Mathematics

A Qualitative Approach to Inverse Electromagnetic Scattering for Inhomogeneous Media

Fioralba Cakoni

Rutgers University
www.math.rutgers.edu/∼fc292/

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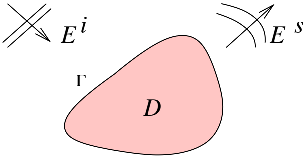
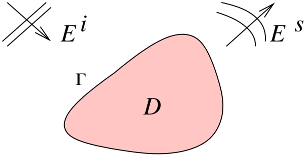
Scattering by an Inhomogeneous Media



$$
\begin{aligned}
\text{curl } E^s - i\omega\mu_0 H^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\text{curl } H^s - i\omega\epsilon_0 E^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\text{curl } E - i\omega\mu(x) H &= 0 && \text{in } D \\
\text{curl } H - (i\omega\epsilon(x) - \sigma(x) E &= 0 && \text{in } D \\
\nu \times E &= \nu \times (E^s + E^i) && \text{on } \partial D \\
\nu \times H &= \nu \times (H^s + H^i) && \text{on } \partial D \\
\lim_{|x|\to\infty} \left(\sqrt{\mu_0} H^s \times x - \sqrt{\epsilon_0}|x| E^s\right) &= 0 \\
\lim_{|x|\to\infty} \left(\sqrt{\epsilon_0} E^s \times x - \sqrt{\mu_0}|x| H^s\right) &= 0
\end{aligned}
$$

- E^i, H^i incident electro-magnetic field (satisfy the equations in the vacuum).
- ϵ_0 and μ_0 electric permittivity and magnetic permeability in the vacuum.
- $\epsilon(x)$, $\mu(x)$ and $\sigma(x)$ electric permittivity, magnetic permeability and conductivity in the homogeneity.

Scattering by an Inhomogeneous Media



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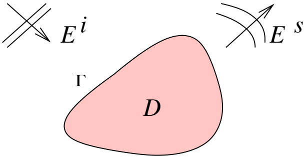
\lim_{|x| \to \infty} (\text{curl}\, E^s \times x - ik|x| E^s) &= 0
\end{aligned}
$$

- $k = \omega\sqrt{\epsilon_0 \mu_0}$ is the wave number.
- $N = \dfrac{\epsilon(x)}{\epsilon_0} + i\dfrac{\sigma(x)}{\omega\epsilon_0}$ (relative permittivity plus conductivity), positive definite 3×3 matrix valued function in $L^\infty(D)$
- $A = \dfrac{\mu(x)}{\mu_0}$ (relative permeability), positive definite 3×3 matrix valued function in $L^\infty(D)$

Scattering by an Inhomogeneous Media



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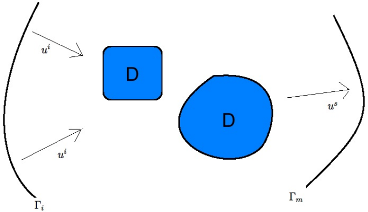
More generally A and N are such that:

- The scattering problem is well-posed.
- The corresponding (so-called) interior transmission problem is Fredholm.

Inverse Scattering Problem



Typical Data: From a knowledge of E^s measured on Γ_m, for several interrogating waves E^i situated on Γ_i and possibly for a range of frequencies

Problem 1: Reconstruct everything i.e. D, N and A.
Weak scattering approximation, optimization techniques . . .

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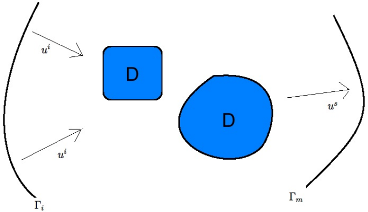

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Problem 1: Reconstruct everything i.e. D, N and A.
Weak scattering approximation, optimization techniques . . .

Problem 2: Obtain partial information such as the support D and estimates on N and A

Qualitative Methods

A class of methods for solving Problem 2 is known as Qualitative Methods

- Linear sampling method (Colton-Kirsch (1996)) . . . and Factorization method (Kirsch (1998)) . . .
- Singular sources method (Potthast (2001)) . . .
- Convex scattering support (Kusiak-Sylvester (2003), Griesmaier-Hanke-Sylvester (2013)) . . .
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F. Cakoni and D. Colton and P. Monk (2011), *The linear Sampling Method in Inverse Electromagnetic Scattering, CBMS-NSF, SIAM Publications, 80*.

