Numerical-Asymptotic Approximation at High Frequency

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University of Reading

Mathematical and Computational Aspects of Maxwell's Equations
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Our group on HF stuff in Reading

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- Ivan Graham, Euan Spence (Bath)
- Jon Hargreaves, Yiu Lam (Salford)
- Marko Lindner (Hamburg)
- Markus Melenk (Vienna)
- Peter Monk (Delaware)

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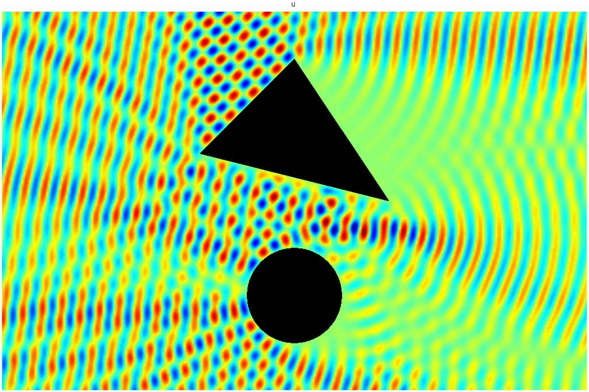
High frequency scattering



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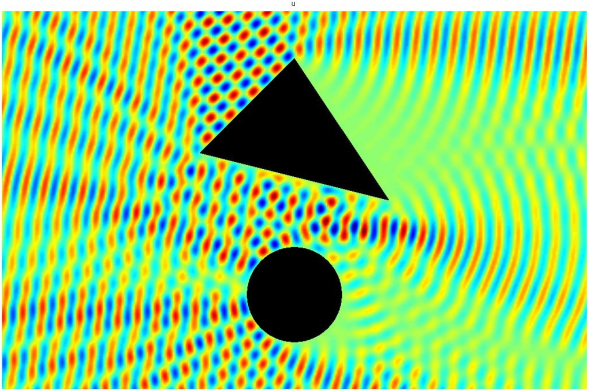

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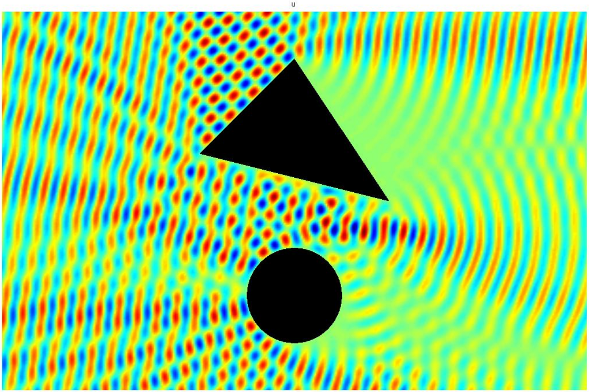

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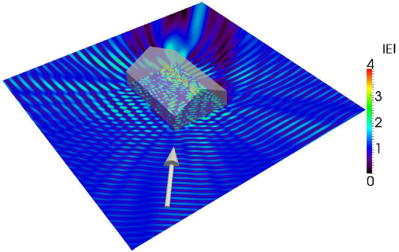
Difficult when k is large

Difficulties at high frequencies

- Solutions oscillate in space with wavelength $\lambda = 2\pi/k$.
- Conventional boundary elements lead to full matrices of dimension at least $N = \mathcal{O}(k^{d-1})$, as $k \to \infty$.
- Domain finite elements lead to sparse matrices but require even larger N.

Can improve BEM, e.g. using FMM, but **cost still grows rapidly** as k increases.

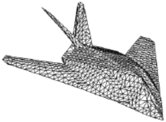
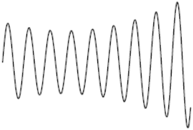
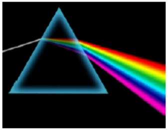


EM scattering by ice crystal solved using BEM++, see www.bempp.org & Groth et al. *J. Quant. Spec. Rad. Trans.* 2015.

The “mid frequency” problem

$$(\Delta + k^2)u = 0$$

increasing k →

Numerical methods (BEM)	What to do here??	Asymptotic methods (GTD)
		
controllably accurate computationally infeasible at large k		computational cost independent of k accurate only as $k \to \infty$

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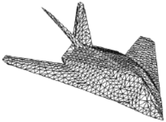
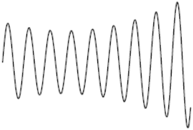
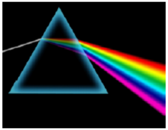


controllably accurate

computationally infeasible at large k



Asymptotic methods (GTD)



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Hybrid Numerical-Asymptotic (HNA) approach

Fuse conventional BEM with high frequency asymptotics to create algorithms that are **controllably accurate** and **computationally feasible** over the whole frequency range.

