Semigroups from digraphs

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Work in progress!

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Outline

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Arcs

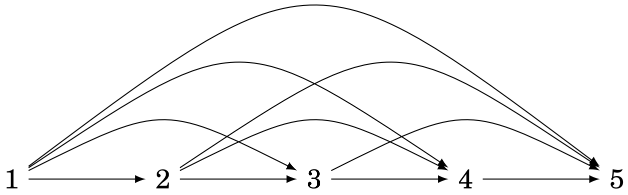
- We are in Sing_n, the semigroup of singular transformations of $[n] = \{1, \dots, n\}$.
- An idempotent of defect one is any transformation of the form $(a \to b)$ for distinct $a, b \in [n]$, such that for any $v \in [n]$:

$$v(a \to b) = \begin{cases} b & \text{if } v = a, \\ v & \text{otherwise.} \end{cases}$$

  We call $(a \to b)$ an arc.
- Let D be a digraph on $[n]$. We then view $D \subseteq \mathrm{Sing}_n$ and we are interested in $\langle D \rangle$.

Example 1

- Let D be the transitive tournament on n vertices.
- Then $\langle D \rangle = \mathrm{OI}_n = \{\alpha : v \le v\alpha\}$.
- E.g. $\alpha = (5, 2, 4, 5, 5) = (1 \to 5)(4 \to 5)(3 \to 4)$.



Example 2

- Let D be the undirected path on n vertices.
- Then $\langle D \rangle = \mathrm{O}_n = \{\alpha : u \leq v \Rightarrow u\alpha \leq v\alpha\}$.

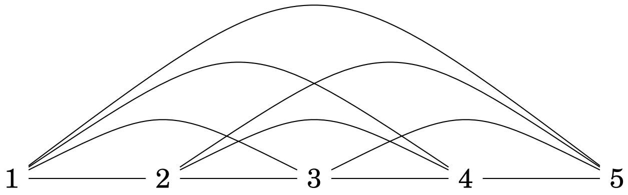
$$1 \longrightarrow\!\!\!\!- \; 2 \; - \; 3 \; - \; 4 \; - \; 5$$

- Let D be the directed path on n vertices.
- Then $\langle D \rangle = \mathrm{C}_n = \{\alpha : v \leq v\alpha, u \leq v \Rightarrow u\alpha \leq v\alpha\}$.

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

Example 3

- Let $D = K_n$ be the clique on n vertices.
- (J. M. Howie '66) Then $\langle D \rangle = \mathrm{Sing}_n$.



Previous results

(T. You + X. Yang '02, X. Yang and H. Yang '06, X. Yang and H. Yang '09)
Different properties of $\langle D \rangle$, such as:

- $\mathrm{Arcs}(\langle D \rangle) = D \cup \{(a \to b) : (b \to a)$ lies in a cycle of $D\}$.
- (J. M. Howie '78) $\langle D \rangle = \mathrm{Sing}_n$ iff D contains a strong tournament.
- Classification of when $\langle D \rangle$ is regular.
- When $\langle D_1 \rangle \cong \langle D_2 \rangle$.

Outline

Improper colourings

Let D be undirected and connected.

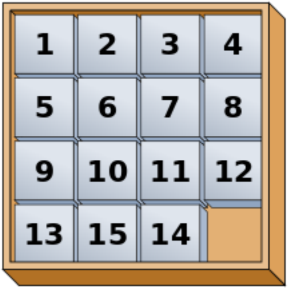
- A colouring of D is $\alpha \in \mathrm{Tran}_n$.
- A colouring is improper if there exist $u \sim v$ with $u\alpha = v\alpha$.
- The improper colourings form a semigroup $\mathrm{IC}(D)$, where

$$\langle D \rangle \leq \mathrm{IC}(D) \leq \mathrm{Sing}_n.$$

Arcs generating improper colourings

- Let $n \geq 8$ and D be 2-connected (i.e., any two vertices lie on a common cycle).
- (PJC + ACR + MRG + JDM)
  If D is non-bipartite, then $\langle D \rangle = \mathrm{IC}(n)$.
  If D is bipartite, then $\langle D \rangle < \mathrm{IC}(n)$ but $\langle D \rangle$ contains all improper colourings of defect 2 or more.
- The proof is based on a theorem in (R. M. Wilson '74):
  "Graph Puzzles, Homotopy, and the Alternating Group."

