Large scale geometry of automorphism groups

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- provide a geometric picture of topological groups as we have of say f.g. groups, Lie groups and Banach spaces,
- identify new computable isomorphic invariants of topological groups.

Overview of the three lectures:

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First lecture:
Coarse geometry of topological groups

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A uniform space is intended to capture the idea of being uniformly close in a topological space and hence gives rise to concepts of Cauchy sequences and completeness.

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$$\mathcal{U}_d = \{E \subseteq X \times X \mid \exists \alpha > 0 \;\; E_\alpha \subseteq E\}.$$

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Again, if (X, d) is a pseudometric space, there is a canonical coarse structure $\mathcal{E}_d$ obtained by

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The main point here is that, for a uniform structure, we are interested in E_α for α small, but positive, while, for a coarse structure, α is often large, but finite.

Morphisms

Recall that a map $\phi\colon (X,\mathcal{U}) \to (M,\mathcal{V})$ between uniform spaces is uniformly continuous if

$$\forall F \in \mathcal{V} \ \exists E \in \mathcal{U}\colon \quad (x,y) \in E \ \Rightarrow \ \big(\phi(x),\phi(y)\big) \in F.$$

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E.g., a map $\phi\colon (X,d) \to (M,\partial)$ is a coarse embedding if

$$\rho\big(d(x,y)\big) \leqslant \partial\big(\phi(x),\phi(y)\big) \leqslant \omega\big(d(x,y)\big)$$

for some $\rho,\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty}\rho(t) = \infty$.

Left-uniform structure on a topological group

If G is a topological group, its left-uniformity $\mathcal{U}_L$ is that generated by entourages of the form

$$E_V = \{(x, y) \in G \times G \mid x^{-1}y \in V\},$$

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where V varies over all identity neighbourhoods in G.

A basic theorem, due essentially to G. Birkhoff (fils) and S. Kakutani, is that

$$\mathcal{U}_L = \bigcup_d \mathcal{U}_d,$$

where the union is taken over all continuous left-invariant écarts d on G, i.e., so that

$$d(zx, zy) = d(x,y).$$

Left-coarse structure on a topological group

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Definition

If G is a topological group, its left-coarse structure $\mathcal{E}_L$ is given by

$$\mathcal{E}_L = \bigcap_d \mathcal{E}_d,$$

where the intersection is taken over all continuous left-invariant écarts d on G.

Relatively OB sets

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for every continuous left-invariant écart d on G.

One may easily show that the class $\mathcal{OB}$ of relatively (OB) subsets is an ideal of subsets of G stable under the operations

$$A \mapsto A^{-1}, \quad (A, B) \mapsto AB \quad \text{and} \quad A \mapsto \overline{A}.$$

Proposition

The left-coarse structure $\mathcal{E}_L$ on a topological group G is generated by entourages of the form

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Though our theory is applicable to all topological groups, given the topic of the conference, we shall mainly focus on automorphism groups or, more generally, on Polish, that is, separable and completely metrisable topological groups.

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• More generally, in a locally compact σ-compact group, they are the relatively compact subsets.

• Similarly, in the underlying additive group $(X, +)$ of a Banach space $(X, \|\cdot\|)$, they are the norm bounded subsets.

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3. *G is locally (OB), i.e., there is a relatively (OB) identity neighbourhood $V \subseteq G$.*

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Alternatively, we may quasiorder the continuous left-invariant écarts on G by

$$\begin{aligned} \partial \lll d \quad &\Leftrightarrow \quad \exists \rho\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ so that } \partial(x,y) \leqslant \rho\big(d(x,y)\big) \\ &\Leftrightarrow \quad \mathrm{id}\colon (G,d) \to (G,\partial) \text{ is bornologous.} \end{aligned}$$

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The coarsely proper écarts are then the maximal elements in this ordering.

The previous theorem can be seen as an extension of a result due to S. Kakutani and K. Kodaira stating that every locally compact σ-compact group carries a continuous left-invariant proper écart, i.e., so that balls are compact.

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Indeed, by a result of B. de Mendonça Braga, every Polish group isomorphically and coarsely embeds into

$$\prod_{n\in\mathbb{N}} \mathrm{Aff}(\mathbb{G}),$$

where $\mathrm{Aff}(\mathbb{G})$ is the group of affine isometries of the Gurarii Banach space, which is coarsely equivalent to $\mathbb{G}$.