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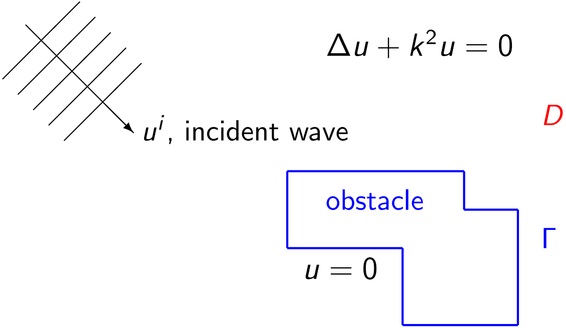
To a large extent this work born in Durham in 2002 ...

... motivated by an inspirational talk by Oscar Bruno in the programme "Computational methods for wave propagation in direct scattering".

$$\Delta u + k^2 u = 0$$



Using Green's representation theorem

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial n}(\mathbf{y})\, \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D,$$

we reformulate the scattering problem as a BIE for $\dfrac{\partial u}{\partial n}$

A typical scattering problem

... in operator notation,

$$\mathcal{A}\frac{\partial u}{\partial n} = f.$$

To solve numerically:

- choose a finite-dimensional approximation space $V_N \subset V$;
- select an approximation v_N to $\partial u/\partial n$ from V_N using the Galerkin method: find $v_N \in V_N$ such that

$$\langle \mathcal{A}v_N, w_N\rangle = \langle f, w_N\rangle, \quad \forall w_N \in V_N.$$

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Key idea: enrich the BEM approximation space with **oscillatory basis functions**

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Key idea: enrich the BEM approximation space with **oscillatory basis functions**

$$\frac{\partial u}{\partial n}(x,k) \approx v_0(x,k) + \sum_{j=1}^{J} v_j(x,k)\,\mathrm{e}^{\mathrm{i}k\phi_j(x)},$$

- v_0 is some **known** leading order **asymptotic** behaviour
- ϕ_j, $j = 1,\ldots,J$ are **specified** phases, from **asymptotics**
- v_j, $m = 1,\ldots,J$ are **unknown** amplitudes, found **numerically**

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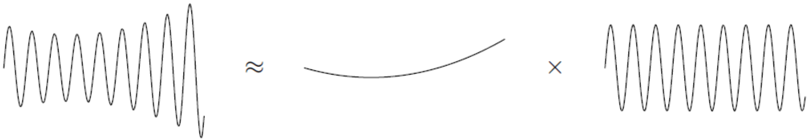

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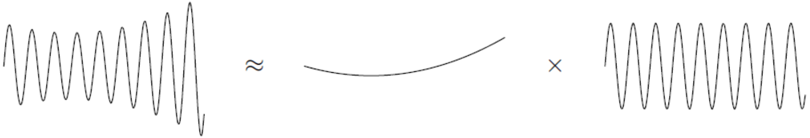

Expectation: If v_0 and ϕ_j are chosen appropriately, v_j, $j = 1, \ldots, J$, will be **slowly varying**, and **less expensive** to approximate than $\partial u/\partial n$.

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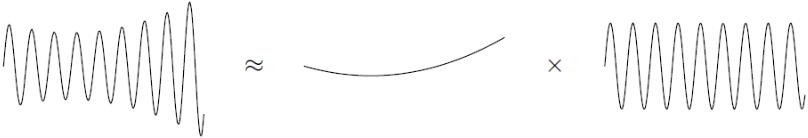

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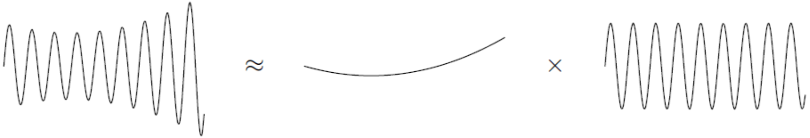

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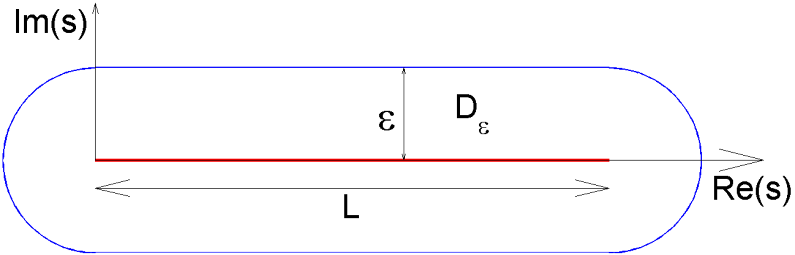
In many cases **we can prove this** by **rigorous HF best approximation estimates** - this talk – & **prove convergence of Galerkin method** by combining with **HF estimates of continuity and coercivity constants** - talks by Spence/Smyshlyaev

Polynomial approximation of analytic functions



Let $\Pi_p = \{\text{polynomials of degree } \leq p\}$. If $v(s)$ is analytic in D_ϵ, the ϵ neighbourhood of $[0, L]$, and

$$|v(s)| \leq M, \quad \text{for } s \in D_\epsilon,$$

then, for some $C, \tau > 0$,

$$\inf_{v_p \in \Pi_p} \|v - v_p\|_{L^2(0,L)} \leq C\, M\, \mathrm{e}^{-\tau p}.$$

