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The solution of the Gevrey smoothing conjecture for the fully nonlinear homogeneous Boltzmann equation

Dirk Hundertmark
joint work with **Jean-Marie Barbaroux, Tobias Ried, Semjon Vugalter**

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INSTITUTE FOR ANALYSIS

Outline

- **Homogeneous Boltzmann equation**
  Boltzmann collision operator, singular angular collision kernel, Maxwell's weak formulation, weak solutions
- **Gevrey spaces**
  fractional heat equation, Gevrey spaces
- **Gevrey smoothing for the homogeneous Boltzmann equation**
  (Maxwellian molecules)
  main results, strategy of the proof
- **Commutator estimates**
  estimates in Fourier space, a Gronwall argument, the impossible imbedding $L^2 \to L^\infty$: extracting L^∞ bounds from L^2 bounds
- **Conclusion**
  The induction scheme

The Homogeneous Boltzmann Equation

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- The Boltzmann equation is one of the most important PDEs in kinetic theory, describing the dynamics of dilute gases
- In the *spatially homogeneous* setting, the time evolution of the distribution function $f : \mathbb{R}_+ \times \mathbb{R}^d \to [0, \infty)$ is governed by

$$\partial_t f = Q(f, f)$$

Boltzmann bilinear operator for Maxwellian molecules

$$Q(g, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \underbrace{b(\cos\theta)}_{\text{angular collision cross-section}} \left(g(v'_*) f(v') - g(v_*) f(v) \right) \, d\sigma dv_*$$

- Elastic collisions $\Rightarrow$ conservation of energy and momentum

$$v' + v'_* = v + v_*$$

$$|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$$

- Non-Maxwellian case: would have the term $|v - v_*|^\gamma b(\cos\theta)$, instead of $b(\cos\theta)$.

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More precisely

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with the parametrization

$$v' := \frac{v - v_*}{2} + \frac{|v - v_*|}{2}\sigma$$

$$v'_* := \frac{v - v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

$$\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$$

Convenient and important: By replacing *b* with a symmetrized version, if necessary, we can w.lo.g. assume $0 \leq \theta \leq \frac{\pi}{2}$.

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Does $Q(g, f)$ have a regularising effect on (weak) solutions?

Definition (Weak Solution)

- $f \in \mathcal{C}(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d))$, $f \geq 0$, $f(0, \cdot) = f_0$
- mass is conserved: $\int_{\mathbb{R}^d} f \, \mathrm{d}v = \int_{\mathbb{R}^d} f_0 \, \mathrm{d}v$
- kinetic energy is conserved: $\int_{\mathbb{R}^d} f \, v^2 \, \mathrm{d}v = \int_{\mathbb{R}^d} f_0 \, v^2 \, \mathrm{d}v$
- entropy is increasing: $H(f) = \int_{\mathbb{R}^d} f \log f \, \mathrm{d}v \leq \int_{\mathbb{R}^d} f_0 \log f_0 \, \mathrm{d}v$
- For all $\varphi \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{C}_0^\infty(\mathbb{R}^d))$ and for all $t \geq 0$ one has $\langle f(t, \cdot), \varphi(t, v) \rangle - \langle f_0, \varphi(0, \cdot) \rangle - \int_0^t \langle f(\tau, \cdot) \partial_\tau \varphi(\tau, \cdot) \rangle \, \mathrm{d}\tau = \int_0^t \langle Q(f, f)(\tau, \cdot), \varphi(\tau, \cdot) \rangle \, \mathrm{d}\tau$

Here $\langle f, g \rangle := \int_{\mathbb{R}^d} \overline{f(x)} g(x) \, \mathrm{d}x$ is the usual L^2 scalar product.

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Existence and Uniqueness of weak solutions: Arkeryd, Mischler, Goudon, Toscani, Villani, Wennberg,...

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Absence of Smoothing in the Grad Cut-off Case

- Simplification: Grad's angular cut-off assumption

$$\int_{\mathbb{S}^{d-1}} b(\cos\theta)\,\mathrm{d}\sigma = a < \infty$$

- Then one can split the collision operator

$$Q(g,f) = \underbrace{Q^+(g,f)}_{\text{gain}} - \underbrace{Q^-(g,f)}_{\text{loss}} = Q^+(g,f) - f(Lg)$$

where $Lg = a\int_{\mathbb{R}^d} g(v)\,\mathrm{d}v$.