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Definition

A map $\phi\colon (M, d_M) \to (N, d_N)$ between pseudometric spaces is said to be a quasi-isometric embedding if there are constants K and C so that

$$\frac{1}{K} \cdot d_M(x, y) - C \leqslant d_N(\phi x, \phi y) \leqslant K \cdot d_M(x, y) + C.$$

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Moreover, ϕ is a quasi-isometry if in addition $\phi[M]$ is cobounded in N, that is, $\sup_{y \in N} d_N(y, \phi[M]) < \infty$.

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From this we define a left-invariant metric on Γ, called the word metric, by

$$\rho_S(x, y) = \ell_S(x^{-1}y).$$

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From this we define a left-invariant metric on Γ, called the word metric, by

$$\rho_S(x, y) = \ell_S(x^{-1}y).$$

The fundamental observation underlying geometric group theory is then that given any two finite symmetric generating sets $S, S' \subseteq \Gamma$, the word metrics ρ_S and $\rho_{S'}$ are quasi-isometric,

Example: Finitely generated groups

Consider a finitely generated group Γ and fix a finite symmetric generating set $S \subseteq \Gamma$.

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The fundamental observation underlying geometric group theory is then that given any two finite symmetric generating sets $S, S' \subseteq \Gamma$, the word metrics ρ_S and $\rho_{S'}$ are quasi-isometric, i.e.,

$$\mathrm{id}\colon (\Gamma, \rho_S) \to (\Gamma, \rho_{S'}) \quad \text{is a quasi-isometry.}$$

Quasimetric structure

To generalise the example of finitely generated groups, we refine the quasiordering $\lll$ of continuous left-invariant écarts as follows.

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Also, any two maximal écarts are necessarily quasi-isometric and thus provide a canonical and well-defined quasimetric structure on G.

Here a quasimetric space is simply a set with a quasi-isometric equivalence class of écarts.

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Theorem

A Polish group G admits a maximal écart if and only if G is (OB) generated, that is, there is a relatively (OB) subset $A \subseteq G$ algebraically generating G.

Our study therefore reduces to investigating Polish groups with the word metric ρ_A induced by some/any relatively (OB) generating set $A \subseteq G$..

Examples

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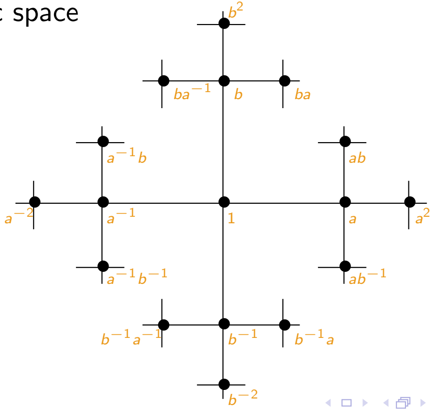
For example, the free non-abelian group $\mathbb{F}_2$ on two generators a, b gives rise to the quasimetric space



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Also, as S_∞ has property (OB), a simple calculation shows that the semidirect product

$$S_\infty \ltimes F$$

is quasi-isometric to F equipped with the word metric

$$\rho_{\{\text{transpositions}\}}.$$

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From these examples we see that the theory presented is a conservative extension of geometric group theory for finitely or compactly generated groups and of the geometric non-linear analysis of Banach spaces.

Homeomorphism groups

Let M be a compact manifold and $\mathcal{V} = \{V_i\}_{i=1}^k$ an open covering of M.

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By fundamental work of Edwards and Kirby, there is an identity neighbourhood U in $\mathrm{Homeo}(M)$ so that every element $h \in U$ can be written as $h = g_1 \cdots g_k$, where

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We may thus define the corresponding fragmentation norm on the identity component $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms by letting

$$\ell_{\mathcal{V}}(h) = \min\big(m \;\big|\; h = g_1 \cdots g_m \;\&\; \mathrm{supp}(g_i) \subseteq V_{j_i} \text{ for some } j_i\big).$$

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From this, we obtain a left-invariant metric by

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Proposition (K. Mann & C.R.)

For all covers $\mathcal{V}$ of a compact manifold M by embedded open balls, the metric $\rho_{\mathcal{V}}$ is quasi-isometric to a maximal metric on $\mathrm{Homeo}_0(M)$.

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In particular, every separable metric space admits a quasi-isometric embedding into $\mathrm{Homeo}_0(M)$.