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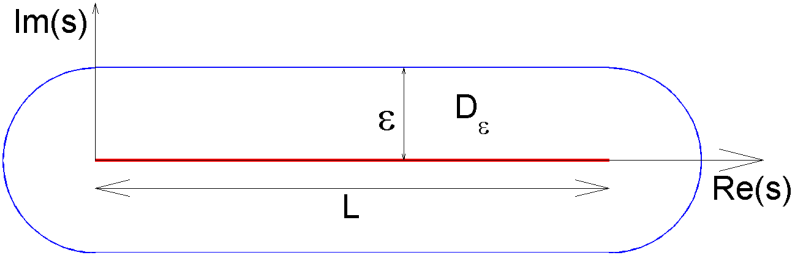

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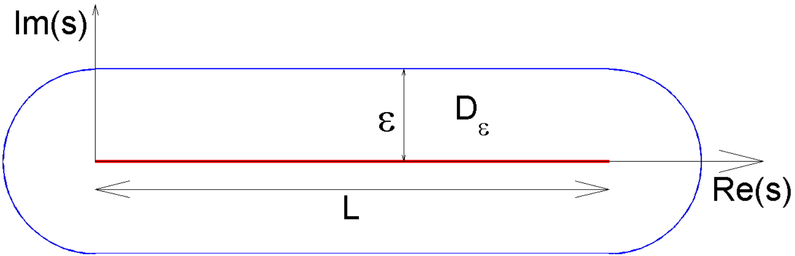
N.B. If v is k-dependent but $M = O(1)$ as $k \to \infty$ then $p = O(1)$ maintains accuracy.



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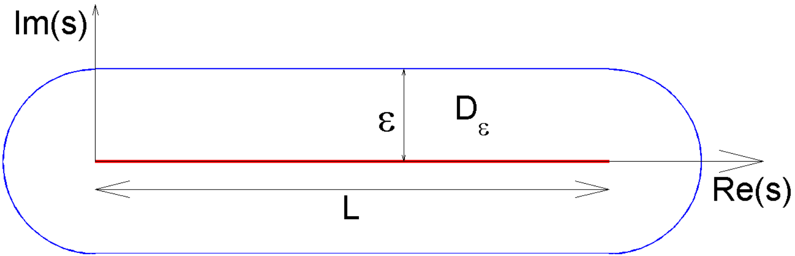
N.B. If v is k-dependent and $M = O(k^m)$ as $k \to \infty$ then $p = O(\log k)$ maintains accuracy.

Polynomial approximation of analytic functions



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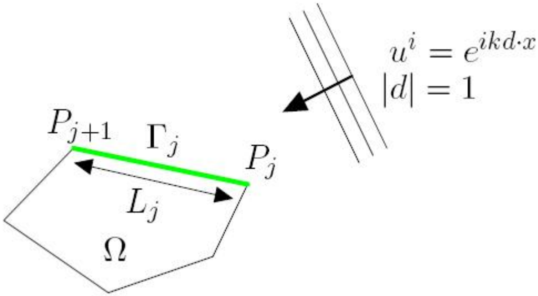
N.B. If $v(s) = \exp(\mathrm{i}ks)$ **then** $M = \exp(k\epsilon)$ **and** $p = O(k)$ **needed to maintain accuracy.** cf. M. Ainsworth (2004)

High frequency asymptotics - convex polygons

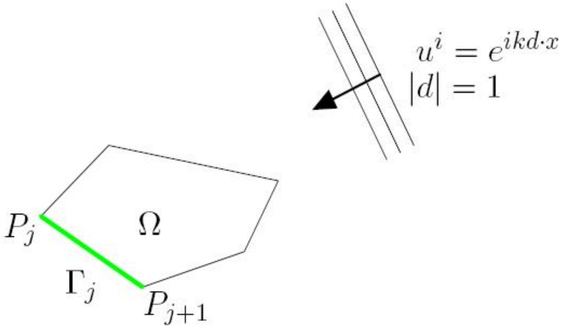


According to GTD, for a **convex** polygon, the leading-order asymptotic behaviour on a "lit" side is

$$\frac{\partial u}{\partial n} \sim 2\frac{\partial u^i}{\partial n} + v^+(s)\mathrm{e}^{iks} + v^-(s)\mathrm{e}^{-iks}, \qquad k \to \infty$$

where s is arc length along the side.