Duhamel Formula

$$f(t,v) = \mathrm{e}^{-\int_0^t Lf(\tau,v)\,\mathrm{d}\tau} f_0(v) + \int_0^t \mathrm{e}^{-\int_s^t Lf(\tau,v)\,\mathrm{d}\tau} \underbrace{Q^+(f,f)}_{\text{smoothing}}(s,v)\,\mathrm{d}s$$

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$\Longrightarrow$ **Propagation of regularity and singularities!**

Long-range interactions: singular angular collision kernels

- The situation is totally different if the angular collision kernel has a non-integrable singularity for small collision angles (*grazing collisions*)
- We will consider the following type of singularity

$$\sin^{d-2}\theta\, b(\cos\theta) \sim \frac{\kappa}{\theta^{1+2\nu}} \quad \theta \to 0$$

for some $\kappa > 0$ and $0 < \nu < 1$. NOT integrable near 0!

- Additional assumption: $\int_0^{\frac{\pi}{2}} \sin^{d-2}\theta\,(1-\cos\theta)\, b(\cos\theta)\, \mathrm{d}\theta = m_b < \infty$. I.e., b is not too bad away from $\cos\theta = 1$. (Finite momentum transfer)
- As soon as one has long–range interactions between the particles, b will have a singularity at 1.

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Observation: $Q(g, f)$ behaves like a singular integral operator with a leading term similar to a fractional Laplacian $(-\Delta)^\nu$.

Quantitatively, this is expressed by the coercivity,

$$\langle f, -Q(g, f)\rangle \geq c_g \langle f, (-\Delta)^\nu f\rangle - l.o.t$$

E.g., Alexandre, Desvillettes, Villani, Wennberg. In terms of compactness properties already earlier in some work of Lions.

Intermezzo: Fractional Heat Equation

- Fractional heat equation ($\nu > 0$)

$$\begin{cases} \partial_t u + (-\Delta)^\nu u &= 0 \\ u\big|_{t=0} &= u_0 \in L^1(\mathbb{R}^d) \end{cases}$$

- in Fourier space

$$\widehat{u}(t,\xi) = \mathrm{e}^{-t|\xi|^{2\nu}} \widehat{u_0}(\xi) \quad \text{with} \quad \widehat{u_0} \in L^\infty(\mathbb{R}^d),$$

so there exists a finite constant $M > 0$ such that

$$\sup_{t>0} \sup_{\xi \in \mathbb{R}^d} \mathrm{e}^{t|\xi|^{2\nu}} |\widehat{u}(t,\xi)| \leq M < \infty.$$

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Gevrey Spaces

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Let $\alpha > 0$. $f \in C^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ belongs to the Gevrey class $G^\alpha(\mathbb{R}^d)$, if there exists ϵ_0, $M > 0$ such that

$$\left\| \xi \mapsto e^{\epsilon_0 \langle \xi \rangle^{\frac{1}{\alpha}}} |\widehat{f}(\xi)| \right\|_{L^2(\mathbb{R}^d)} \leq M < \infty. \qquad \left(\langle \xi \rangle = \sqrt{1 + |\xi|^2} \right)$$

- $\alpha = 1$ real analytic functions C^ω
- $0 < \alpha < 1$ ultra-analytic functions
- $\alpha > 1$ Gevrey-α functions

Gevrey spaces *interpolate between* C^∞ *and* C^ω

Heat equation $\partial_t u + (-\Delta)^\nu u = 0$ with initial condition in $L^1(\mathbb{R}^d)$

$\Rightarrow$ solution $u(t) \in G^{\frac{1}{2\nu}}(\mathbb{R}^d)$ for $t > 0$ (and not better).

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Gevrey Smoothing Conjecture

Conjecture

Any weak solution of the non-cutoff homogeneous Boltzmann equation with a singular cross section kernel of order ν and with initial datum in $L^1_2(\mathbb{R}^d) \cap L\log L(\mathbb{R}^d)$, i.e., finite mass, energy and entropy, belongs to the Gevrey class $G^{\frac{1}{2\nu}}(\mathbb{R}^d)$ for strictly positive times.

