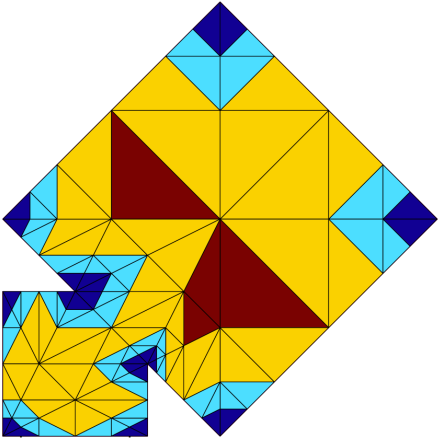

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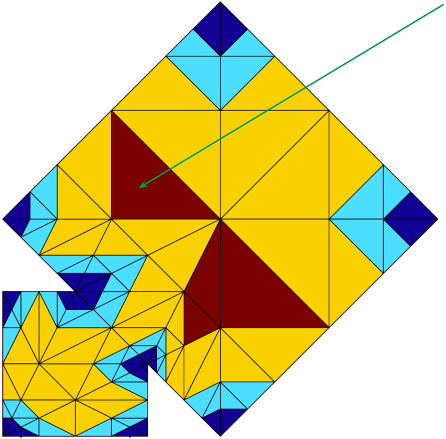

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far from corners
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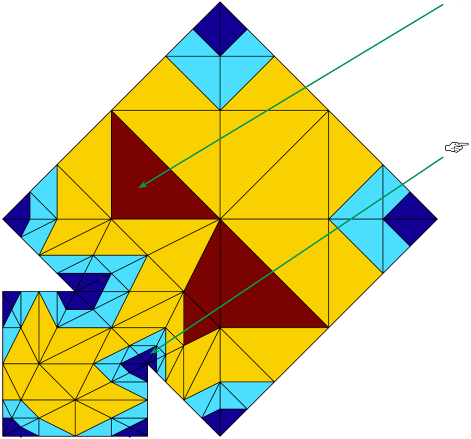

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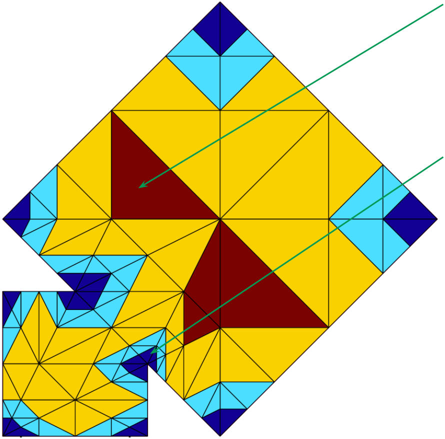

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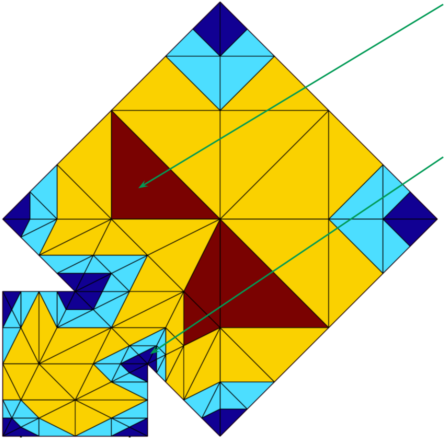

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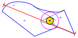

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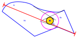

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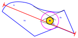

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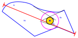

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$$\|w\|^2_{\mathcal{F}_h} := \omega^{-1} \left\|\beta^{1/2} [\nabla_h w]_N\right\|^2_{0,\mathcal{F}^I_h} + \omega \left\|\alpha^{1/2} [w]_N\right\|^2_{0,\mathcal{F}^I_h} + \text{b.t.}$$

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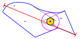

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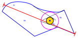

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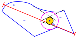

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the challenge!

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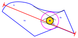

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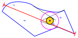

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$$\|w\|_{0,\Omega} \lesssim \frac{(|\mathcal{F}_h^I| + |\Gamma_R|)d_\Omega^2}{|\Omega|} \left(\frac{1}{\omega h_{\max}} + d_\Omega \omega + (d_\Omega \omega)^3 \right) \|w\|_{\mathcal{F}_h}$$

for *any* $\mathcal{T}_h$-p.w. Trefftz function w. ($|\mathcal{F}_h^I| \mathrel{\widehat{=}}$ length of interior edges)

hp-PWDG Trial Space

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Geometrically graded layer meshes (grading parameter $0 < \sigma < 1$)

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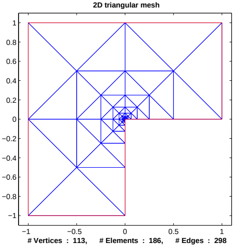

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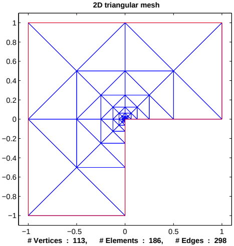

