Coherent Structures and Shocks in Periodic Nonlinear Maxwell Equations

Gideon Simpson
Joint with Dmitry E. Pelinovsky (McMaster),
Michael I. Weinstein (Columbia)

Department of Mathematics
Drexel University

July 15, 2016

Shock Inhibition of Hyperbolic Equations

Classical Regularizations

- Diffusive,

$$v_t + vv_x = \mu v_{xx}$$

- Dispersive,

$$v_t + vv_x + \alpha v_{xxx} = 0$$

Shock Inhibition of Hyperbolic Equations

Classical Regularizations

- Diffusive,

$$v_t + v v_x = \mu v_{xx}$$

- Dispersive,

$$v_t + v v_x + \alpha v_{xxx} = 0$$

Regularizations for Maxwell (Zero Dispersion Limit)

$$\partial_t^2 \left(n(z)^2 E + \chi E^3 \right) = \partial_z^2 E$$

Shock Inhibition of Hyperbolic Equations

Classical Regularizations

- Diffusive,

$$v_t + vv_x = \mu v_{xx}$$

- Dispersive,

$$v_t + vv_x + \alpha v_{xxx} = 0$$

Regularizations for Maxwell (Zero Dispersion Limit)

$$\partial_t^2 \left(n(z)^2 E + \chi E^3 \right) = \partial_z^2 E$$

Dispersion from Heterogeneity in Nonlinear Medium

- Can the heterogeneity of a medium inhibit shocks?
- Can it create (stable) localized states?

Spatial Variations & Dispersion

Periodically Varying System of Conservation Laws

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}(x, \mathbf{v}) = 0$$
$$\mathbf{f}(x + \mathbb{P}, \mathbf{v}) = \mathbf{f}(x, \mathbf{v})$$

Spatial Variations & Dispersion

Periodically Varying System of Conservation Laws

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}(x, \mathbf{v}) = 0$$
$$\mathbf{f}(x + \mathbb{P}, \mathbf{v}) = \mathbf{f}(x, \mathbf{v})$$

Floquet-Bloch Theory

Linearizing about $\mathbf{v} = \mathbf{0}$,

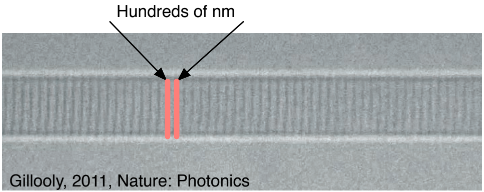
$$\partial_t \mathbf{V} + \partial_x \left(D_{\mathbf{v}} \mathbf{f}(x, \mathbf{0}) \mathbf{V} \right) = 0$$

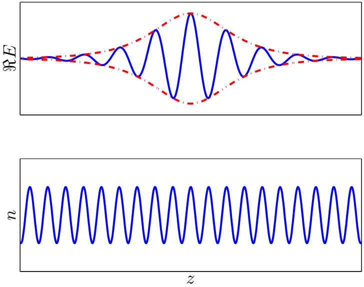
the solution is given by

$$\mathbf{V}(x,t) = \sum_{j=1}^{\infty} \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]} \langle \mathbf{W}_j(\cdot; k), \mathbf{V}_0 \rangle \, e^{-i\omega_j(k) t} \mathbf{W}_j(x; k) dk$$

Optical Fibers

Bragg Gratings – Zero Dispersion Point





$$\partial_t^2 \left(n(z)^2 E + \chi E^3 \right) = \partial_z^2 E$$

Optical Dispersion

Chromatic Dispersion (Lorentzian Model)

$$\partial_t^2 D = \partial_z^2 E$$
$$D = E + P$$
$$\omega_0^{-2} \partial_t^2 P + P - \phi P^3 = (n^2 - 1) E$$

Optical Dispersion

Chromatic Dispersion (Lorentzian Model)

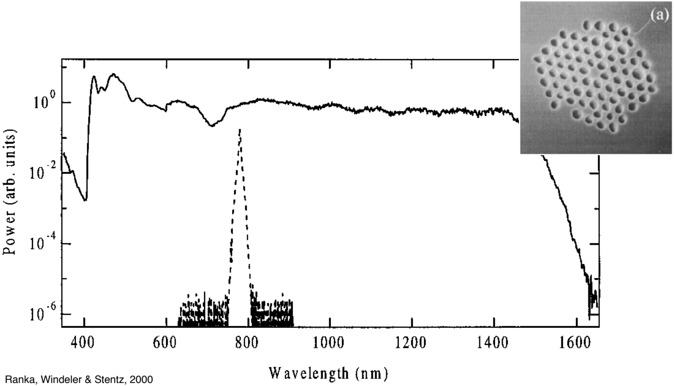
$$\partial_t^2 D = \partial_z^2 E$$
$$D = E + P$$
$$\omega_0^{-2} \partial_t^2 P + P - \phi P^3 = (n^2 - 1) E$$

Material Dispersion

$$\partial_t^2 \left(n(z)^2 E + \chi E^3 \right) = \partial_z^2 E$$

Optical Shocks

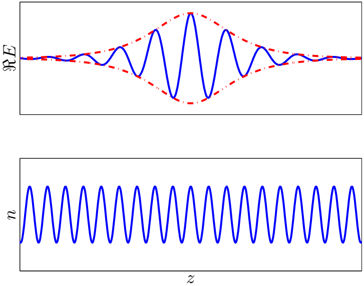
Experimental Generation of an "Optical Continuum" – Zero Dispersion Point



Ranka, Windeler & Stentz, 2000

- Cladding moves zero dispersion point moved to 767 nm
- Pulse initially concentrated about 790 nm
- Continuous output power spectrum from 390 – 1600 nm

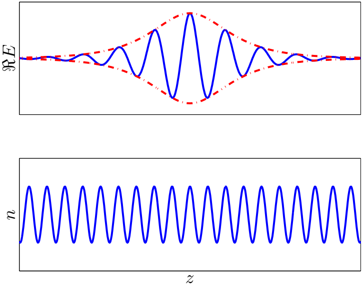
Maxwell & Coupled Mode Equations



Nonlinear Maxwell in a Periodic Medium

$$\partial_t^2 \left(n(z)^2 E + \chi E^3 \right) = \partial_z^2 E$$
$$n(z) = 1 + \epsilon N(z), \quad 0 < \epsilon \ll 1$$