A. Kirsch and N. Grinberg (2008), *The Factorization Method for Inverse Problems, Oxford University Press*.

F. Cakoni and D. Colton and H. Haddar (2016), *Inverse Scattering Theory and Transmission Eigenvalues, CBMS-NSF, SIAM Publications*.

Qualitative Methods

To fix the ideas we take a plane wave incident field

$$E^i(x, d, p) := ik(d \times p) \times de^{ikx \cdot d}$$

propagating in the direction $d \in \mathbb{S}^2$ with polarization $p \in \mathbb{R}^3$

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The scattered field E^s has the asymptotic behavior

$$E^s(x; d, p, k) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}; d, p, k) + O\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$ uniformly with respect $\hat{x} = x/|x|$.

$E_\infty(\hat{x}, d, p, k)$ is the far field pattern of the scattered field E^s.

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Scattering Data

$E_\infty(\hat{x}; d, p, k)$, for $d \in \mathbb{S}^2_i \subset \mathbb{S}^2$, $\hat{x} \in \mathbb{S}^2_m \subset \mathbb{S}^2$ and (possibly) $k \in [k_1,\ k_2]$.

Far Field Operator

The far field operator $F : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ is defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}; d, g(d), k) ds_d.$$

- Fg is the far field pattern of the scattered field corresponding to the incident field

$$E_g(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds_d \qquad g \in L^2_t(\mathbb{S}^2) \quad (g(\hat{x}) \cdot \hat{x} = 0)$$

  known as a electric Herglotz wave function .

- F is related to the scattering operator $\mathcal{S}$ by

$$\mathcal{S} = I + \frac{ik}{2\pi} F$$

Far Field Operator

Theorem

$F : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ is injective and has dense range if and only if there does not exist a nontrivial solution to the homogeneous interior transmission problem

$$\begin{aligned}
\text{curl curl } E_0 - k^2 E_0 = 0 \quad & \text{in} \quad D \\
\text{curl } A \,\text{curl } E - k^2 N E = 0 \quad & \text{in} \quad D \\
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\nu \times A \,\text{curl } E = \nu \times \text{curl } E_0 \quad & \text{on} \quad \partial D
\end{aligned}$$

such that $E_0 := E_g$ is an electric Herglotz wave function.

Values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem has non trivial solution are called transmission eigenvalues.

TE and Non-Scattering Frequencies

If k is a transmission eigenvalue and the eigenfunction E_0 that solves curl curl $E_0 - k^2 E_0 = 0$ in D can be extended outside D as a solution $\tilde{E}_0$ of the same equation, then the scattered field due to $\tilde{E}_0$ as an incident wave is identically zero.

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In general such an extension of E_0 does not exist. For example in the case of scalar Helmholtz equation corners always scatter!

BLASTEN-PÄIVÄRINTA-SYLVESTER (2013),*Comm. Math. Phys.*

PÄIVÄRINTA-SALO-VESALEINEN (2014), *Rev. Mat. Iberoamericana*

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Since electric Herglotz wave functions are dense in

$$\left\{U \in \mathbb{H}(D) : \text{ curl curl } U - k^2 U = 0 \quad \text{in } D\right\},$$

at a transmission eigenvalue it is possible to superimpose plane waves such that this superposition produces an arbitrarily small scattered field.

Determination of the Support

(Uniqueness)

If D_1, A_1, N_1 and D_2, A_2, N_2 give rise to the same far field data, i.e $E_\infty^{(1)}(\hat{x}; d, p, k) = E_\infty^{(2)}(\hat{x}; d, p, k)$, for $d \in \mathbb{S}_i^2 \subset \mathbb{S}^2$, $\hat{x} \in \mathbb{S}_m^2 \subset \mathbb{S}^2$, three linearly independent polarizations and fixed k, then $D_1 = D_2$.

CAKONI-COLTON (2003) - *Proc. Edinburgh Math. Soc. 46*

The conditions on A and N for the above theorem to be valid are those that guarantee that the transmission eigenvalue problem is a compact perturbation of an invertible operator.