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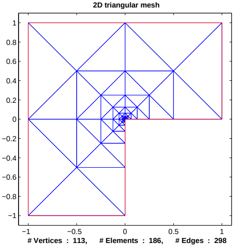

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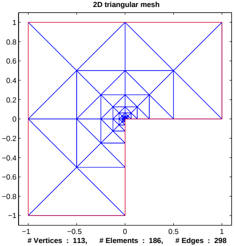

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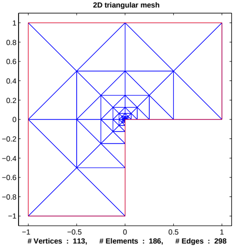

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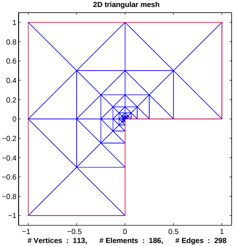

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Lemma. $\exists \eta > 0$ independent of u and L (but not of ω!) such that u is analytic in

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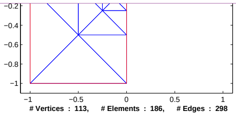

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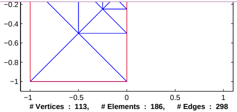

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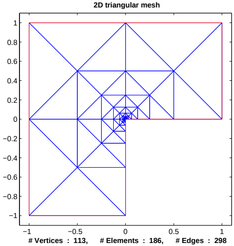

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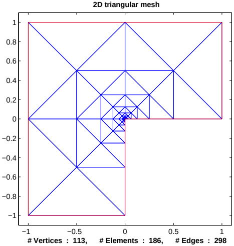

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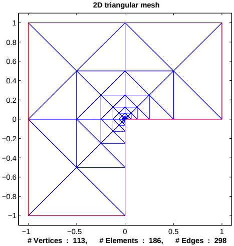

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▶ No. of local PW directions increases with level L

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Duality estimates **+**

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Practical *hp*-PWDG: vulnerable to plane wave instability!

What Next ?

1. Motivation: Classical FEM – Approximation Challenges
2. Operator Adapted Trial Spaces
3. Trefftz-Discontinuous Galerkin Discretization
4. *h*-Version of PWDG: Convergence
5. *p*-Version of PWDG: Convergence
6. *hp*-PWDG: A Priori Error Estimates
7. Miscellaneous Issues and Open Problems

Adaptive Plane Wave Approximation

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An a priori/inherent adaptive strategy:

Find dominant propagation directions by

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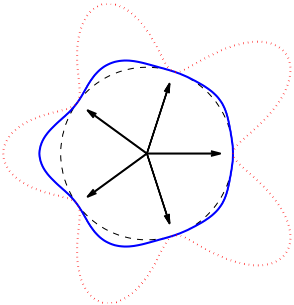

◁ dependence of dispersion (🟥), dissipation (🟦) on propagation direction

dispersion/dissipation vanish in directions $\mathbf{d}_j$ of plane wave basis functions

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E. GILADI AND J. KELLER, *A hybrid numerical asymptotic method for scattering problems*, J. Comp. Phys., 174 (2001), pp. 226–247. [PUM]

T. BETCKE AND J. PHILLIPS, *Approximation by dominant wave directions in plane wave methods*, Preprint University College London, UK, 2012. [PWDG]

M. AMARA, S. CHAUDHRY, J. DIAZ, R. DJELLOULI, AND S. L. FIEDLER, *A local wave tracking strategy for efficiently solving mid- and high-frequency Helmholtz problems*, Comput. Methods Appl. Mech. Engrg., 276 (2014), pp. 473–508. [LSQ]

C. GITTELSON, *Plane wave discontinuous Galerkin methods*, MSc thesis, SAM, ETH Zürich, 2008.

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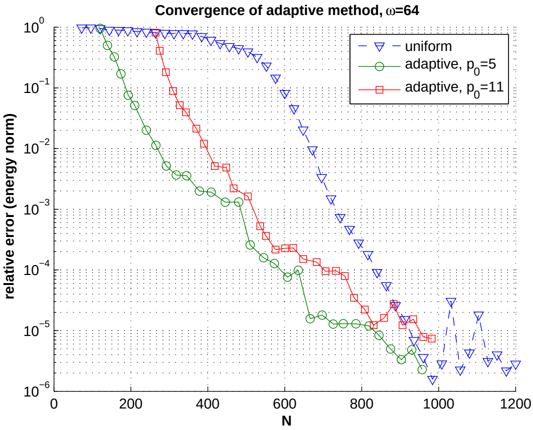
❸ Add $\boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d}_e \cdot \boldsymbol{x})$ to plane wave basis on T