Maxwell & Coupled Mode Equations



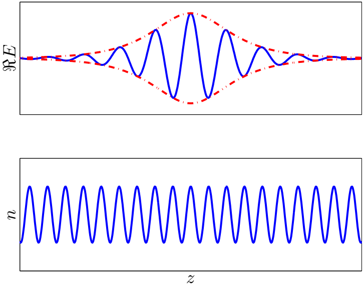
Nonlinear Maxwell in a Periodic Medium

$$\partial_t^2 \left(n(z)^2 E + \chi E^3 \right) = \partial_z^2 E$$
$$n(z) = 1 + \epsilon N(z), \quad 0 < \epsilon \ll 1$$

Ansatz

$$E = \epsilon^{\frac{1}{2}} [\mathcal{E}^+(\epsilon z, \epsilon t) e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t) e^{-i(z+t)} + \mathrm{c.c.} + \mathrm{O}(\epsilon)]$$

Maxwell & Coupled Mode Equations



Nonlinear Maxwell in a Periodic Medium

$$\partial_t^2 \left(n(z)^2 E + \chi E^3 \right) = \partial_z^2 E$$
$$n(z) = 1 + \epsilon N(z), \quad 0 < \epsilon \ll 1$$

Ansatz

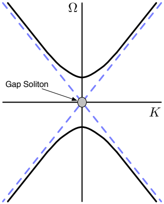
$$E = \epsilon^{\frac{1}{2}} [\mathcal{E}^+(\epsilon z, \epsilon t) e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t) e^{-i(z+t)} + \mathrm{c.c.} + \mathrm{O}(\epsilon)]$$

The Nonlinear Coupled Mode Equations (NLCME)

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = i N_2 \mathcal{E}^- + i\Gamma \left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2 \right) \mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i \bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$

Properties of NLCME

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = iN_2 \mathcal{E}^- + i\Gamma \left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2 \right) \mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$

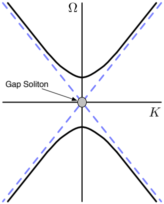


- Dispersive, $\Omega^2 = K^2 + |N_2|^2$,

Properties of NLCME

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = iN_2 \mathcal{E}^- + i\Gamma \left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2 \right) \mathcal{E}^+,$$

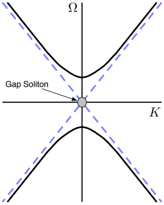
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$



- Dispersive, $\Omega^2 = K^2 + |N_2|^2$,
- Possess explicit, gap solitons (Aceves–Wabnitz ’89),

Properties of NLCME

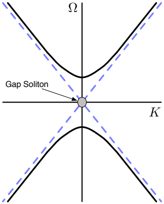
$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = iN_2 \mathcal{E}^- + i\Gamma \left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2 \right) \mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$



- Dispersive, $\Omega^2 = K^2 + |N_2|^2$,
- Possess explicit, gap solitons (Aceves–Wabnitz '89),
- Rigorously formulated as an approximation to nonlinear Maxwell + chromatic dispersion by Goodman, Weinstein & Holmes '01,

Properties of NLCME

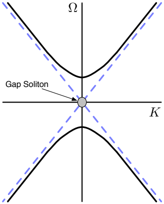
$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = i N_2 \mathcal{E}^- + i\Gamma \left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2 \right) \mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i \bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$



- Dispersive, $\Omega^2 = K^2 + |N_2|^2$,
- Possess explicit, gap solitons (Aceves–Wabnitz '89),
- Rigorously formulated as an approximation to nonlinear Maxwell + chromatic dispersion by Goodman, Weinstein & Holmes '01,
- GWP in H^1– No Shocks,

Properties of NLCME

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = iN_2 \mathcal{E}^- + i\Gamma \left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2 \right) \mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$



- Dispersive, $\Omega^2 = K^2 + |N_2|^2$,
- Possess explicit, gap solitons (Aceves–Wabnitz '89),
- Rigorously formulated as an approximation to nonlinear Maxwell + chromatic dispersion by Goodman, Weinstein & Holmes '01,
- GWP in H^1– No Shocks,
- Mathematically inconsistent, *i.e.* letting $\tilde{\mathcal{E}}$ be the correction,

$$\left(\partial_t^2 - \partial_z^2\right) \tilde{\mathcal{E}} = \left(\mathcal{E}^+\right)^3 e^{3i(z-t)} + \left(\mathcal{E}^-\right)^3 e^{-3i(z+t)} + \ldots$$

Secular growth in t

Simulating Nonlinear Maxwell

- Conservation law form of Maxwell,

$$\partial_t \begin{pmatrix} D \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E(D,z) \end{pmatrix} = 0$$

$$D = n(z)^2 E + \epsilon \chi E^3, \quad n(z) = 1 + \epsilon N(z), \quad N(z) = N(z + 2\pi)$$

Simulating Nonlinear Maxwell

- Conservation law form of Maxwell,

$$\partial_t \begin{pmatrix} D \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E(D,z) \end{pmatrix} = 0$$

$$D = n(z)^2 E + \epsilon \chi E^3, \quad n(z) = 1 + \epsilon N(z), \quad N(z) = N(z + 2\pi)$$

- Finite volume methods, `http://www.clawpack.org`,

Simulating Nonlinear Maxwell

- Conservation law form of Maxwell,

$$\partial_t \begin{pmatrix} D \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E(D,z) \end{pmatrix} = 0$$

$$D = n(z)^2 E + \epsilon\chi E^3, \quad n(z) = 1 + \epsilon N(z), \quad N(z) = N(z + 2\pi)$$

- Finite volume methods, `http://www.clawpack.org`,
- Requires an entropy condition for the Riemann Problem,

Simulating Nonlinear Maxwell

- Conservation law form of Maxwell,

$$\partial_t \begin{pmatrix} D \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E(D,z) \end{pmatrix} = 0$$

$$D = n(z)^2 E + \epsilon \chi E^3, \quad n(z) = 1 + \epsilon N(z), \quad N(z) = N(z + 2\pi)$$

- Finite volume methods, http://www.clawpack.org,
- Requires an entropy condition for the Riemann Problem,
- Correct entropy condition for Maxwell?; diffusive regularization

Simulating Nonlinear Maxwell

- Conservation law form of Maxwell,

$$\partial_t \begin{pmatrix} D \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E(D,z) \end{pmatrix} = 0$$

$$D = n(z)^2 E + \epsilon\chi E^3, \quad n(z) = 1 + \epsilon N(z), \quad N(z) = N(z + 2\pi)$$

- Finite volume methods, `http://www.clawpack.org`,
- Requires an entropy condition for the Riemann Problem,
- Correct entropy condition for Maxwell?; diffusive regularization
- Cubic nonlinearity: system is non-convex.