That is, the homogeneous non-cutoff Boltzmann equation for Maxwellian molecules enjoys the same smoothing properties as the fractional heat equation.

In particular, if $\nu \geq \frac{1}{2}$ the solution should become instantaneously analytic.

Since $\nu < 1$ can be very close to 1, one might even have nearly Gaussian decay of $\widehat{f}$.

E.g., Desvillettes-Wennberg 2004.

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Known results

- Existence of Gevrey regular solutions for nice, in particular, Gevrey, initial conditions (Ukai 1984)
- Propagation of Gevrey regularity (Desvillettes-Furioli-Terraneo 2009).
- H^∞ smoothing (Alexandre-El Safadi 2004, Morimoto-Ukai-Xu-Yang 2009)
- Several results for the linearized Boltzmann equation (Morimoto-et-al 2009, Xu, Lerner-Morimoto-Pravda-Starov-Xu 2014 (radially symmetric)).
- Similar results for the Kac equation, under some higher moments assumption (Lekrine-Xu 2009, Glangetas-Najeme 2013)

Main Results

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Theorem 1 [Barbaroux, 43 £, Ried, Vugalter (2015)]

Let $d \geq 2$. Let f be a weak solution of the Cauchy problem

$$\begin{cases} \partial_t f &= Q(f,f) \\ f|_{t=0} &= f_0 \end{cases} \tag{1}$$

with initial datum $0 \leq f_0 \in L\log L(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$.

Then, for all $0 < \alpha \leq \min\left\{\frac{\log(5/3)}{\log 2}, \nu\right\}$,

$$f(t,\cdot) \in G^{\frac{1}{2\alpha}}(\mathbb{R}^d)$$

for all $t > 0$.

In particular, since $\frac{\log(5/3)}{\log 2} \simeq 0.73696$, the weak solution is *real analytic* if $\nu = \frac{1}{2}$ and *ultra-analytic* if $\nu > \frac{1}{2}$ in *any dimension*.

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In particular, since $\frac{\log(5/3)}{\log 2} \simeq 0.73696$, the weak solution is *real analytic* if $\nu = \frac{1}{2}$ and *ultra-analytic* if $\nu > \frac{1}{2}$ in *any dimension*.

Main Results

Under slightly stronger assumptions on the kernel b (bounded on $[0, 1-\delta]$ $\forall\delta > 0$) we can improve this to

Theorem 2 [Barbaroux, 43 £, Ried, Vugalter (2015)]

For initial conditions $f_0 \geq 0$, $f_0 \in L\log L(\mathbb{R}^d) \cap L^1_m(\mathbb{R}^d)$ with an integer

$$m \geq \max\left(2, \frac{2^\nu - 1}{2 - 2^\nu}\right),$$

any weak solution of the Cauchy problem (1) belongs to the Gevrey class $G^{\frac{1}{2\nu}}(\mathbb{R}^d)$ for strictly positive times.

For $\nu \leq \log(9/5)/\log(2) \simeq 0,847996$ we have $m = 2$ and the theorem does not require anything except the physically reasonable assumptions of finite mass, energy, and entropy.

If $\log(9/5)/\log(2) < \nu < 1$ and $f_0 \in L\log L \cap L^1_2$, then we can still conclude that the solution is in $G^{\frac{\log 2}{2\log(9/5)}}$, in particular it is ultra-analytic.

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Strategy of the Proof

- By known H^∞-smoothing result we can assume $f_0 \in L^2(\mathbb{R}^d)$.
- Take growing weights $G(\eta) = e^{\beta t \langle \eta \rangle^{2\alpha}}$ and cutoff $\mathbb{1}_\Lambda(\eta) := \mathbb{1}_{|\eta| \le \Lambda}$ and set

$$G_\Lambda(t,\eta) := G(t,\eta)\mathbb{1}_\Lambda(\eta)$$

- Need to control the Fourier multiplier $\|G_\Lambda(t, D_v) f(t,\cdot)\|_{L^2}$ as $\Lambda \to \infty$.
- Take $\varphi(t,\cdot) := G_\Lambda(t, D_v) f(t,\cdot)$ as a test function in the weak formulation.