In the scalar case using transmission eigenvalues uniqueness results for D with one incident wave are

HU-SALO-VESALAINEN (2016) - *SIAM J. Math. Anal. 48*

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HU-SALO-VESALAINEN (2016) - *SIAM J. Math. Anal. 48*

Problem

Determine the support D without any a priory knowledge of A and N

Linear Sampling Method

The linear sampling method is based on solving the far field equation

$$(Fg)(\hat{x}) = E_\infty(\hat{x}, z, q, k) \qquad \text{for } g \in L^2_t(\mathbb{S}^2)$$
$$\hat{x} \in \mathbb{S}^2,\ q, z \in \mathbb{R}^3 \text{ and } k \text{ fixed}$$

where $E_\infty(\hat{x}, z, q, k) := \dfrac{ik}{4\pi}(\hat{x} \times q) \times \hat{x} e^{-ik\hat{x}\cdot z}$.

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k is not a transmission eigenvalue, and let $g_\epsilon := g_{z,k,\epsilon,q}$ be such that

$$\|Fg_\epsilon - E_\infty(\hat{x}, z, q, k)\|_{L^2_t(\mathbb{S}^2)} < \epsilon$$

- for $z \in D$ there is a g_ϵ such that $\lim_{\epsilon\to 0} \|E_{g_\epsilon}\|_{\mathbb{H}(D)}$ exists
- for $z \notin D$ $\lim_{\epsilon\to 0} \|E_{g_\epsilon}\|_{\mathbb{H}(D)} = \infty$

The underlying mathematical tools are the 1) Fredholm property of the interior transmission problem and 2) approximation by E_g.

Linear Sampling Method

In practice, one solves the regularized far field equation

$$\inf_g \left\{ \|Fg - E_\infty(\hat{x}, z, q, k)\|^2_{L^2} + \alpha \|g\|^2_{L^2_t} \right\} \quad \text{or} \quad (\alpha I + F^* F)g = F^* E_\infty$$

Indicator Function

$$\mathbb{I}(z) = \frac{1}{\|g_{z,k,\alpha,q_1}\|} + \frac{1}{\|g_{z,k,\alpha,q_2}\|} + \frac{1}{\|g_{z,k,\alpha,q_3}\|}$$

$$\partial D \text{ is the surface } \mathbb{I}(z) = C$$

for some C that is a ad hoc value transitioning from small and large.

Question: Is the Tikhonov regularized solution $g_{z,k,\alpha,q}$ as $\alpha \to 0$ the one given by the above theoretical statement?

Factorization Method

The answer to this question led to the development of the Factorization Method.

For the case when $A = I$ and under the assumption that $\Re(N) > I$ or $\Re(N) < I$ uniformly on D

$$z \in D \Longleftrightarrow E_\infty(\hat{x}, z, q, k) \in Range\,(F_\#)^{1/2}$$

where $F_\# = |\Re(F)| + |\Im(F)|$

For Maxwell's equations the factorization method was done in

- A. Kirsch (2004) - *Inverse Problems, 20*
- A. Lechleiter (2009) - *Inverse Problems and Imaging, 3*

Factorization Method

- $F = H^* \mathbf{T} H$
- $\overline{R(H)} = \{U \in L^2(D) : \text{ curl curl } U - k^2 U = 0 \quad \text{in } D\}$
- $z \in D \Longleftrightarrow E_\infty(\hat{x}, z, q, k) \in \text{Range}(H^*)$
- **T** which is roughly the solution operator of the forward problem, must satisfy a list of properties that restrict the class of the problems where the factorization method can be justified.

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- $\mathbf{T}$ which is roughly the solution operator of the forward problem, must satisfy a list of properties that restrict the class of the problems where the factorization method can be justified.

Observation: If $\mathbf{T}$ was coercive then

$$|(Fg, g)_{L^2}| \sim \|Hg\|_{L^2(D)}$$

However this is not the case but one can use instead of F a different operator B known in terms of $B := H^* \mathbf{T}_b H$ with $\mathbf{T}_b$ coercive (in appropriate spaces). Then

$$J_\alpha(g, E_\infty^z) := \alpha\, |(Bg, g)_{L^2}| + \|Fg - E_\infty^z\|$$

Generalized Linear Sampling Method

GLSM rigorously characterizes D in terms of a minimizing sequence g_α of the functional $J_\alpha(\cdot, E_\infty)$.