High frequency asymptotics - convex polygons



On an "unlit" side it is just

$$\frac{\partial u}{\partial n} \sim v^+(s)\mathrm{e}^{iks} + v^-(s)\mathrm{e}^{-iks}, \qquad k \to \infty.$$

Regularity results - convex polygons

Theorem (Hewett, Langdon and Melenk (2013))

Let Ω be a convex polygon. Then on any side Γ_j

$$\frac{\partial u}{\partial n}(x) = \Psi(x) + \mathrm{e}^{iks} v_j^+(s) + \mathrm{e}^{-iks} v_j^-(L_j - s), \qquad x \in \Gamma_j,$$

where

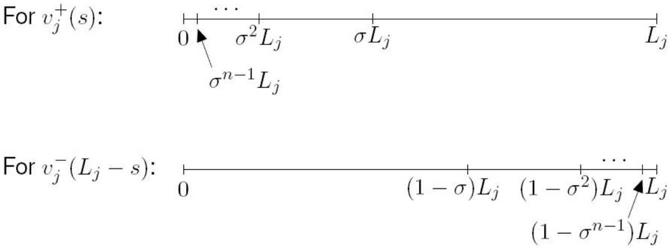
- $\Psi := 2\frac{\partial u^i}{\partial n}$ *if* Γ_j *is lit and* $\Psi := 0$ *otherwise;*
- *The functions* $v_j^{\pm}(s)$ *are* **analytic** *in* $\mathrm{Re}[s] > 0$*, with:*

$$|v_j^+(s)| \leq C \begin{cases} k^{3/2}\log^{1/2}(2+k)|ks|^{\pi/\Omega_j - 1}, & 0 < |s| \leq 1/k, \\ k^{3/2}\log^{1/2}(2+k)|ks|^{-1/2}, & |s| > 1/k, \end{cases}$$

where Ω_j *is the exterior angle at the vertex* P_j*.*

hp approximation space V_N

Approximate $v_j^{\pm}$ by piecewise polynomials of order p on overlapping geometric meshes, graded towards the corner singularities



Here σ is a grading parameter - typically $\sigma \approx 0.15$.

Error estimate - convex polygons

Theorem (Hewett, Langdon and Melenk (2013))

For $k \geq k_0 > 0$, there exist constants $C, \tau > 0$, such that

$$\left\| \frac{\partial u}{\partial n} - v_N \right\|_{L^2(\Gamma)} \leq C k^{5/2} \mathrm{e}^{-p\tau}.$$

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We can achieve any required accuracy with N growing like $\log^2 k$ as $k \to \infty$, rather than like k, as for a standard BEM.

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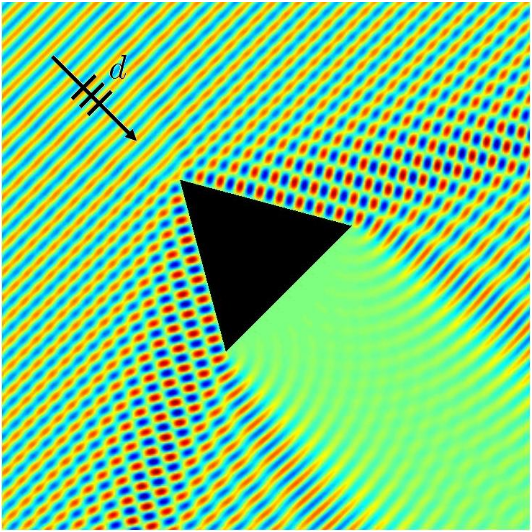
We can achieve any required accuracy with N growing like $\log^2 k$ as $k \to \infty$, rather than like k, as for a standard BEM.
Method is essentially frequency independent.

Numerical results - equilateral triangle

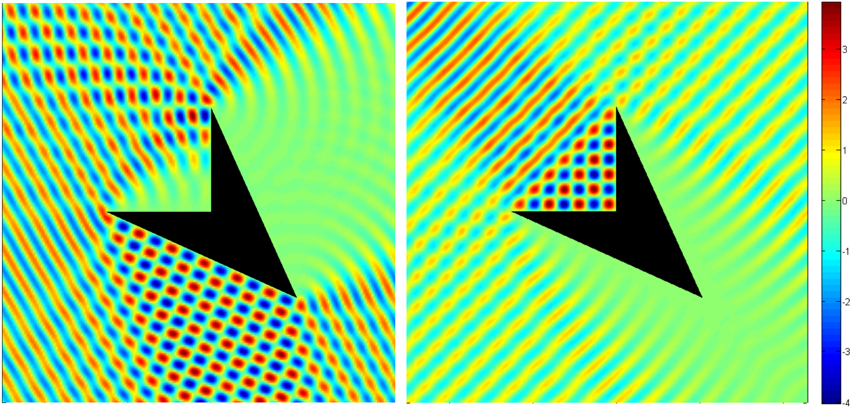


Numerical results, fixed $N = 300$, triangle

k	$\frac{N}{L/\lambda}$	$(1/k)\|\partial u/\partial n - v_{300}\|_{L^2(\Gamma)}$	COND	rel. cpt(s)
5	20.00	1.96×10^{-1}	3.50×10^{2}	1.00
10	10.00	1.48×10^{-1}	2.77×10^{1}	0.99
20	5.00	1.12×10^{-1}	3.51×10^{1}	0.97
40	2.50	8.50×10^{-2}	4.60×10^{1}	1.11
80	1.25	6.44×10^{-2}	6.12×10^{1}	1.07
160	0.63	4.88×10^{-2}	8.27×10^{1}	1.04
320	0.31	3.70×10^{-2}	1.12×10^{2}	1.20
640	0.16	2.80×10^{-2}	1.53×10^{2}	1.20
1280	0.08	2.16×10^{-2}	2.08×10^{2}	1.23
2560	0.04	1.65×10^{-2}	2.83×10^{2}	1.33
5120	0.02	1.26×10^{-2}	3.85×10^{2}	1.33