  After some technicalities, this yields the L^2 reformulation of the homogeneous Boltzmann equation

$$\frac{1}{2}\|G_\Lambda(t,D_v)f(t,\cdot)\|_{L^2}^2 - \frac{1}{2}\int_0^t \Big\langle f(\tau,\cdot), \Big(\partial_\tau G_\Lambda^2(\tau,D_v)\Big) f(\tau,\cdot)\Big\rangle \, \mathrm{d}\tau$$
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- Want to use the sub-elliptic estimate (coercivity) by Alexandre, Desvillettes, Villani, Wennberg [ADVW00]

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$\Longrightarrow$ **Need good estimates on the commutator**

$$\langle G_\Lambda Q(f, f) - Q(f, G_\Lambda f), G_\Lambda f\rangle$$

What if there were no commutator?

In this case,

$$\frac{1}{2}\|G_\Lambda(t,D_v)f(t,\cdot)\|_{L^2}^2 - \frac{1}{2}\int_0^t \left\langle f(\tau,\cdot), \left(\partial_\tau G_\Lambda^2(\tau,D_v)\right) f(\tau,\cdot)\right\rangle \mathrm{d}\tau$$
$$\leq \frac{1}{2}\|\mathbb{1}_\Lambda(D_v)f_0\|_{L^2}^2 - C_{f_0}\int_0^t \|G_\Lambda f(\tau,\cdot)\|_{H^\nu}^2 \,\mathrm{d}\tau + C\|f_0\|_{L^1_2}\int_0^t \|G_\Lambda f(\tau,\cdot)\|_{L^2}^2\,\mathrm{d}\tau.$$

Note

$$\partial_\tau G_\Lambda^2(\tau,\eta) = \partial_\tau e^{2\beta\tau\langle\eta\rangle^{2\alpha}}\mathbb{1}_\Lambda(\eta) = 2\beta\langle\eta\rangle^{2\alpha}G_\Lambda^2(\tau,\eta)$$

so

$$\int_0^t \left\langle f(\tau,\cdot), \left(\partial_\tau G_\Lambda^2(\tau,D_v)\right) f(\tau,\cdot)\right\rangle \mathrm{d}\tau \leq 2\beta\int_0^t \|G_\Lambda f(\tau,\cdot)\|_{H^\alpha}\,\mathrm{d}\tau$$

which, since $\nu \geq \alpha$, is controlled by the H^ν norm, just choose β small enough.

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with Gronwall's bound we conclude

$$\|G_\Lambda(t,D_v)f(t,\cdot)\|_{L^2}^2 \leq \|\mathbb{1}_\Lambda(D_v)f_0\|_{L^2}^2 e^{2Ct}$$

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i.e, have Gevrey smoothing.

Possible catch: We get at most an L^2 bound for $G_\Lambda(t,D_v)f(t,\cdot)$!

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Bound on the commutator

By Bobylev's identity

$$|\langle Q(f, G_\Lambda f) - G_\Lambda Q(f,f), G_\Lambda f\rangle|$$
$$\leq \int_{\mathbb{R}^d} \mathrm{d}\eta \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma\, b\left(\frac{\eta}{|\eta|}\cdot\sigma\right) |\widehat{f}(\eta^-)|\, |\widehat{f}(\eta^+)||G(\eta^+) - G(\eta)|G_\Lambda(\eta)|\widehat{f}(\eta)|$$

- Here $\eta^\pm = \frac{1}{2}(\eta \pm |\eta|\sigma)$
- Note $|\eta|^2 = |\eta^-|^2 + |\eta^+|^2$, because of the support assumption on b:

$$0 \leq |\eta^-| \leq |\eta^+| \quad \text{and} \quad \frac{|\eta|^2}{2} \leq |\eta^+|^2 \leq |\eta|^2.$$

- $b\left(\frac{\eta}{|\eta|}\cdot\sigma\right)$ has a non-integrable blow up when σ points into the direction of η, i.e., when η^+ is close to η, but then $G(\eta^+) - G(\eta)$ should be small (keep fingers crossed....).