NLCME Soliton Data

Seed NLCME Soliton $(\mathcal{E}^+, \mathcal{E}^-)$ into Maxwell,

NLCME Soliton Data

Seed NLCME Soliton $(\mathcal{E}^+, \mathcal{E}^-)$ into Maxwell,

$$E(z,t) = \mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$
$$B(z,t) = -\mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$

NLCME Soliton Data

Seed NLCME Soliton $(\mathcal{E}^+, \mathcal{E}^-)$ into Maxwell,

$$E(z,t) = \mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$

$$B(z,t) = -\mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$

$$D = n(z)^2 E + \epsilon \chi E^3, \quad n(z) = 1 + \epsilon \frac{4}{\pi}\cos(2z)$$

NLCME Soliton Data

Seed NLCME Soliton $(\mathcal{E}^+, \mathcal{E}^-)$ into Maxwell,

$$E(z,t) = \mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$

$$B(z,t) = -\mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$

$$D = n(z)^2 E + \epsilon \chi E^3, \quad n(z) = 1 + \epsilon \frac{4}{\pi}\cos(2z)$$

(Loading...)

Side pulses absent from NLCME

NLCME Soliton Data

Seed NLCME Soliton $(\mathcal{E}^+, \mathcal{E}^-)$ into Maxwell,

$$E(z,t) = \mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$

$$B(z,t) = -\mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}$$

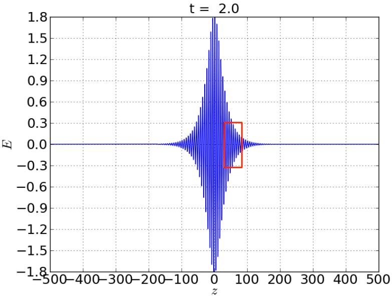
$$D = n(z)^2 E + \epsilon\chi E^3, \quad n(z) = 1 + \epsilon\frac{4}{\pi}\cos(2z)$$

(Loading...) (Loading...)

Side pulses absent from NLCME Periodicity turned off

Carrier Shocks

Zoom of Maxwell Simulation with NLCME data

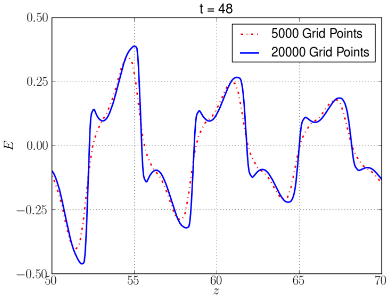


(Loading...)

- Violation of the monochromatic slowly varying envelope approximation essential to NLCME

Carrier Shocks

Resolution Comparison



- Violation of the monochromatic slowly varying envelope approximation essential to NLCME

Remarks on Maxwell Simulations

- Simulations reveal disagreement between NLCME and Maxwell

Remarks on Maxwell Simulations

- Simulations reveal disagreement between NLCME and Maxwell
- Nonlinear Maxwell can preserve localized structures of "prepared" data

Remarks on Maxwell Simulations

- Simulations reveal disagreement between NLCME and Maxwell
- Nonlinear Maxwell can preserve localized structures of "prepared" data
- Shocks show violation of the monochromatic slowly varying envelope approximation essential to NLCME

Remarks on Maxwell Simulations

- Simulations reveal disagreement between NLCME and Maxwell
- Nonlinear Maxwell can preserve localized structures of "prepared" data
- Shocks show violation of the monochromatic slowly varying envelope approximation essential to NLCME
- Motivates searching for:
    - A refined approximation
    - Solitons
    - Methods for shock inhibition

Revised Asymptotic Expansion

Hunter–Keller 83, Majda–Rosales 84, Hunter–Majda–Rosales 86,...

Generalized Ansatz

$$\partial_t \begin{pmatrix} n(z)^2 E + \epsilon \chi E^3 \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E \end{pmatrix} = 0$$

$$\partial_t \mathbf{G}(\mathbf{u}, \mathbf{z}) + \partial_z \mathbf{H}(\mathbf{u}) = 0.$$

$$\begin{pmatrix} E \\ B \end{pmatrix} = \mathbf{u}(z,t) = \mathbf{u}^{(0)}(z,t,Z,T) + \epsilon \mathbf{u}^{(1)}(z,t,Z,T) + \epsilon^2 \mathbf{u}^{(2)}(z,t,Z,T) + \dots.$$

Revised Asymptotic Expansion

Hunter–Keller 83, Majda–Rosales 84, Hunter–Majda–Rosales 86,...

Generalized Ansatz

$$\partial_t \begin{pmatrix} n(z)^2 E + \epsilon \chi E^3 \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E \end{pmatrix} = 0$$

$$\partial_t \mathbf{G}(\mathbf{u}, \mathbf{z}) + \partial_z \mathbf{H}(\mathbf{u}) = 0.$$

$$\begin{pmatrix} E \\ B \end{pmatrix} = \mathbf{u}(z,t) = \mathbf{u}^{(0)}(z,t,Z,T) + \epsilon \mathbf{u}^{(1)}(z,t,Z,T) + \epsilon^2 \mathbf{u}^{(2)}(z,t,Z,T) + \dots .$$

Leading Order

$$\mathbf{u}^{(0)} = E^{+}(z-t, Z, T)\mathbf{r}_{+} + E^{-}(z+t, Z, T)\mathbf{r}_{-},$$

$$E^{(0)} = E^{+}(z-t, Z, T) + E^{-}(z+t, Z, T)$$

Revised Asymptotic Expansion

Hunter–Keller 83, Majda–Rosales 84, Hunter–Majda–Rosales 86,...