Non-convex polygons

The leading-order asymptotic behaviour on Γ is more complicated:

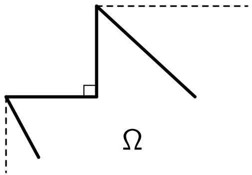


Partial illumination Re-reflections

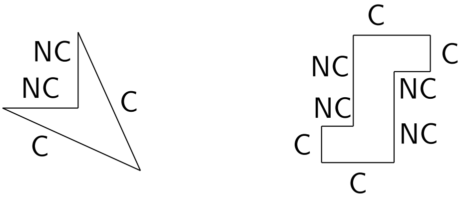
Restrict attention to a particular class of nonconvex polygons

Assume that:

1. Each exterior angle is either a right angle or greater than π.
2. At each right angle, the obstacle lies within the dashed lines:



Examples:



On a "convex" (C) side, $\partial u / \partial n$ behaves as in convex case

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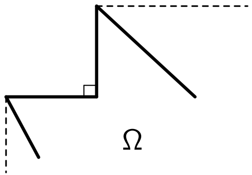

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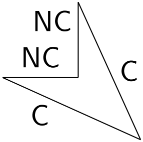
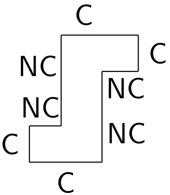
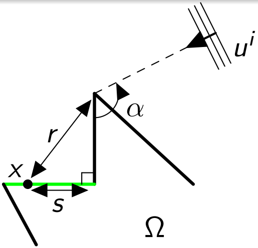




On a "convex" (C) side, $\partial u/\partial n$ behaves as in convex case

<u>Question</u>: What happens on a "nonconvex" (NC) side?

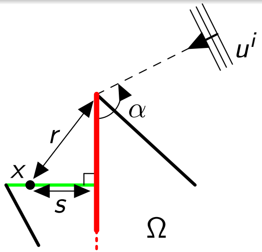
Regularity results on a nonconvex side



For $x \in \Gamma_j$ the following representation holds

$$\frac{\partial u}{\partial n}(x) = \Psi(x) + v_j^+(L_j + s)e^{iks} + v_j^-(L_j - s)e^{-iks} + \tilde{v}_j(s)e^{ikr}$$

Regularity results on a nonconvex side



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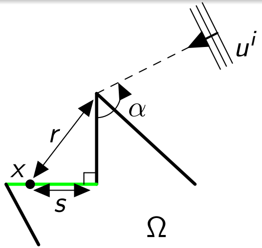
Leading order behaviour

$$\Psi(x) := \begin{cases} 2\frac{\partial u^d}{\partial n}(x), & \frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

where u^d is the known solution of a canonical diffraction problem.

Regularity results on a nonconvex side



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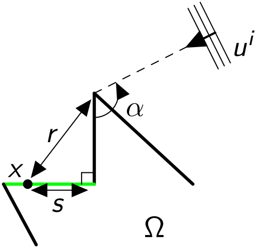
Theorem

The functions $v_j^\pm$ have the same properties as those for the convex sides, in particular are analytic in the right hand complex plane.

Regularity results on a nonconvex side



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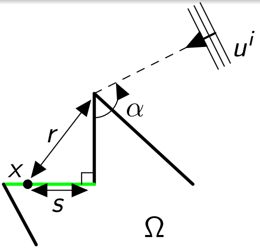
Theorem

The function $\tilde{v}_j$ is analytic in a complex k-independent ϵ-neighbourhood D_ϵ of the side Γ_j with

$$|\tilde{v}_j(s)| \leq Ck\log^{1/2}(2+k), \quad s \in D_\epsilon, \quad k \geq k_1.$$

Regularity results on a nonconvex side



For $x \in \Gamma_j$ the following representation holds

$$\frac{\partial u}{\partial n}(x) = \Psi(x) + v_j^+(L_j + s)e^{iks} + v_j^-(L_j - s)e^{-iks} + \tilde{v}_j(s)e^{ikr}$$

Approximation space:

- Replace v_j^- by a piecewise polynomial supported on a geometric mesh.
- Replace v_j^+ and $\tilde{v}_j$ by polynomials supported on the whole side.

