Large scale geometry of automorphism groups

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Permutation groups and transformation semigroups,
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Second lecture:
Geometry of automorphism groups


Amalgamated Bank of Chicago

Applications to model theory

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The topology on $\mathrm{Aut}(\mathbf{M})$ is always that obtained by declaring pointwise stabilisers

$$V_{\overline{a}} = \{g \in \mathrm{Aut}(\mathbf{M}) \mid g(\overline{a}) = \overline{a}\}$$

of finite tuples $\overline{a}$ in $\mathbf{M}$ to be open.

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Provided this holds, then, up to quasi-isometry,

$$\rho_S \text{ is independent of the choice of } S$$

so defines an isomorphic invariant of the group, the quasi-isometry type.

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$$\mathcal{O}(\bar{a}) = \mathcal{O}(\bar{b}) \ \Leftrightarrow\ \mathrm{tp}^{\mathbf{M}}(\bar{a}) = \mathrm{tp}^{\mathbf{M}}(\bar{b}),$$

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2. similarly, provide realisations of and tools for analysing the large scale geometry of $\mathrm{Aut}(\mathbf{M})$,
3. show how the geometry of $\mathrm{Aut}(\mathbf{M})$ interacts with the algebraic and dynamical structure of the group and with the structure $\mathbf{M}$.

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Without loss of generality, we may assume that $\mathcal{S}$ consists of types of the form $p = \mathrm{tp}^{\mathbf{M}}(\overline{b}, \overline{c})$, where

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We define a graph $\mathbf{X}_{\bar{a},\mathcal{S}}$ on the set $\mathcal{O}(\bar{a})$ of realisations of $\mathrm{tp}^{\mathbf{M}}(\bar{a})$ in $\mathbf{M}$ by connecting distinct $\bar{b}, \bar{c} \in \mathcal{O}(\bar{a})$ by an edge if and only if

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Observe that, since

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for all tuples $\overline{b}$, $\overline{c}$ and automorphisms $g \in \mathrm{Aut}(\mathbf{M})$, the diagonal action of $\mathrm{Aut}(\mathbf{M})$ on $\mathcal{O}(\overline{a})$ is an action by automorphisms on the graph $\mathbf{X}_{\overline{a},\mathcal{S}}$.

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We thus have a transitive isometric action $\mathrm{Aut}(\mathbf{M}) \curvearrowright \big(\mathbf{X}_{\bar{a},\mathcal{S}}, \rho_{\bar{a},\mathcal{S}}\big)$.

Theorem

Let **M** *be a countable* ω*-homogeneous structure.*
Then $\mathrm{Aut}(\mathbf{M})$ *is (OB) generated if and only if there is a finite tuple* $\overline{a}$ *in* **M** *satisfying the following two requirements.*

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Condition (2), which in itself is equivalent to the pointwise stabiliser $V_{\bar{a}}$ being relatively (OB) in $\mathrm{Aut}(\mathbf{M})$, may require some amount of work to verify.

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Theorem (Milnor–Schwarz Theorem)

For $\bar{a}$ and $\mathcal{R}$ as above, the map

$$g \in \mathrm{Aut}(\mathbf{M}) \mapsto g \cdot \bar{a} \in \mathbf{X}_{\bar{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{M})$ and $\mathbf{X}_{\bar{a},\mathcal{R}}$.

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Secondly, let $\mathcal{R} = \{E\}$ consist of the single type which is the edge relation E. Then, since $\mathbf{X}_{a,\mathcal{R}} = \mathbf{T}$ is connected, Condition (1) is also verified.

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By the Milnor–Švarc Theorem, we see that the map

$$g \in \mathrm{Aut}(\mathbf{T}) \mapsto g(a) \in \mathbf{T}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{T})$ and $\mathbf{X}_{a,\mathcal{R}} = \mathbf{T}$.

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However, the initial data given, namely $\mathrm{Aut}(\mathbf{M})$ as an abstract group, is an incredibly detailed piece of information.

Instead the result here says that $\mathbf{T}$ is recoverable up to quasi-isometry from much coarser topological-algebraic information about $\mathrm{Aut}(\mathbf{T})$, namely the quasi-isometry type of a word metric ρ_S with respect to some relatively (OB) generating set S.