L. AUDIBERT (2016), *Qualitative Methods for Heterogeneous Media, Ph.D Thesis, Ecole Polytechnique*.

For Maxwell's equations GLSM is discussed in the article by H. Haddar in

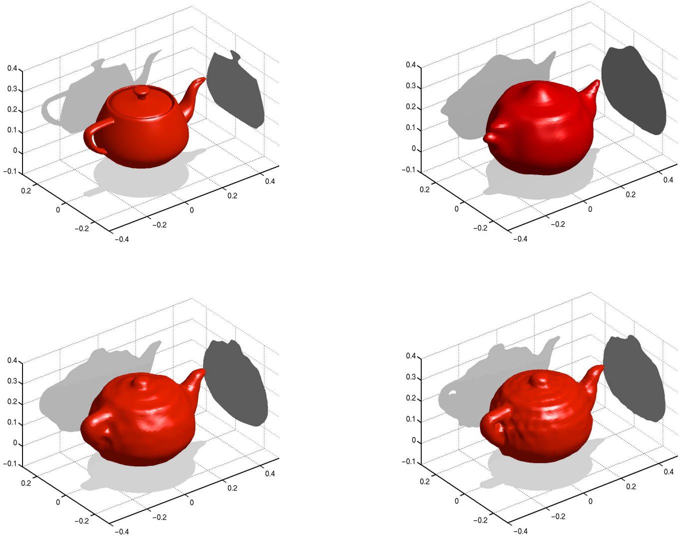
H. HADDAR, R. HIPTMAIER, P. MONK AND R. RODRIGUEZ (2015), *Computational Electromagnetism, CIME Foundation Subseries, Springer*.

Links between all this methods are discussed in

F. CAKONI AND D. COLTON AND H. HADDAR (2016), *Inverse Scattering Theory and Transmission Eigenvalues, CBMS-NSF, SIAM Publications*.

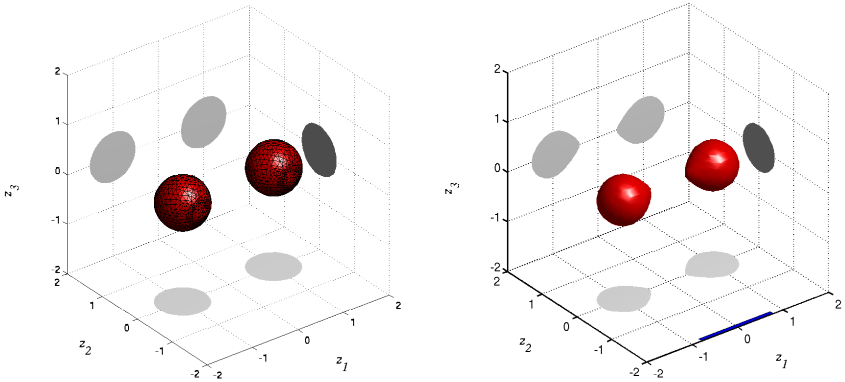
Shape Reconstruction

COLLINO-FARES-HADDAR (2003) – 252 directions, π/k (ka) are 0.224 (12), 0.112 (24) and 0.075 (42)

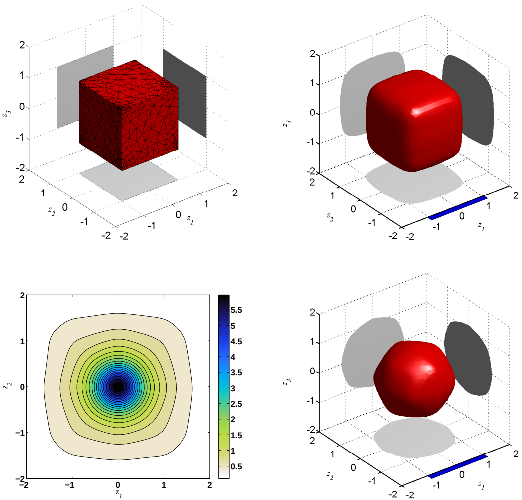


Shape Reconstruction

DUE TO P. MONK



Limited Aperture



Example from

Cakoni-Monk (2006) - *NMAA, Proceedings ENUMATH*

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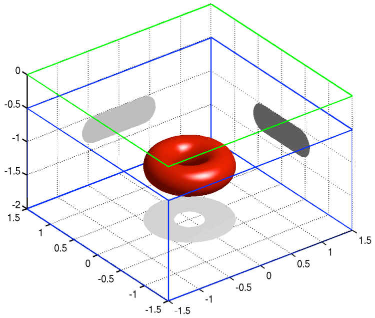
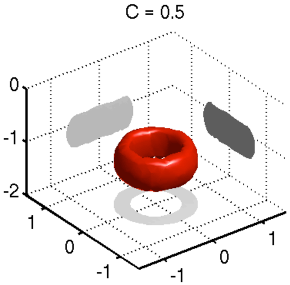