Generalized Ansatz

$$\partial_t \begin{pmatrix} n(z)^2 E + \epsilon\chi E^3 \\ B \end{pmatrix} + \partial_z \begin{pmatrix} -B \\ -E \end{pmatrix} = 0$$

$$\partial_t \mathbf{G}(\mathbf{u}, \mathbf{z}) + \partial_z \mathbf{H}(\mathbf{u}) = 0.$$

$$\begin{pmatrix} E \\ B \end{pmatrix} = \mathbf{u}(z,t) = \mathbf{u}^{(0)}(z,t,Z,T) + \epsilon \mathbf{u}^{(1)}(z,t,Z,T) + \epsilon^2 \mathbf{u}^{(2)}(z,t,Z,T) + \ldots.$$

Leading Order

$$\mathbf{u}^{(0)} = E^+(z-t,Z,T)\mathbf{r}_+ + E^-(z+t,Z,T)\mathbf{r}_-,$$

$$E^{(0)} = E^+(z-t,Z,T) + E^-(z+t,Z,T)$$

NLCME ansatz,

$$E^{(0)} = \mathcal{E}^+(Z,T)e^{i(z-t)} + \mathcal{E}^-(Z,T)e^{-i(z+t)}$$

Effective Equations

Simpson & Weinstein 2011, MMS

NLGO Closure: Sublinear Growth in Correction

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L \left\| \mathbf{u}^{(1)} \right\| (t) dt = 0.$$

Effective Equations

Simpson & Weinstein 2011, MMS

NLGO Closure: Sublinear Growth in Correction

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L \left\|\mathbf{u}^{(1)}\right\|(t)dt = 0.$$

Integro-Differential equations for $E^{\pm}(\phi, Z, T)$

$$\begin{aligned}\partial_T E^+ + \partial_Z E^+ &= \partial_\phi \left\langle N(\phi_+ + s)E^-(\phi_+ + 2s\right\rangle_s \\ &\quad + \Gamma\partial_\phi \left[\frac{1}{3}\left(E^+\right)^3 + E^+\left\langle \left(E^-\right)^2\right\rangle\right], \\ \partial_T E^- - \partial_Z E^- &= -\partial_\phi \left\langle N(\phi_- - s)E^+(\phi_- - 2s)\right\rangle_s \\ &\quad - \Gamma\partial_\phi \left[\left\langle\left(E^+\right)^2\right\rangle E^- + \frac{1}{3}\left(E^-\right)^3\right]\end{aligned}$$

Effective Equations

Simpson & Weinstein 2011, MMS

NLGO Closure: Sublinear Growth in Correction

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L \left\|\mathbf{u}^{(1)}\right\|(t)dt = 0.$$

Integro-Differential equations for $E^{\pm}(\phi, Z, T)$

$$\begin{aligned}\partial_T E^+ + \partial_Z E^+ &= \partial_\phi \left\langle N(\phi_+ + s)E^-(\phi_+ + 2s\right\rangle_s \\ &\quad + \Gamma\partial_\phi\left[\frac{1}{3}\left(E^+\right)^3 + E^+\left\langle\left(E^-\right)^2\right\rangle\right], \\ \partial_T E^- - \partial_Z E^- &= -\partial_\phi \left\langle N(\phi_- - s)E^+(\phi_- - 2s)\right\rangle_s \\ &\quad - \Gamma\partial_\phi\left[\left\langle\left(E^+\right)^2\right\rangle E^- + \frac{1}{3}\left(E^-\right)^3\right]\end{aligned}$$

$$\phi_\pm = z \mp t,$$

Effective Equations

Simpson & Weinstein 2011, MMS

NLGO Closure: Sublinear Growth in Correction

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L \left\| \mathbf{u}^{(1)} \right\| (t) dt = 0.$$

Integro-Differential equations for $E^{\pm}(\phi, Z, T)$

$$\begin{aligned}
\partial_T E^+ + \partial_Z E^+ &= \partial_\phi \left\langle N(\phi_+ + s) E^-(\phi_+ + 2s \right\rangle_s \\
&\quad + \Gamma \partial_\phi \left[\frac{1}{3} \left(E^+\right)^3 + E^+ \left\langle \left(E^-\right)^2 \right\rangle \right], \\
\partial_T E^- - \partial_Z E^- &= -\partial_\phi \left\langle N(\phi_- - s) E^+(\phi_- - 2s) \right\rangle_s \\
&\quad - \Gamma \partial_\phi \left[\left\langle \left(E^+\right)^2 \right\rangle E^- + \frac{1}{3} \left(E^-\right)^3 \right]
\end{aligned}$$

$$\phi_\pm = z \mp t, \quad \langle f \rangle_s = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(s) ds$$

Periodicity & the extended Nonlinear Coupled Mode Equations (xNLCME)

Simpson & Weinstein 2011, MMS

Periodically Varying Index of Refraction

$$N(z) = N(z + 2\pi), \Rightarrow N(z) = \sum N_p e^{ipz}, \quad N_0 = 0$$

Periodicity & the extended Nonlinear Coupled Mode Equations (xNLCME)

Simpson & Weinstein 2011, MMS

Periodically Varying Index of Refraction

$$N(z) = N(z + 2\pi), \Rightarrow N(z) = \sum N_p e^{ipz}, \quad N_0 = 0$$

Harmonic Decomposition – xNLCME

$$E^{\pm}(\phi, Z, T) = \sum E_p^{\pm}(Z, T) e^{ip\phi}.$$

Periodicity & the extended Nonlinear Coupled Mode Equations (xNLCME)

Simpson & Weinstein 2011, MMS

Periodically Varying Index of Refraction

$$N(z) = N(z+2\pi), \Rightarrow N(z) = \sum N_p e^{ipz}, \quad N_0 = 0$$

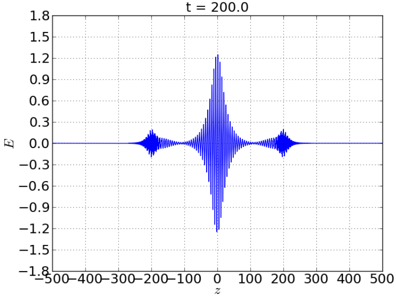
Harmonic Decomposition – xNLCME

$$E^{\pm}(\phi, Z, T) = \sum E_p^{\pm}(Z,T) e^{ip\phi}.$$

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \mathrm{i}p\frac{\Gamma}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum \left|E_q^-\right|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \mathrm{i}p\frac{\Gamma}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum \left|E_q^+\right|^2\right) E_p^-\right]$$