Error estimate - nonconvex polygons

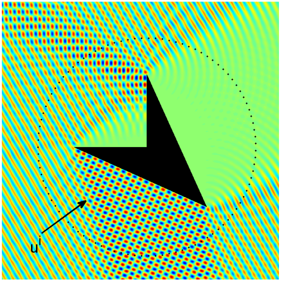
Theorem (C-W, Hewett, Langdon and Twigger (2015))

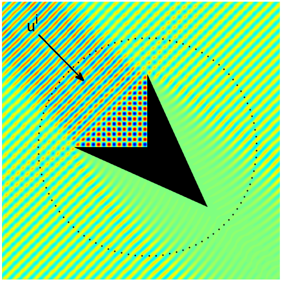
For $k \geq k_0 > 0$, there exist constants $C, \tau > 0$, such that

$$\left\| \frac{\partial u}{\partial n} - v_N \right\|_{L^2(\Gamma)} \leq C k^{5/2} \mathrm{e}^{-p\tau},$$

Total number of degrees of freedom $N = O\left(p^2\right)$.

Again, we can provably achieve any required accuracy with N growing like $\log^2 k$ as $k \to \infty$, rather than like k, as for a standard BEM.

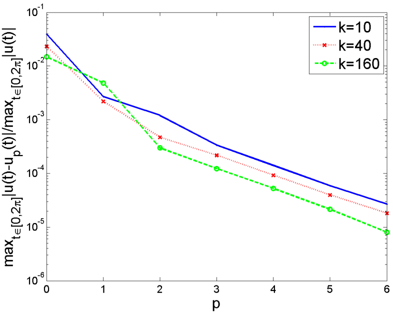




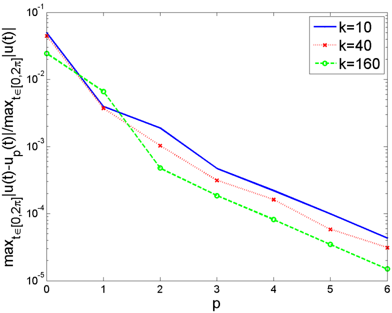
Partial illumination

Re-reflections

Relative max. error on circle in domain



Partial illumination



Re-reflections

3D screen (Hargreaves, Hewett, Lam, Langdon 2015)

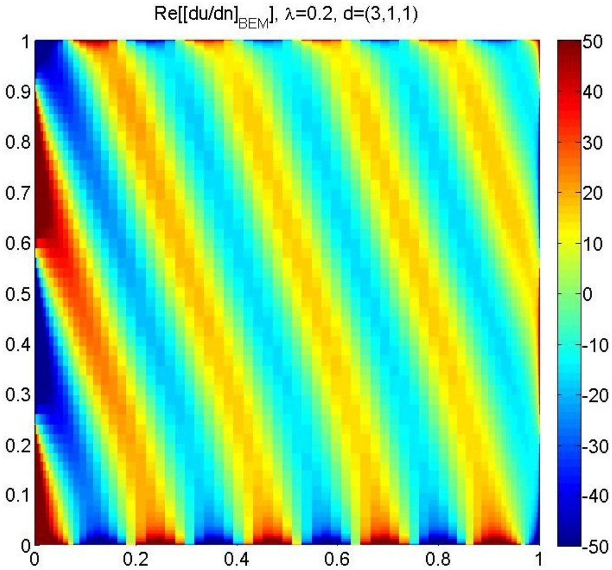
Recall ansatz:

$$\phi(x) \approx V_0(x,k) + \sum_{m=1}^{M} V_m(x,k)\mathrm{e}^{ik\phi_m(x)}$$

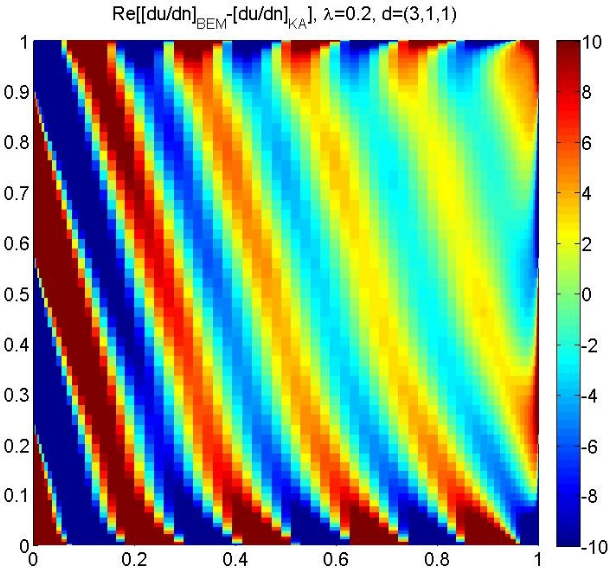
Leading order behaviour is much more complicated than for 2D

- Much harder to identify M and ϕ_m, $m = 1, \ldots, M$, so that corresponding amplitudes V_m are not oscillatory.
- "Edge waves" and "corner waves", diffracted by edges and corners respectively, travel in many directions across surface of screen.
- These waves are rediffracted infinitely often by the other edges and corners of the screen, taking a different direction of travel after each rediffraction.