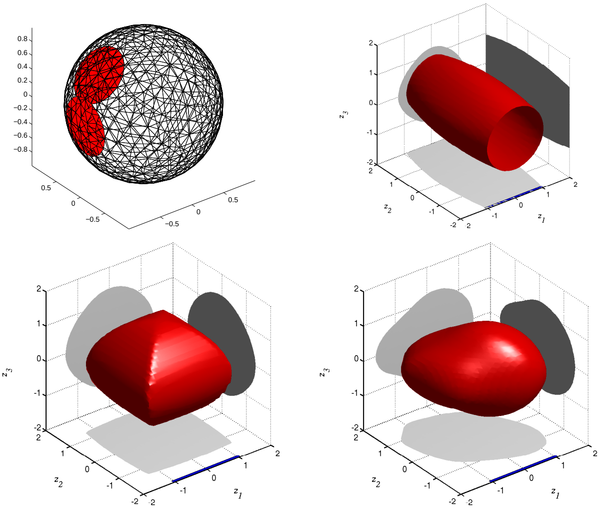


Inhomogeneous background. Example from

CAKONI-FARES-HADDAR (2006) - *Inverse Problems*

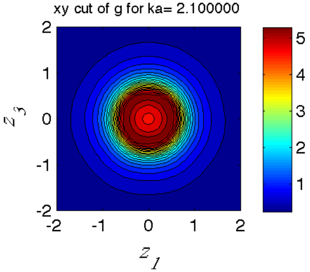
Limited Aperture

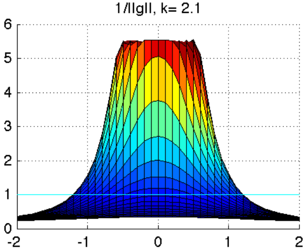
DUE TO P. MONK

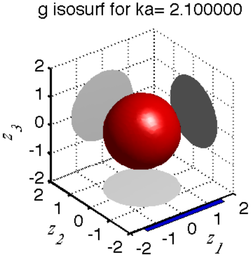


Examples of Reconstruction

$N = 16I$, k is not TE







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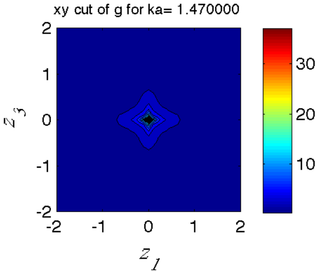
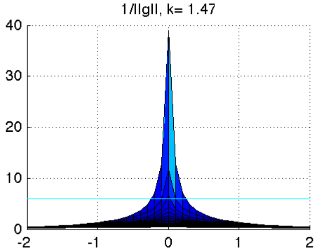
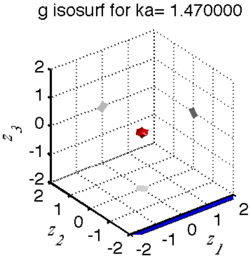






Transmission Eigenvalue Problem

Having determined the support D without knowing anything about the material properties we would like to get some information about the constitutive parameters A and N.

For this we appeal to the transmission eigenvalue problem:

$$\begin{aligned}
\text{curl curl } E_0 - k^2 E_0 &= 0 && \text{in} \quad D \\
\text{curl } A \text{ curl } E - k^2 N E &= 0 && \text{in} \quad D \\
\nu \times E &= \nu \times E_0 && \text{on} \quad \partial D \\
\nu \times A \text{ curl } E &= \nu \times \text{curl } E_0 && \text{on} \quad \partial D
\end{aligned}$$

Question: Can real transmission eigenvalues be determined from scattering data?

Linear Sampling Method

CAKONI-COLTON-HADDAR (2010) *C. R. Math. Acad. Sci. Paris*

We can use again the far field equation for $z \in D$

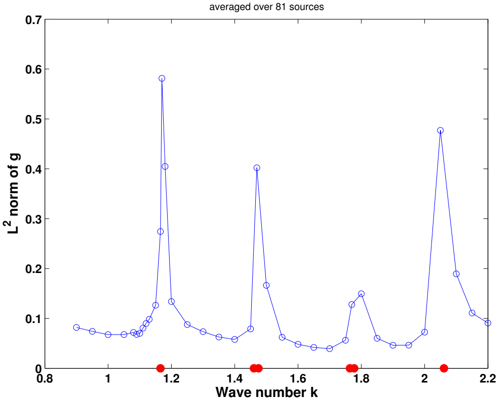
$$(Fg)(\hat{x}) = E_\infty(\hat{x}, z, q, k) \qquad \text{for } g \in L^2_t(\mathbb{S}^2) \text{ and } k \in [k_0,\ k_1]$$

Assume that A and N are such that the interior transmission problem is Fredholm, and let g_α be the regularize solution of the far field equation.