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Bound on $G(\eta^+) - G(\eta)$

Let $\widetilde{G}(s) := e^{\beta t(1+s)^{\alpha/2}}$, $s = |\eta|^2$, and $s_+ := |\eta^+|^2$, $s_- := |\eta^-|^2$.
Then $s = s_+ + s_-$ and

$$0 \le G(\eta) - G(\eta^+) = \widetilde{G}(s) - \widetilde{G}(s_+) = \int_{s_+}^{s} \frac{d}{dr} e^{\beta t(1+r)^\alpha}\, \mathrm{d}r$$

$$\le 2\alpha\beta t(1+s^+)^{\alpha-1}(s-s^+)e^{\beta t(1+s)^\alpha}$$

since $s/2 \le s_+ \le s$

$$\le 2^{2-\alpha}\alpha\beta t(1+s)^{\alpha-1}(s-s_+)e^{\beta t(1+s)^\alpha}$$

since $(1+s)^\alpha = (1+s_-+s_+)^\alpha \le (1+s_-)^\alpha + (1+s_+)^\alpha$ (subadditivity)

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Bound on commutator II

So we get

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- Good news: The term $\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right)$ kills the singularity of b.
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- Bad news: The term $G(\eta^-)|\widehat{f}(\eta^-)|$ is potentially very strongly growing.

If we had $0 \leq t \leq T$ (which we can always assume) and if

$$G(\eta^-)|\widehat{f}(\eta^-)| \leq M$$

for some maybe large constant M, then one could conclude

$$\begin{aligned}
&|\langle Q(f, G_\Lambda f) - G_\Lambda Q(f,f), G_\Lambda f\rangle| \\
&\leq 2\alpha\beta t M \int_{\mathbb{R}^d} \mathrm{d}\eta \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma\, b\left(\tfrac{\eta}{|\eta|}\cdot\sigma\right)\left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) \\
&\qquad\qquad \times G_\Lambda(\eta^+)|\widehat{f}(\eta^+)|\, G_\Lambda(\eta)|\widehat{f}(\eta)|\, \langle\eta^+\rangle^{2\alpha} \\
&\lesssim \beta T M \|G_\Lambda f\|^2_{H^\alpha}.
\end{aligned}$$

By simply choosing β small enough we would conclude as before (without commutator) that

$$\|G_\Lambda(t, D_v) f(t,\cdot)\|^2_{L^2} \leq \|\mathbb{1}_\Lambda(D_v) f_0\|^2_{L^2} e^{2Ct}$$

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The impossible catch

You have to assume Gevrey in order to deduce Gevrey!

Even worse, the norms are incompatible: Need Gevrey on an L^∞ level in order to conclude Gevrey on an L^2 level.

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Why is H^∞ smoothing so much simpler?

If one assumes that the weight is polynomial, i.e., G is replaced by

$$M_\Lambda(t,\eta) := e^{\beta t \log<\eta>} \mathbb{1}_\Lambda(\eta)$$

then a similar calculation gives

$$H(\eta) - H(\eta^+) \lesssim \beta t 2^{\beta t} \left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) H(\eta^+)$$

(no $H(\eta^-)$ term) and the commutation error is bounded by

$$\begin{aligned}
&|\langle Q(f, H_\Lambda f) - H_\Lambda Q(f,f), H_\Lambda f\rangle| \\
&\lesssim \beta t 2^{\beta t} \int_{\mathbb{R}^d} \mathrm{d}\eta \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma\, b\left(\tfrac{\eta}{|\eta|}\cdot\sigma\right) \left(1 - \frac{|\eta^+|^2}{|\eta|^2}\right) |\widehat{f}(\eta^-)| \\
&\qquad \times H_\Lambda(\eta^+)|\widehat{f}(\eta^+)|\, H_\Lambda(\eta)|\widehat{f}(\eta)|\, \langle \eta^+\rangle^{2\alpha}
\end{aligned}$$

So there is no growing weight on the bad term $|\widehat{f}(\eta^-)|$, which can be simply controlled by

$$|\widehat{f}(\eta^-)| \le \|f\|_1 = 1.$$

Why is H^∞ smoothing so much simpler?

Then

$$
\begin{aligned}
&|\langle Q(f, H_\Lambda f) - H_\Lambda Q(f,f), H_\Lambda f\rangle| \\
&\lesssim \beta t 2^{\beta t} \|H_\Lambda f\|_{L^2}
\end{aligned}
$$

and, as before, Gronwall applies to get

$$
\|f\|_{H^\beta} = \lim_{\Lambda\to\infty} \|H_\Lambda f\|_{L^2}^2 \le \|f_0\|_{L^2} e^{A(\beta,t)}
$$

thus $f \in H^\beta$ for all $\beta > 0$.