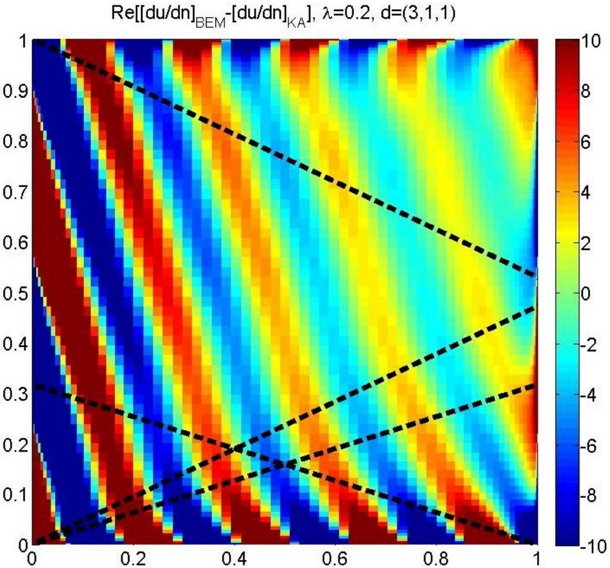
Solution behaviour



Solution behaviour without leading order

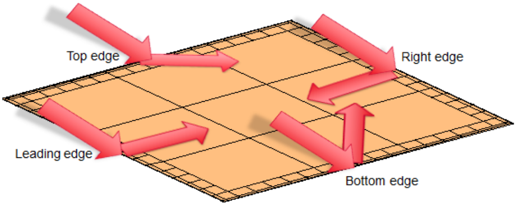


Shadow boundaries associated with edge waves



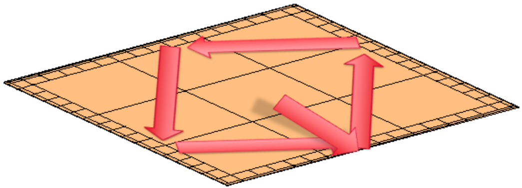
Hybrid approximation space

- Subtract leading order oscillatory behaviour (incident field).
- Small conventional elements around the rim (to represent singular behaviour at edge).
- Large hybrid elements in the centre; basis functions are plane waves multiplied by polynomial basis functions (order p).
- Phase functions on hybrid elements correspond to first order diffraction directions ("edge plane waves").

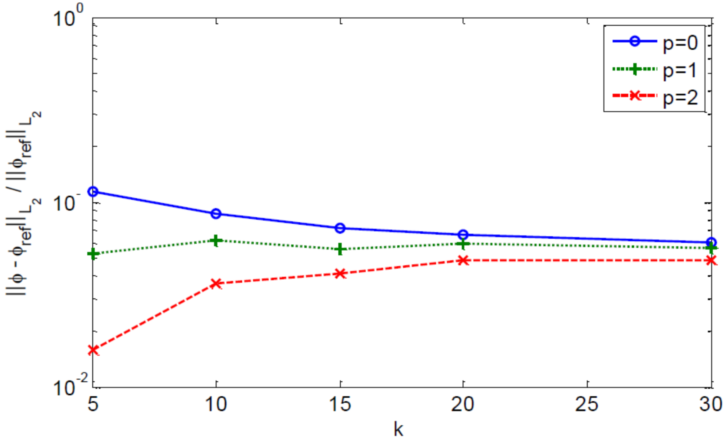


Hybrid approximation space

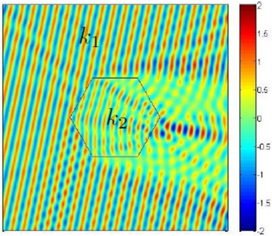
- Subtract leading order oscillatory behaviour (incident field).
- Small conventional elements around the rim (to represent singular behaviour at edge).
- Large hybrid elements in the centre; basis functions are plane waves multiplied by polynomial basis functions (order p).
- Phase functions on hybrid elements correspond to first order diffraction directions ("edge plane waves"). Also, reflections of EPWs.



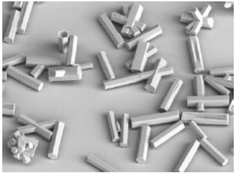
Preliminary numerical results

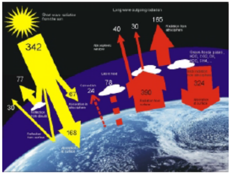


Scattering by penetrable obstacles (Groth, Hewett, Langdon)



Motivating application from Met Office: scattering by ice crystals in cirrus clouds

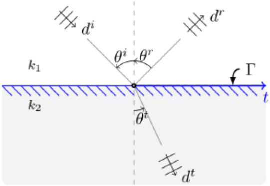




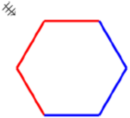
Challenge: infinitely many phases to consider, even for a convex scatterer!