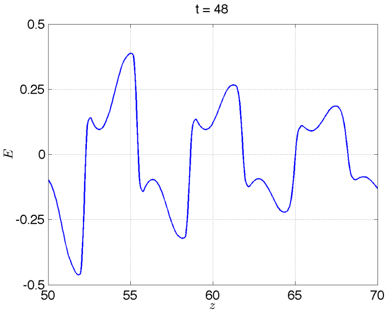
Inclusion of third harmonic ($E^{\pm}_{\pm 3}$), resolves side pulses



Maxwell

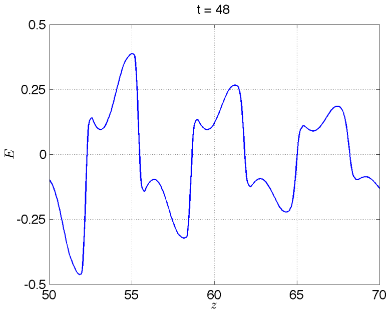
(Loading...)

Higher harmonics suggest shock formation

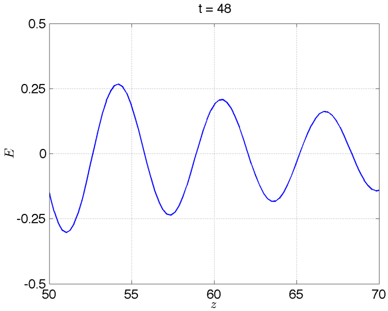


Maxwell

Higher harmonics suggest shock formation

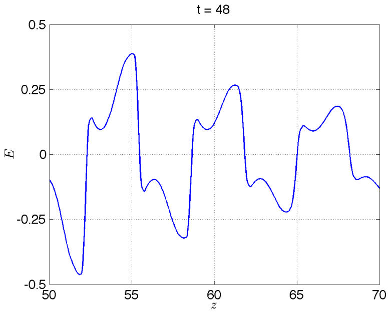


Maxwell

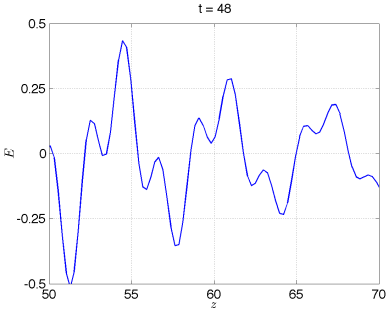


Odd modes $|p| \leq 2$

Higher harmonics suggest shock formation

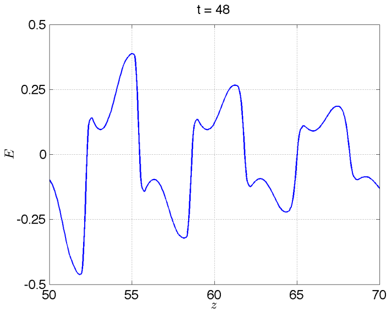


Maxwell

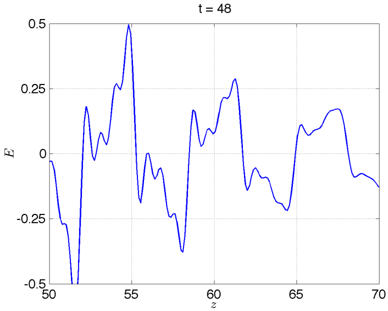


Odd modes $|p| \leq 4$

Higher harmonics suggest shock formation

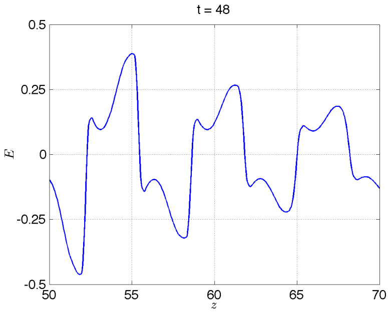


Maxwell

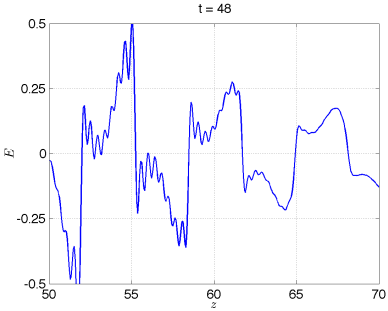


Odd modes $|p| \leq 8$

Higher harmonics suggest shock formation



Maxwell



Odd modes $|p| \leq 16$

Spectral Gaps in xNLCME

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i} p N_{2p} E_p^- + \mathrm{i} p \frac{\Gamma}{3} \left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3 \left(\sum \left| E_q^- \right|^2 \right) E_p^+ \right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i} p \bar{N}_{2p} E_p^+ + \mathrm{i} p \frac{\Gamma}{3} \left[\sum E_q^- E_r^- E_{p-q-r}^- + 3 \left(\sum \left| E_q^+ \right|^2 \right) E_p^- \right]$$

Spectral Gaps in xNLCME

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i} p N_{2p} E_p^-$$
$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i} p \bar{N}_{2p} E_p^+$$

$$\begin{pmatrix} E_p^+ \\ E_p^- \end{pmatrix} = e^{ip(KZ-\Omega T)} \begin{pmatrix} 1 \\ \gamma \end{pmatrix}$$

Decoupled Dispersion Relations: $\Omega^2 = K^2 + |N_{2p}|^2$

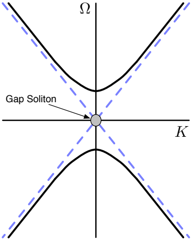
Spectral Gaps in xNLCME

$$\begin{aligned} \partial_T E_p^+ + \partial_Z E_p^+ &= \mathrm{i}pN_{2p}E_p^- \\ \partial_T E_p^- - \partial_Z E_p^- &= \mathrm{i}p\bar{N}_{2p}E_p^+ \end{aligned}$$

$$\begin{pmatrix} E_p^+ \\ E_p^- \end{pmatrix} = e^{ip(KZ-\Omega T)} \begin{pmatrix} 1 \\ \gamma \end{pmatrix}$$