Adaptivity: Numerical Experiment



- Fast initial convergence

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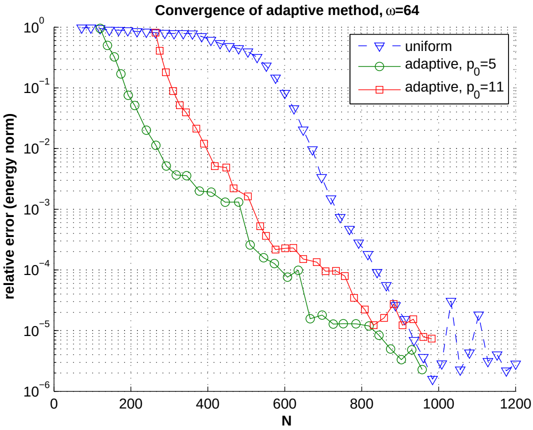

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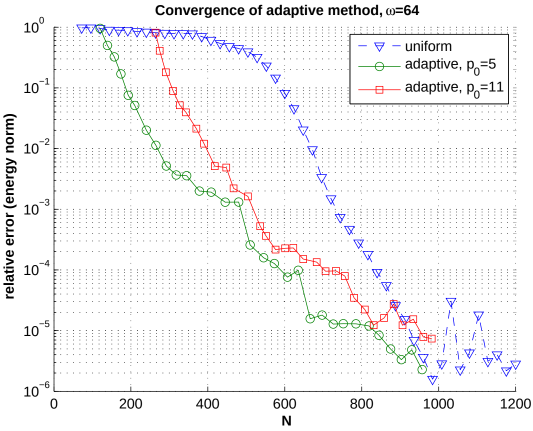

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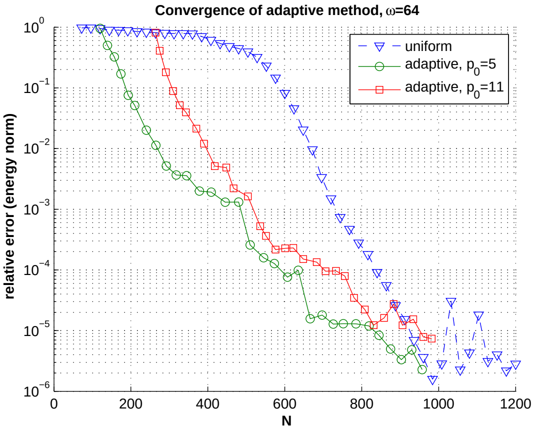

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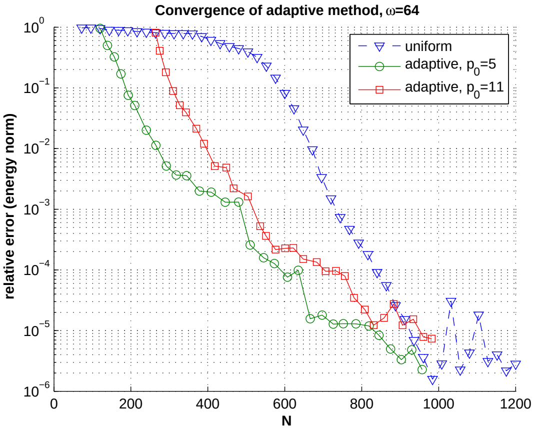

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Outlook and Conclusion

PWDG research problems

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T. Luostari, T. Huttunen, and P. Monk, *Improvements for the ultra weak variational formulation*, International Journal for Numerical Methods in Engineering, 94 (2013), pp. 598–624.

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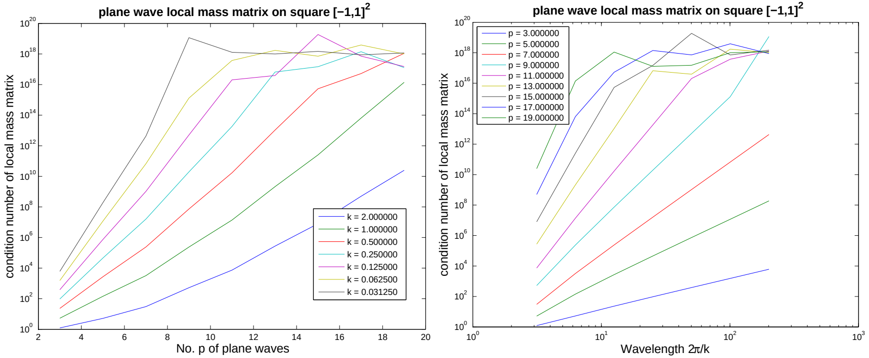

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L. Yuan and Q. Hu, *A solver for Helmholtz system generated by the discretization of wave shape functions*, Adv. Appl. Math. Mech., 5 (2013), pp. 791–808.

P. Monk, J. Schöberl, and A. Sinwel, *Hybridizing Raviart-Thomas elements for the Helmholtz equation*, RICAM Report 22-08 (2008).

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THANK YOU

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