Geometrical optics

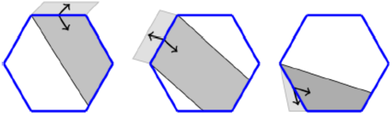
Rays refract according to Snell's law:



GO computed by beam tracing:



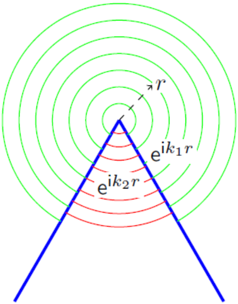
Incident wave

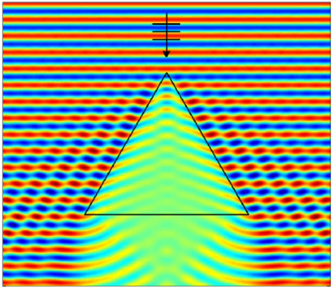


Primary beams from first reflection/refraction event

Diffraction by a penetrable wedge

Look to this canonical problem for information about the diffracted field.
There is no known exact or asymptotic solution, however we only require the phase information which we may glean from heuristics based on the impenetrable case.



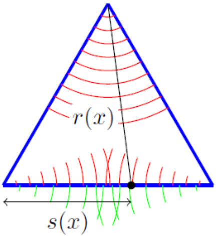


Heuristically, the GTD approximation should contain components of the form

$$D_1(\theta)e^{ik_1 r} \quad \text{and} \quad D_2(\theta)e^{ik_2 r}.$$

HNA approximation space

- There should be infinitely many phases due to reflections within the scatterer.
- **Simplify**: neglect internal reflections of diffracted waves.



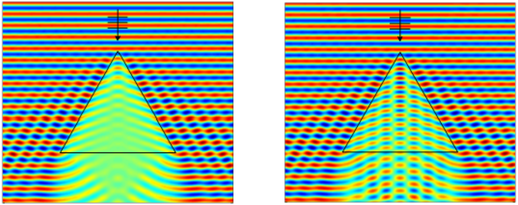
We approximate v on the bottom side (and other sides similarly) as

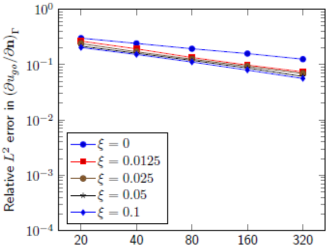
$$\begin{aligned} v(x,k) \quad \approx \quad & v_0(x,k) + v_1^+(x)\mathrm{e}^{\mathrm{i}k_1 s(x)} + v_2^+(x)\mathrm{e}^{\mathrm{i}k_2 s(x)} + v_1^-(x)\mathrm{e}^{-\mathrm{i}k_1 s(x)} \\ & + v_2^-(x)\mathrm{e}^{-\mathrm{i}k_2 s(x)} + v^r \mathrm{e}^{\mathrm{i}k_2 r(x)}, \end{aligned}$$

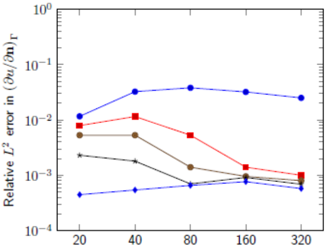
where $v_m^\pm$ and v^r are piecewise polynomials on overlapping meshes.

Numerical results: triangle

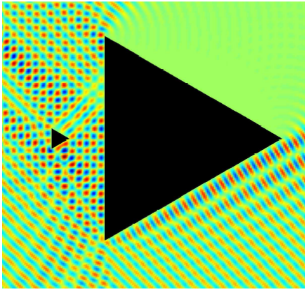
- Scattering by a triangle of refractive index $\mu = 1.5 + \xi\mathrm{i}$ for $\xi = 0.1, 0.05, 0.025, 0.0125, 0$. Number of DOF fixed at 205
- Compare accuracy with that of GO



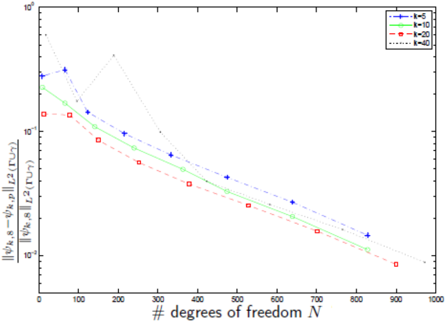




Multiple scattering configurations (with Gibbs, Langdon, Moiola)



Scattering configuration



Summary

- Standard BEM (e.g. BEM++) is error controllable and adaptable, but cost grows with frequency;
- Asymptotic methods are fast, but inaccurate when frequency is not sufficiently large;
- HNA BEM combines best features of BEM and asymptotic methods, but is limited to certain classes of problems;
- Much more to be done to extend method as a computational tool to wider geometries - see open problems session
- Much deep mathematics needed to prove error estimates more broadly, especially in 3D - see open problem session

Further reading:

C-W, Graham, Langdon & Spence, *Acta Numerica* **21** (2012), pp. 89–305.

C-W & Langdon, *Acoustic scattering: high frequency boundary element ...* in Unified transform for BVPs: applications and advances, A S Fokas & B Pelloni (eds.), SIAM, 2015, pp. 181–226.