Decoupled Dispersion Relations: $\Omega^2 = K^2 + |N_{2p}|^2$

$$\Omega_0 \equiv \inf_{p \in Z_{\mathrm{odd}}} |N_{2p}|$$



Spectral Gaps in xNLCME

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^-$$
$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+$$

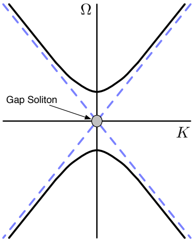
$$\begin{pmatrix} E_p^+ \\ E_p^- \end{pmatrix} = e^{ip(KZ-\Omega T)} \begin{pmatrix} 1 \\ \gamma \end{pmatrix}$$

Decoupled Dispersion Relations: $\Omega^2 = K^2 + |N_{2p}|^2$

$$\Omega_0 \equiv \inf_{p \in Z_{\mathrm{odd}}} |N_{2p}|$$

$\Omega_0 > 0$ Gaps present in all p; $N(z)$ is rough, *i.e.* a periodic array of delta functions

$\Omega_0 = 0$ Gaps vanish in p; $N(z)$ more regular



Design of $N(z)$ could permit or inhibit soliton components

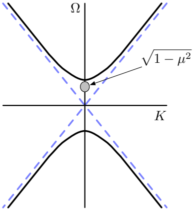
Dirac Delta Medium

"NLS" System

Band Edge

Assume $N_{2p} = 1$ for all p,

$$E_p^{\pm}(Z,T) = \pm \mu e^{-ip\Omega T} U_p(\mu Z) + \mathrm{O}(\mu^2)$$
$$\Omega = \sqrt{1-\mu^2}, \quad 0 < \mu \ll 1.$$



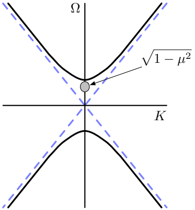
Dirac Delta Medium

"NLS" System

Band Edge

Assume $N_{2p} = 1$ for all p,

$$E_p^{\pm}(Z,T) = \pm\mu e^{-ip\Omega T} U_p(\mu Z) + \mathrm{O}(\mu^2)$$
$$\Omega = \sqrt{1-\mu^2}, \quad 0 < \mu \ll 1.$$



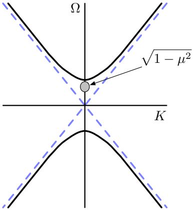
Dirac Delta Medium

"NLS" System

Band Edge

Assume $N_{2p} = 1$ for all p,

$$E_p^{\pm}(Z,T) = \pm \mu e^{-ip\Omega T} U_p(\mu Z) + \mathrm{O}(\mu^2)$$
$$\Omega = \sqrt{1-\mu^2}, \quad 0 < \mu \ll 1.$$



Leading Order – ∞ Many Coupled Stationary Modes

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \Gamma \left(3U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r}\right) = \mathrm{O}(\mu).$$

Artificial Continuation Parameter

xNLS System

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \Gamma \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

Artificial Continuation Parameter

xNLS System

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \Gamma \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

xNLS$^\epsilon$ System

$$U_p''(\zeta) - p^2 U_p + 6 p^2 \Gamma U_p^3 + \tfrac{2}{3} p^2 \epsilon \Gamma \left(3 U_p {\sum}' |U_q|^2 + {\sum}' U_q U_r U_{p-q-r} \right) = 0$$

Artificial Continuation Parameter

xNLS System

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \Gamma \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

xNLS$^\epsilon$ System

$$U_p''(\zeta) - p^2 U_p + 6p^2 \Gamma U_p^3 + \tfrac{2}{3} p^2 \epsilon \Gamma \left(3 U_p {\sum}' |U_q|^2 + {\sum}' U_q U_r U_{p-q-r} \right) = 0$$

- At $\epsilon = 0$, decoupled, monochromatic, solitons are solutions

$$U_p^{\epsilon=0}(\zeta) = \frac{1}{\sqrt{3\Gamma}} \operatorname{sech}(p\zeta)$$

Artificial Continuation Parameter

xNLS System

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \Gamma \left(3U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

xNLS$^\epsilon$ System

$$U_p''(\zeta) - p^2 U_p + 6p^2 \Gamma U_p^3 + \tfrac{2}{3} p^2 \epsilon \Gamma \left(3U_p {\sum}' |U_q|^2 + {\sum}' U_q U_r U_{p-q-r} \right) = 0$$

- At $\epsilon = 0$, decoupled, monochromatic, solitons are solutions

$$U_p^{\epsilon=0}(\zeta) = \frac{1}{\sqrt{3\Gamma}} \mathrm{sech}(p\zeta)$$

- Will they persist for $\epsilon > 0$? – Polychromatic Solitons

Persistence of Monochromatic Solitons

Theorem (Pelinovsky, S. & Weinstein, SIADS 2012)

Let $\mathcal{U}$ *have* $\mathcal{U}_{\pm 1} =$ *a soliton, and* $\mathcal{U}_{p \neq \pm 1} = 0$*.* $\exists \epsilon_0 > 0$ *such that for* $0 < \epsilon < \epsilon_0$*, a unique localized state,* U^ϵ *exists, close to* $\mathcal{U}$*:*

$$\left\{ \sum_{p \in \mathbb{Z}_{\mathrm{odd}}} \int_{\mathbb{R}} (p^2 + \xi^2)^s \left| \widehat{\mathcal{U}}_p - \widehat{U}_p^\epsilon \right|^2 d\xi \right\}^{1/2} = \|\mathcal{U} - U^\epsilon\|_{X^s} \leq C\epsilon$$

for $s > 1$*.*

Persistence of Monochromatic Solitons

Theorem (Pelinovsky, S. & Weinstein, SIADS 2012)

Let $\mathcal{U}$ have $\mathcal{U}_{\pm 1}$ = a soliton, and $\mathcal{U}_{p \neq \pm 1} = 0$. $\exists \epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, a unique localized state, U^ϵ exists, close to $\mathcal{U}$:

$$\left\{ \sum_{p \in \mathbb{Z}_{\mathrm{odd}}} \int_{\mathbb{R}} (p^2 + \xi^2)^s \left| \widehat{\mathcal{U}}_p - \widehat{U}^\epsilon_p \right|^2 d\xi \right\}^{1/2} = \|\mathcal{U} - U^\epsilon\|_{X^s} \leq C\epsilon$$

for $s > 1$.