For any ball $B \subset D$ and all $z \in B$, $\|E_{g_\alpha}\|_{\mathbb{H}(D)}$ is bounded as $\alpha \to 0$ if and only if k is not a transmission eigenvalue

Computation of Real Transmission Eigenvalues

Results for an isotropic sphere of unit radius. DUE TO P. MONK.



$N = 16I$

Solving the far-field equation for several source points z inside the sphere gives obvious peaks at the first transmission eigenvalue. Red dots indicate exact transmission eigenvalues.

TE and the Far Field Operator

Real transmission eigenvalues can be characterized in terms of the eigenvalues of the far field operator $F_k : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$.

Lechleiter-Rennoch (2015) - *SIAM J. Math Anal.*

Assume that $A = I$ and either $N - I > 0$ or $N - I < 0$

Facts on the compact operator F_k (recall $\mathcal{S}_k = I + \frac{ik}{2\pi} F_k$).

- F_k is normal, i.e. $F_k F_k^* = F_k^* F_k$. Thus, $\mathcal{S}_k$ is unitary, i.e. $\mathcal{S}_k \mathcal{S}_k^* = \mathcal{S}_k^* \mathcal{S}_k = I$.
- As such F_k has an infinite number of eigenvalues $\lambda_j(k)$ accumulating to 0: they lie on the circle in $\mathbb{C}$

$$|\lambda|^2 - \frac{4\pi}{k} \Im(\lambda) = 0.$$

Write $\lambda_j(k) = r_j(k) e^{i\vartheta_j(k)}$

Inside-Outside Duality

Essential is the symmetric factorization of the far field operator

$$F_k = H_k^* \mathbf{T}_k H_k$$

- $\Im(\mathbf{T}_k u, u) \geq 0$
- $(F_k g, g)_{L^2(D)} = (\mathbf{T}_k H_k g, H_k g)_{L^2(D)} = (\mathbf{T}_k u, u)_{\mathbb{H}(D)}$
- Fix $N > I$

  If k is not a transmission eigenvalue, then $\Re(\lambda_j(k)) > 0$ for $j \in \mathbb{N}$ large enough thus

$$\vartheta_j(k) \to 0 \; as \; j \to \infty$$

Inside-Outside Duality

- The largest phase eigenvalue $\lambda_*(k)$ is well defined, i.e.

$$\vartheta_*(k) := \max_j \{\vartheta_j(k) \in [0, \pi)\} .$$

- If k is not a transmission eigenvalue,

$$\cot \vartheta_*(k) = \min_{\overline{R(H_k)}^{L^2(D)}} \frac{\Re(\mathbf{T}_k u, u)_{\mathbb{H}(D)}}{\Im(\mathbf{T}_k u, u)_{\mathbb{H}(D)}}$$

- k is a transmission eigenvalue if and only if there is $u_0 \in \overline{R(H_k)}^{L^2(D)}$ such that $\Im(\mathbf{T}_k u_0, u_0) = 0$

Inside-Outside Duality

If $N - I > 0$ and

$$\lim_{k \to k_0} \vartheta_*(k) = \pi$$

then $k_0 > 0$ is a transmission eigenvalue.

Transmission Eigenvalue Problem

Important questions in the context of inverse scattering:

- Fredholm property of the transmission eigenvalue problem. It arises in important questions such as uniqueness of inverse problems for inhomogenous media or justification of linear sampling methods.
- Discreteness of transmission eigenvalues. Methods for solving the inverse problem for inhomogeneous media such as the linear sampling method and factorization method fail at a transmission eigenvalue. Connection to uniqueness in thermo-acoustic tomography.
- Existence of transmission eigenvalues
  - Real transmission eigenvalues can be determined from the scattering data.
  - Transmission eigenvalues carry information about material properties.

Historical Overview

- The transmission eigenvalue problem in scattering theory was introduced by KIRSCH (1986) and COLTON-MONK (1988)
- Research was focused on the discreteness of transmission eigenvalues for variety of scattering problems: COLTON-KIRSCH-PÄIVÄRINTA (1989) and RYNNE-SLEEMAN (1991).
- The first proof of existence of at least one transmission eigenvalues for large contrast PÄIVÄRINTA-SYLVESTER (2009).
- The existence of an infinite set of real transmission eigenvalues was first proven by CAKONI-GINTIDES-HADDAR (2010).
- Completeness+Weyl estimates first given by LAKSHTANOV-VAINBERG (2012) and ROBBIANO (2013).
- Since the appearance of these papers there has been an explosion of interest in the transmission eigenvalue problem ...