The 15-puzzle and (R. M. Wilson '74)



$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[d, dash] \\
5 \arrow[r, dash] \arrow[d, dash] & 6 \arrow[r, dash] \arrow[d, dash] & 7 \arrow[r, dash] \arrow[d, dash] & 8 \arrow[d, dash] \\
9 \arrow[r, dash] \arrow[d, dash] & 10 \arrow[r, dash] \arrow[d, dash] & 11 \arrow[r, dash] \arrow[d, dash] & 12 \arrow[d, dash] \\
13 \arrow[r, dash] & 14 \arrow[r, dash] & 15 \arrow[r, dash] & 16
\end{tikzcd}$$

- G_v is the "puzzle group" of all permutations of $[n] \setminus \{v\}$ obtained by sliding tiles.
- For any v, $G_v \cong G_n$.
- If D is non-bipartite, then $G_n = \mathrm{Sym}_{n-1}$; otherwise, $G_n = \mathrm{Alt}_{n-1}$.

Outline

Length of words

- We study the length of words $w \in D^*$ that express $\alpha \in \langle D \rangle$.
- For any D and $\alpha \in \langle D \rangle$, let

$$l_D(\alpha) := \min \left\{ \text{length}(w) : w \in D^*, w = \alpha \right\}.$$

- We are interested in the longest elements:

$$l_D(r) := \max \{ l_D(\alpha) : \alpha \in \langle D \rangle, \text{rk}(\alpha) = r \}.$$

Results for the clique K_n

- (N. Iwahori '77, J. M. Howie '80)

  $$l_{K_n}(\alpha) = n - \mathrm{fix}(\alpha) + \mathrm{cycl}(\alpha),$$

  where $\mathrm{fix}(\alpha) = \{v : v\alpha = v\}$ and $\mathrm{cycl}(\alpha)$ is the number of cyclic components of α.
- Easy to maximise:

  $$l_{K_n}(r) = n + \left\lfloor \frac{r-2}{2} \right\rfloor,$$
  $$l_{K_n}(n-1) = \left\lfloor \frac{3n-3}{2} \right\rfloor.$$

- Note that $l_{K_n}(r)$ increases with r.

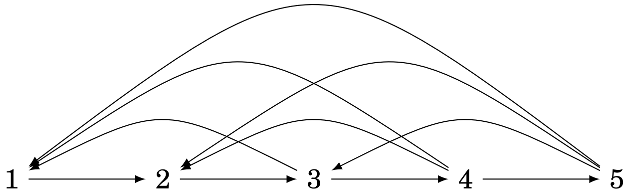
Strong tournaments

- We now restrict ourselves to strong tournaments.
- They are the "almighty" ones: the minimal arc generating sets of Sing_n.
- Question: How does $l_D(r)$ behave with D?
- Two more pieces of notation:

$$l_{\max}(r) := \max\{l_D(r) : D \text{ is a strong tournament on } [n]\},$$
$$l_{\min}(r) := \min\{l_D(r) : D \text{ is a strong tournament on } [n]\}.$$

The "bad" tournament

Let π_n be the tournament below.



Conjecture (PJC + ACR + MRG + JDM)

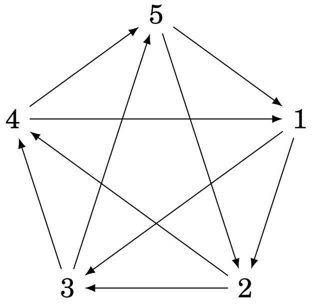
For any n and $r \leq n-1$, $l_{\pi_n}(r) = l_{\max}(r)$. Moreover,

$$l_{\pi_n}(n-1) = \frac{n^2+3n-6}{2},$$

which is achieved by $\alpha = (n, n-1, \ldots, 2, n)$.

The “good” tournament

Let $n = 2m+1$ and κ_n be the circulant tournament $\{(v \to v + [m])\}$.



Conjecture (PJC + ACR + MRG + JDM)

For any n odd and $r \le n-1$, $l_{\kappa_n}(r) = l_{\min}(r)$. Moreover,

$$l_{\kappa_n}(2) = n + 1.$$

What we've got so far

Preliminary results from (PJC + ACR + MRG + JDM):

$$\forall D \quad l_D(1) = n - 1.$$
$$l_{\pi_n}(r), l_{\max}(r) = \Theta(rn).$$
$$l_{\kappa_n}(r), l_{\min}(r) = n + \Theta(r).$$

Idea behind the last two results:

- Let
$$\Delta_D(r) := \max\left\{\sum_{i=1}^{r} d_D(u_i, v_i)\right\},$$
where $u_1, \ldots, u_r$ are all pairwise distinct, and so are $v_1, \ldots, v_r$.
- Then $\Delta_{\pi_n}(r) = \Delta_{\max}(r)$ and $\Delta_{\kappa_n}(r) = \Delta_{\min}(r)$.