- Proof by Lyapunov-Schmidt reduction of modes $U_{p \neq \pm 1}$ on to $U_{\pm 1}$, followed by implicit function theorem
- Generalizes to any finite, but not infinite, collection of decoupled solitons comprising $\mathcal{U}$
- Multicomponent, Polychromatic, Solitons

Band Edge Solitons in xNLCME

Theorem (Pelinovsky, S. & Weinstein, SIADS 2012)

If a soliton, U, exists in xNLS, then for small $\mu = \sqrt{1-\Omega^2}$ (sufficiently close to the band edge), a soliton exists in xNLCME:

$$E_p^+(Z,T) = e^{-ip\Omega T} A_p(Z), \quad E_p^- = e^{-ip\Omega T} B_p(Z)$$
$$\|A - \mu U(\mu\cdot,\cdot)\|_{X_s} + \|B + \mu U(\mu\cdot,\cdot)\|_{X_s} \leq C\mu^2$$

- Proof by implicit function theorem

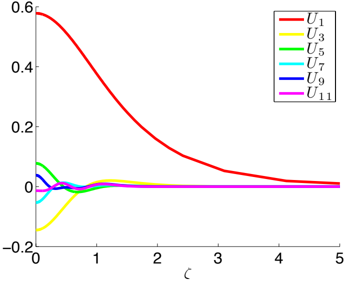
Direct Solution of Truncated NLS System

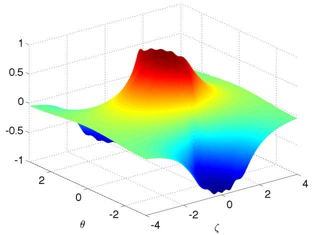
Polychromatic Solitons – Numerical Continuation to $\epsilon = 1$

NLS System

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3}p^2\Gamma\left(3U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r}\right) = 0$$

Alternating Signs & # of Nodes, $|p| \leq 12$





$$U(\theta,\zeta) = \sum U_p(\zeta) e^{ip\theta}$$

Persistence fo Coupled NLS Solitons in xNLCME

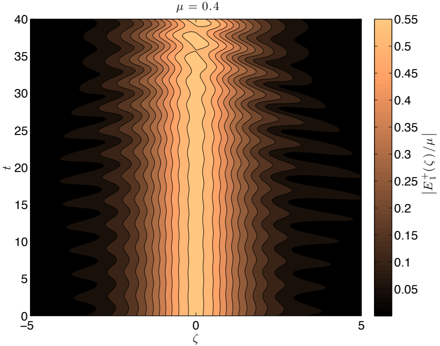
Resolves odd $|p| \leq 8$

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i} p N_{2p} E_p^- + \mathrm{i} p \frac{\Gamma}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i} p \bar{N}_{2p} E_p^+ + \mathrm{i} p \frac{\Gamma}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right]$$

$$E_p^{\pm}(Z,0) = \pm \mu U_p(\mu Z)$$

$$p = 1$$



Persistence fo Coupled NLS Solitons in xNLCME

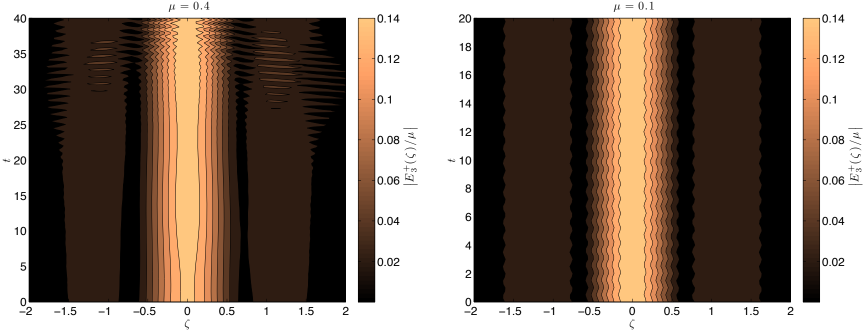
Resolves odd $|p| \leq 8$

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \mathrm{i}p\frac{\Gamma}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum \left|E_q^-\right|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \mathrm{i}p\frac{\Gamma}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum \left|E_q^+\right|^2\right) E_p^-\right]$$

$$E_p^{\pm}(Z,0) = \pm\mu U_p(\mu Z)$$

$$p = 3$$



Persistence fo Coupled NLS Solitons in xNLCME

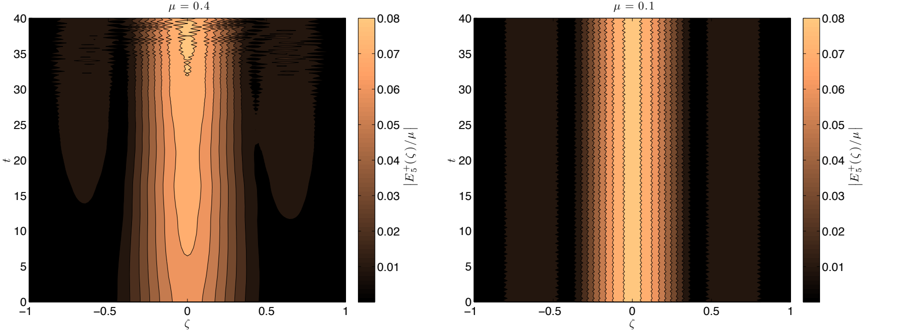
Resolves odd $|p| \leq 8$

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \mathrm{i}p\frac{\Gamma}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \mathrm{i}p\frac{\Gamma}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right]$$