Special issue of Inverse Problems on Transmission Eigenvalues, October 2013

Transmission Eigenvalues

$$\begin{aligned} \text{curl curl } E_0 - k^2 E_0 &= 0 && \text{in} && D \\ \text{curl } A \text{ curl } E - k^2 N E &= 0 && \text{in} && D \\ \nu \times E &= \nu \times E_0 && \text{on} && \partial D \\ \nu \times A \text{ curl } E &= \nu \times \text{curl } E_0 && \text{on} && \partial D \end{aligned}$$

In a "natural" variational form this problem reads

$$\int_D (\text{curl } AE) \cdot (\text{curl } \overline{E}') \, dx - \int_D (\text{curl } E_0) \cdot (\text{curl } \overline{E}_0') \, dx$$
$$-k^2 \int_D NE \cdot \overline{E}' \, dx + k^2 \int_D E_0 \cdot \overline{E}_0' \, dx = 0$$

for all $E', E_0' \in X(D)$, where

$$X(D) := \{(w, v) \in H(curl, D) \times H(curl, D) \,|\, \nu \times w = \nu \times v \text{ on } \Gamma\}.$$

Transmission Eigenvalues

L. Chesnel (2013) - *Inverse Problems*

proved the discreteness of transmission eigenvalues+Fredholm property, provided $A - I$ and $N - I$ are bounded away from zero and have same sign in a neighborhood of ∂D using $\top$- coercivity.

F. Cakoni and A. Kirsch (2010) - *Int. Jour. Comp. Sci. Math.*

proved the existence of infinitely many real transmission eigenvalues for $A = a_0 I$ constant different from one and $N - I$ have the same sign uniformly in D.

- Hoai-Minh Nguyen has recently obtained spectral results for the scalar case for much less regular A and N and various combinations of contrast sign.

Transmission Eigenvalues

Consider $A = I$, letting $k^2 := \tau$ and assume that $N - I > 0$.
It is possible to write

$$\begin{aligned} \text{curl curl } E - \tau N E = 0 \quad & \text{in} \quad D \\ \text{curl curl } E_0 - \tau E_0 = 0 \quad & \text{in} \quad D \\ \nu \times E = \nu \times E_0 \quad & \text{on} \quad \Gamma \\ \nu \times \text{curl } E = \nu \times \text{curl } E_0 \quad & \text{on} \quad \Gamma \end{aligned}$$

$E, E_0 \in L^2(D)$, for the difference $W = E - E_0 \in H_0(\text{curl}^{\,2}, D)$ as

$$(\nabla \times \nabla \times -\tau)(N - I)^{-1}(\nabla \times \nabla \times -\tau N)W = 0$$

and in the variational form, for all $W' \in H_0(cur^2, D)$

$$\int_D (N - I)^{-1}(\nabla \times \nabla \times W - \tau N W)(\nabla \times \nabla \times \overline{W}' - \tau \overline{W}')\, dx = 0$$

$$H_0(\text{curl}^{\,2}, D) = \{u \in H(\text{curl}\,, D), \text{curl } u \in H(\text{curl}\,, D), \nu \times u = 0, \nu \times \text{curl } u = 0 \text{ on } \partial D\}$$

Existence of Real Transmission Eigenvalues

$$(\mathbb{A}_\tau - \tau\mathbb{B})u = 0 \quad \text{in} \;\; H_0(\text{curl}^{\,2}, D)$$

$$\begin{aligned}
(\mathbb{A}_\tau W, W') &= \int_D (N - I)^{-1}(\text{curl}\,\text{curl}\, W - \tau W)\cdot(\text{curl}\,\text{curl}\,\overline{W'} - \tau\overline{W'})\,dx \\
&+ \tau^2 \int_D W\cdot\overline{W'}\,dx \\
(\mathbb{B}W, W') &= \int_D \text{curl}\, W \cdot \text{curl}\,\overline{W'}\,dx
\end{aligned}$$

The mapping $\tau \to \mathbb{A}_\tau$ is continuous from $(0, +\infty)$ to the set of self-adjoint coercive operators from $H_0(\text{curl}^{\,2}, D) \to H_0(\text{curl}^{\,2}, D)$.