Orbital independence relations

The verification that $\mathrm{Aut}(\mathbf{M})$ is locally (OB) often relies on identifying an appropriate independence relation $\underset{A}{\perp}$ between finite subsets of $\mathbf{M}$ relative to a fixed finite subset $A \subseteq \mathbf{M}$ or tuple $\bar{a}$ in $\mathbf{M}$.

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Independence notions similar the those above have recently been studied by K. Tent and M. Ziegler in connection with questions of simplicity of automorphism groups.

Theorem

Suspose $\mathbf{M}$ *is a countable structure,* $A \subseteq \mathbf{M}$ *a finite subset and* $\mathop{\smile\!\!\!\!\!\!\,|}_A$ *an orbital A-independence relation. Then the pointwise stabiliser subgroup* V_A *has property (OB) (relative to itself).*

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Thus, if $A = \emptyset$*, the automorphism group* $\mathrm{Aut}(\mathbf{M}) = V_\emptyset$ *is quasi-isometric to a point and, if* $A \neq \emptyset$*,* $\mathrm{Aut}(\mathbf{M})$ *is locally (OB).*

Functorial amalgamations

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For our purposes, we require a stronger scheme.

Definition

Given an Fraïssé class $\mathcal{K}$ with limit $\mathbf{K}$ and a finite substructure $\mathbf{A} \subseteq \mathbf{K}$, we say that $\mathcal{K}$ satisfies functorial amalgamation over $\mathbf{A}$ if there is a way of choosing the amalgamations over $\mathbf{A}$ in the class $\mathcal{K}$ to be functorial with respect to embeddings.

In terms of arrows

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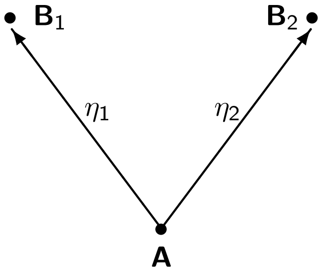

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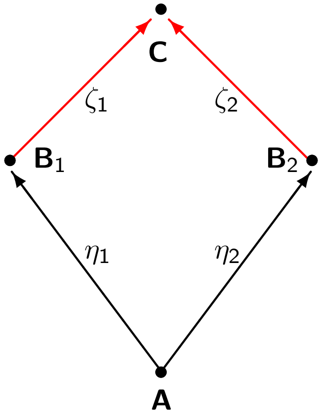

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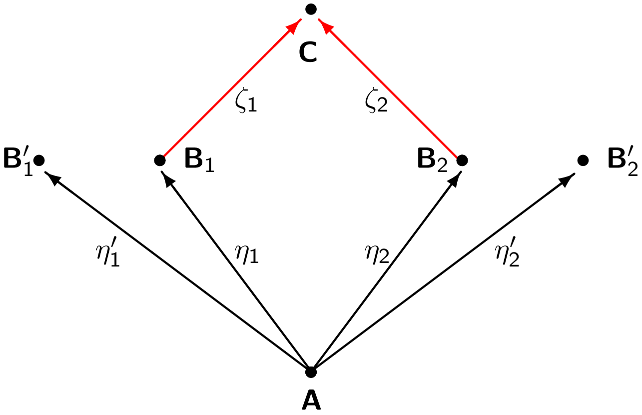

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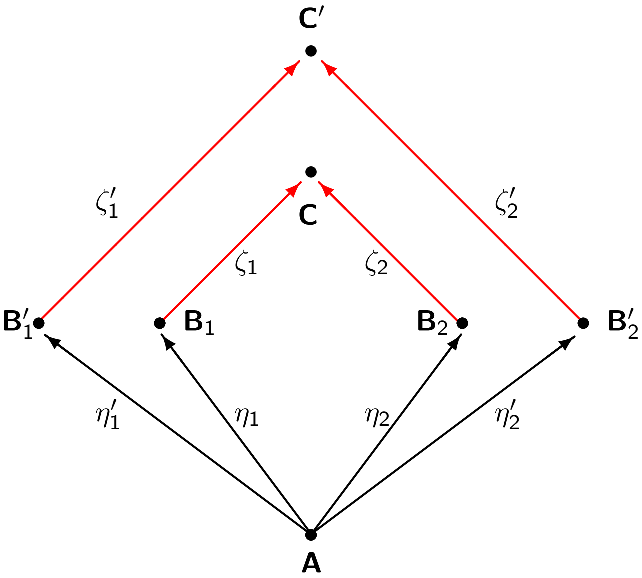

$$\begin{tikzcd}
 & & \mathbf{C}' & & \\
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\mathbf{B}_1' \arrow[uurr, "\zeta_1'"] & \mathbf{B}_1 \arrow[l, "\iota_1"'] \arrow[ur, "\zeta_1"] & & \mathbf{B}_2 \arrow[r, "\iota_2"'] \arrow[ul, "\zeta_2"'] & \mathbf{B}_2' \arrow[uull, "\zeta_2'"'] \\
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The rational Urysohn metric space

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That is, let B and C be two finite metric spaces with only a single point a in common.