$$E_p^\pm(Z,0) = \pm\mu U_p(\mu Z)$$

$$p = 5$$



Persistence fo Coupled NLS Solitons in xNLCME

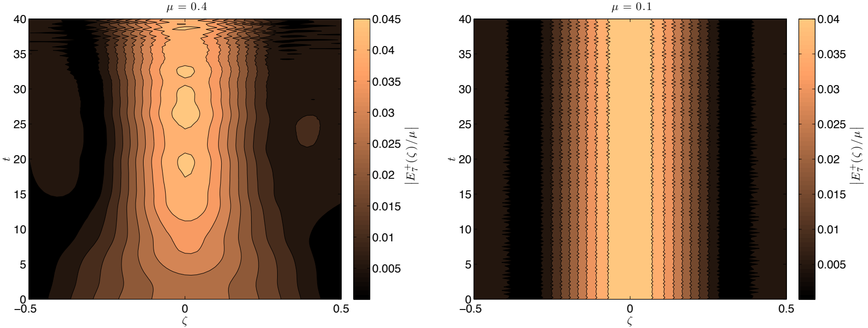
Resolves odd $|p| \leq 8$

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i} p N_{2p} E_p^- + \mathrm{i} p \frac{\Gamma}{3} \left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3 \left(\sum \left| E_q^- \right|^2 \right) E_p^+ \right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i} p \bar{N}_{2p} E_p^+ + \mathrm{i} p \frac{\Gamma}{3} \left[\sum E_q^- E_r^- E_{p-q-r}^- + 3 \left(\sum \left| E_q^+ \right|^2 \right) E_p^- \right]$$

$$E_p^{\pm}(Z, 0) = \pm \mu U_p(\mu Z)$$

$$p = 7$$



Remarks on Polychromatic Solitons

- Polychromatic structures exist in a neighborhood of the decoupled problem with small continuation parameter

Remarks on Polychromatic Solitons

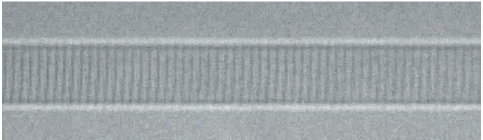
- Polychromatic structures exist in a neighborhood of the decoupled problem with small continuation parameter
- Computation permits us to extend the continuation parameter to $\epsilon = 1$, obtaining polychromatic solitons

Remarks on Polychromatic Solitons

- Polychromatic structures exist in a neighborhood of the decoupled problem with small continuation parameter
- Computation permits us to extend the continuation parameter to $\epsilon = 1$, obtaining polychromatic solitons
- Analytical challenge to prove existence for infinitely many modes

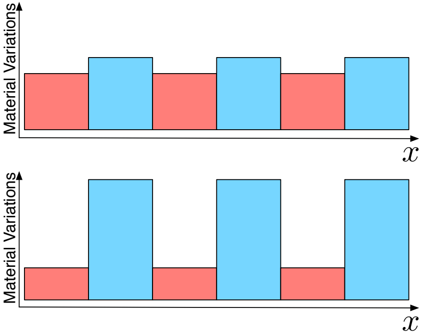
Remarks on Polychromatic Solitons

- Polychromatic structures exist in a neighborhood of the decoupled problem with small continuation parameter
- Computation permits us to extend the continuation parameter to $\epsilon = 1$, obtaining polychromatic solitons
- Analytical challenge to prove existence for infinitely many modes
- Tuning $N(z)$ should vary the polychromatic soliton structure


Gillooly, 2011, Nature: Photonics

High & Low Contrast Media

- Our simulations had spatial variations of $\mathrm{O}(\epsilon)$
- High contrast media is more strongly dispersive, LeVeque 2002, LeVeque–Yong 2003



Maxwell with High Contrast

Large Variations & Initial Conditions

$$\text{Constitutive Relation: } D(E,z) = (4 + 2\cos(2z))E + E^3$$
$$\text{Initial Condition: } D = 2\,\mathrm{sech}(\epsilon z), \quad B = -D$$

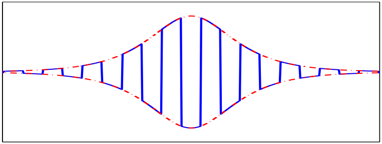
(Loading...)

Maxwell with High Contrast, Zooms

No Shocks

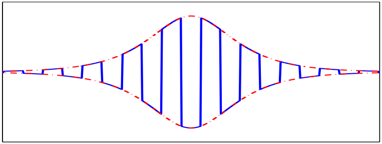
(Loading...) (Loading...)

Remarks & Open Problems



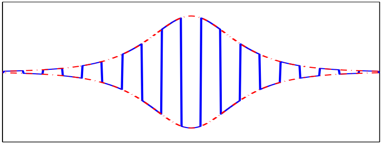
- Periodic Nonlinear Maxwell supports coherent, localized structures,

Remarks & Open Problems



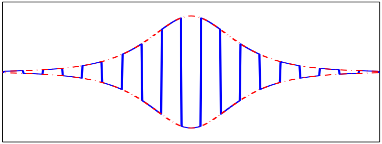
- Periodic Nonlinear Maxwell supports coherent, localized structures,
- xNLCME system corrects deficiencies of NLCME,

Remarks & Open Problems



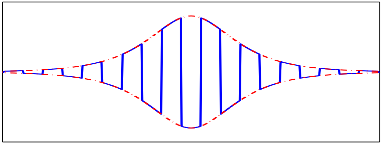
- Periodic Nonlinear Maxwell supports coherent, localized structures,
- xNLCME system corrects deficiencies of NLCME,
- Outstanding challenge to prove existence of solitons in (truncated) the system up to $\epsilon = 1$

Remarks & Open Problems



- Periodic Nonlinear Maxwell supports coherent, localized structures,
- xNLCME system corrects deficiencies of NLCME,
- Outstanding challenge to prove existence of solitons in (truncated) the system up to $\epsilon = 1$
- Carrier shocks present in low contrast case – what about high constrast?

Remarks & Open Problems



- Periodic Nonlinear Maxwell supports coherent, localized structures,
- xNLCME system corrects deficiencies of NLCME,
- Outstanding challenge to prove existence of solitons in (truncated) the system up to $\epsilon = 1$
- Carrier shocks present in low contrast case – what about high constrast?
- Applications for polychromatic solitons?

Acknowledgements

Collaborators Dmitry E. Pelinovsky & Michael I. Weinstein

Publications Simpson & Weinstein, MMS 2011, Pelinovsky, Simpson & Weinstein, SIADS 2012

Thanks D. Ketcheson, R.J. LeVeque, M. Pugh, R. Rosales, C. Sulem

Funding NSF, DOE, NSERC

http://www.math.drexel.edu/~simpson/

Drexel UNIVERSITY