The way out

Main observation: We always have $|\eta^-|^2 \leq |\eta|^2/2 \leq \Lambda^2/2$!

so the uniform bound above on the 'bad term' $G(\eta^-)|\widehat{f}(\eta^-)|$ is only needed on the ball of radius $\Lambda/\sqrt{2}$.

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So if

$$\sup_{|\zeta|\le\Lambda} G(\zeta)|\widehat{f}(\zeta)| \le M,$$

then the Gronwall argument, with Λ replaced by $\sqrt{2}\Lambda$ yields

$$\|G_{\sqrt{2}\Lambda}(t,D_v)f(t,\cdot)\|_{L^2}^2 \le \|\mathbb{1}_{\sqrt{2}\Lambda}(D_v)f_0\|_{L^2}^2 e^{2Ct} \le \|f_0\|_{L^2}^2 e^{2Ct}$$

and maybe this enables an inductive procedure?

Possible catch: Need to get uniform bounds from L^2 bounds. This is usually impossible ;-)

Possible good news: Need to get this uniform bounds only on smaller balls, in between Λ and $\sqrt{2}\Lambda$.

Can assume that $\widehat{f}$ is nice, at least $\widehat{f} \in C^2$ since $f \in L^1_2$.

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Lemma

Let $H \in \mathcal{C}^m(\mathbb{R}^n)$. Then there exists a constant $L_{m,n} < \infty$ (depending only on m, n, $\|H\|_{L^\infty(\mathbb{R}^n)}$ and, $\|D^m H\|_{L^\infty(\mathbb{R}^n)}$) such that

$$|H(x)| \leq L_{m,n} \left(\int_{Q_x} |H(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{m}{2m+n}}$$

where Q_x is a cube in $\mathbb{R}^n$ of side length 2, pointing away from x, with x being one of the corners.

It's proof is easy for $m = 1$ and much, much trickier for $m \geq 2$!

Proof (for $m = 1$).

In dimension $n = 1$, use that for $p \geq 1$,

- $|H(x)|^p - \int_x^{x+1} |H(y)|^p \, \mathrm{d}y = \int_x^{x+1} |H(x)|^p - |H(y)|^p \, \mathrm{d}y \leq p\|H'\|_{L^\infty(\mathbb{R})} \int_x^{x+1} |H(y)|^{p-1} \, \mathrm{d}y$
- Also $\int_x^{x+1} |H(y)|^p \, \mathrm{d}y \leq \|H\|_{L^\infty(\mathbb{R})} \int_x^{x+1} |H(y)|^{p-1} \, \mathrm{d}y$
- Put together, one has

$$|H(x)|^p \leq (p\|H'\|_{L^\infty(\mathbb{R})} + \|H\|_{L^\infty(\mathbb{R})}) \int_x^{x+1} |H(y)|^{p-1} \, \mathrm{d}y$$

Then iterate in each coordinate direction and choose $p = n + 2$.

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For $m > 1$: Kolmogorov-Landau inequality are used to improve exponent by using higher derivatives.

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Immediate Consequence

Since $\widehat{f} \in \mathcal{C}_b^2(\mathbb{R}^d)$ and G is radially increasing, we get

$$|\widehat{f}(\eta)| \leq L_{2,d}\left(\int_{Q_\eta} G(\xi)^{-2}G(\xi)^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{2}{4+d}}$$

$$\leq L_{2,d}G(\eta)^{-\frac{4}{4+d}}\left(\int_{Q_\eta} G(\xi)^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{2}{4+d}}$$

and thus

$$G(\eta)^{\frac{4}{4+d}}|\hat{f}(\eta)| \leq L_{2,d}\|G_{\sqrt{2}\Lambda}f\|_{L^2(\mathbb{R}^d)}^{\frac{4}{4+d}} \quad \text{for all} \quad |\eta| \leq \widetilde{\Lambda} = \frac{1+\sqrt{2}}{2}\Lambda$$

Good news: Uniform control of $G^{\frac{4}{4+d}}|\hat{f}|$ only with the help of $\|G_{\sqrt{2}\Lambda}f\|_{L^2}$.