The free amalgam of B and C over a is the union $B \cup C$ with

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An important fact here is that, unless we bound the diameters of the metric spaces in question, there is no functorial amalgamation of the empty set.

Given a Fraïssé class $\mathcal{K}$ with limit $\mathbf{K}$ and a functorial amalgamation scheme over some finite $\mathbf{A} \subseteq \mathbf{K}$, we obtain an orbital $\mathbf{A}$-independence relation $\mathop{\smile\!\!\!\!\!\!|}_{\mathbf{A}}$ on $\mathbf{K}$ by setting

$$\mathbf{B} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{A}} \mathbf{C} \quad \Leftrightarrow \quad \mathbf{B} \,\&\, \mathbf{C} \text{ are functorially amalgamated over } \mathbf{A}$$

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Theorem

Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ and assume that $\mathbf{A}$ is a finite substructure of $\mathbf{K}$ so that $\mathcal{K}$ admits a functorial amalgamation over $\mathbf{A}$. Then $V_{\mathbf{A}}$ has property (OB) and thus $\mathrm{Aut}(\mathbf{K})$ is locally (OB).

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To show that the automorphism group $\mathrm{Isom}(\mathbb{QU})$ is (OB) generated and to compute the quasi-isometry type, we seek a finite set $\mathcal{R}$ of parameter-free complete types, so that the graph

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For this, set $\mathcal{R} = \{d(x,y) = 1\}$ and note that any two points $x, y \in \mathbb{QU}$ can be connected by a path in $\mathbf{X}_{a,\mathcal{R}}$ of length

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Therefore, $\mathbf{X}_{a,\mathcal{R}}$ is quasi-isometric to $\mathbb{QU}$ and we conclude that the map

$$g \in \mathrm{Isom}(\mathbb{QU}) \mapsto g(a) \in \mathbb{QU}$$

is a quasi-isometry.

Groups with trivial geometry

In many familiar cases, though we are able to identify the large scale geometry of a topological group, it turns out that this is trivial.

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Let $\mathbf{M}$ *be an* $\aleph_0$*-categorical countable structure.*
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Similarly, using forking calculus and the associated independence relation, we may show the same conclusion for saturated ω-stable structures.

Theorem

Let $\mathbf{M}$ *be a saturated countable model of an* ω*-stable theory.*
Then $\mathrm{Aut}(\mathbf{M})$ *is quasi-isometric to a point.*

Recall that a structure $\mathbf{M}$ is atomic if every complete type is isolated.

Tame geometry from model theoretical considerations

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It follows that, if $\mathcal{R}$ is a finite collection of types, then, for every n, the relation on $\overline{b}$ and $\overline{c}$,

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is definable in $\mathbf{M}$.

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is definable in **M**.

Definition (J.-L. Krivine and B. Maurey)

A metric d on a set X is said to be stable if, for all d-bounded sequences (x_n) and (y_m) in X, we have

$$\lim_{n\to\infty}\lim_{m\to\infty} d(x_n, y_m) = \lim_{m\to\infty}\lim_{n\to\infty} d(x_n, y_m),$$

whenever both limits exist.

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- *We say that T is κ-stable if, for all models $\mathbf{M} \models T$ and subsets $B \subseteq \mathbf{M}$ with $|B| \leqslant \kappa$, we have $|S_n^{\mathbf{M}}(B)| \leqslant \kappa$.*

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2. *if $\mathrm{Aut}(\mathbf{M})$ is (OB) generated, it admits a maximal stable metric.*

Noting the independence relations present in models of stable theories, one could be hopeful that the assumption that $\mathrm{Aut}(\mathbf{M})$ be locally (OB) would be superfluous.

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However, this is not so.

Theorem (J. Zielinski)

There is a countable atomic model $\mathbf{M}$ *of an* ω*-stable theory so that* $\mathrm{Aut}(\mathbf{M})$ *is not locally (OB).*