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Improving the subadditivity

Recall the subaddivity

$$(1+s)^\alpha = (1+s_- + s_+)^\alpha \le (1+s_-)^\alpha + (1+s_+)^\alpha$$

which holds for all $s_-, s_+ \ge 0$, and it is sharp. So it cannot be improved.

But we do NOT need it for all $s_-, s_+ \ge 0$, we only need it for $0 \le s_- \le s_+$ and this gives room for improvement!
Indeed,

$$\begin{aligned}(1+s_- + s_+)^\alpha &= s_-^\alpha \left(1 + \frac{1+s_+}{s_-}\right)^\alpha \\ &= s_-^\alpha \left[\left(1+\frac{1+s_+}{s_-}\right)^\alpha - \left(\frac{1+s_+}{s_-}\right)^\alpha\right] + (1+s_+)^\alpha.\end{aligned}$$

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Now note for $0 < \alpha \leq 1$ the map $r \mapsto (1+r)^\alpha - r^\alpha$ is decreasing, so with $r = (1+s_+)/s_- \geq 1$, one has for all $0 \leq s_- \leq s_+$

$$\begin{aligned}(1+s_-+s_+)^\alpha &= s_-^\alpha\left[(1+r)^\alpha - r^\alpha\right] + (1+s_+)^\alpha \\ &\leq s_-^\alpha\left[2^\alpha - 1\right] + (1+s_+)^\alpha \\ &\leq \varepsilon(\alpha)(1+s_-)^\alpha + (1+s_+)^\alpha\end{aligned}$$

with $\varepsilon(\alpha) := 2^\alpha - 1 < 1$.

The Induction Scheme

Induction Hypothesis:

$$\mathrm{Hyp}_\Lambda(M): \quad \sup_{|\zeta|\le\Lambda} G(t,\zeta)^{\epsilon(\alpha)}|\widehat{f}(t,\zeta)| \le M$$

for all $t \in [0,T]$

Step 0: $\mathrm{Hyp}_\Lambda(M)$ is true for some suitably chosen Λ_0

Step 1:

$$\mathrm{Hyp}_\Lambda(M) \Rightarrow \|G_{\sqrt{2}\Lambda} f\|_{L^2} \le C \quad \text{via } \textit{Gronwall}.$$

Step 2 (*$L^2 \to L^\infty$ bound):* If $\epsilon(\alpha) = 2^\alpha - 1 \le \frac{4}{4+d}$, then

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So setting $\Lambda_n := \left(\frac{1+\sqrt{2}}{2}\right)^n \Lambda_0$, we can let $n \to \infty$ and see that

$$\sup_{\zeta \in \mathbb{R}^d} G(t,\zeta)^{\epsilon(\alpha)} |\widehat{f}(t,\zeta)| \leq M$$

for all $t \in [0, T]$, which gives the strong decay of $\widehat{f}(t, \cdot)$ for arbitrarily small $t > 0$.

Essential for this to work:

- M does not increase during the induction procedure!
- This can be accomplished by choosing β small enough at the very beginning. Trust me, :-)

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For some of the nice ;-) details, see

J.-M. Barbaroux, D. Hundertmark, T. Ried, S. Vugalter, **Gevrey smoothing for weak solutions of the fully nonlinear homogeneous Boltzmann and Kac equations without cutoff for Maxwellian molecules**, *October 2015*

References

R. Alexandre, A review of Boltzmann equation with singular kernels, *Kinetic and Related Models* **2** (2009), 551–646. http://dx.doi.org/10.3934/krm.2009.2.551.

R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg, Entropy dissipation and long-range interactions, *Archive for Rational Mechanics and Analysis* **152** (2000), 327–355. http://dx.doi.org/10.1007/s002050000083.

L. Desvillettes, About the use of Fourier transform for the Boltzmann equation, *Rivista di Matematica della Università di Parma (7)* **2*** (2003), 1–99. Available at http://www.rivmat.unipr.it/vols/2003-2s/indice.html.

C. Villani, A review of mathematical topics in collisional kinetic theory, in *Handbook of Mathematical Fluid Dynamics* (S. Friedlander and D. Serre, eds.), **Vol. 1**, Elsevier Science B.V., Amsterdam, 2002, pp. 71–305. http://dx.doi.org/10.1016/S1874-5792(02)80004-0.