$\mathbb{B} : H_0(\text{curl}^{\,2}, D) \to H_0(\text{curl}^{\,2}, D)$ is self-adjoint, compact and non-negative.

Existence of Real Transmission Eigenvalues

Now we consider the generalized eigenvalue problem

$$(\mathbb{A}_\tau - \lambda(\tau)\mathbb{B})u = 0 \quad \text{in} \;\; H_0^2(\text{curl}^{\,2}, D)$$

For a fixed $\tau > 0$ there exists an increasing sequence of eigenvalues $\lambda_j(\tau)_{j\geq 1}$ such that $\lambda_j(\tau) \to +\infty$ and they satisfy Courant-Fisher max-min principle.

τ is a transmission eigenvalue if and only $\lambda_j(\tau) = \tau$

Existence of Real Transmission Eigenvalues

- For $0 < \tau_0 < \dfrac{\lambda_1(D)}{N_{max}}$, we have that $\mathbb{A}_{\tau_0} - \tau_0 \mathbb{B}$ is positive on $H_0^2(D)$, where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D.
- There exists τ_1 such that $\mathbb{A}_{\tau_1} - \tau_1 \mathbb{B}$ is non positive on an m dimensional subspace of $H_0^2(D)$. This can be done for m arbitrarily large

Max-min principle for $\lambda_j(\tau)$ implies each $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, m$, has at least one solution in $[\tau_0, \tau_1]$ meaning that there exists m transmission eigenvalues counting multiplicity within the interval $[\tau_0, \tau_1]$.

CAKONI-GINTIDES-HADDAR (2010) - *SIAM J. Math. Anal.*

Existence of Real Transmission Eigenvalues

Theorem

Assume that $N_{min} > 1$. Then, there exists an infinite discrete set of real transmission eigenvalues τ_j accumulating at $+\infty$ and satisfying

$$\tau_j(N_{max}, B_1) \leq \tau_j(N_{max}, D) \leq \tau_j(N(x), D) \leq \tau_j(N_{min}, D) \leq \tau_j(N_{min}, B_2)$$

where $B_2 \subset D \subset B_1$.

- For $N := nI$ constant, the first transmission eigenvalue $\tau_1(n)$ is strictly monotonically decreasing and continuous with respect to n.
- In particular, the first transmission eigenvalue uniquely determines the constant index of refraction.
- Inverse spectral problem is solved for spherically stratified media, AKTOSUN, COLTON, GINTIDES, LEUNG, PAPANICOLAOU ...

Changing Sign Contrast

- Similar results as above can be obtained for the case when $0 < N_{min} \le N(x) \le N_{max} < 1$.
- The analysis holds for media with voids $D_0 \subset D$ where $N \equiv I$.

  CAKONI-COLTON-HADDAR (2012) - *SIAM J. Math. Anal.*

The general case of sign-changing contrast $N - I$ is considered under the assumption that either $N - I > 0$ or $N - I < 0$ only in a neighborhood of ∂D.

F. CAKONI, H. HADDAR, S. MENG (2015) - *J. Int. Eqns Appl.*

where the discreteness is proven using integral equations method.

- Full spectrum is recently analyzed by H. HADDAR AND S. MENG for $N \in C^\infty$ using Agmon's theory for non-selfadjoint operators.

Non-destructive Testing and TE

Given the measured $k_1(D, N(x))$, we now compute a constant n such that $k_1(D, N(x)) = k_1(D, n)$. Then the monotonicity result implies that

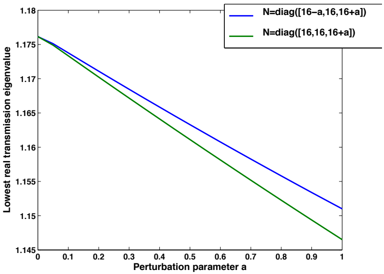
$$N_{min} \leq n \leq N_{max}$$

In the isotropic case $N(x) := n(x)I$, the above constant n gives

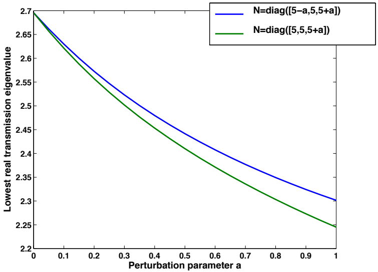
$$n \approx \frac{1}{|D|} \int_D n(x)\, dx$$

Numerical Examples

We can compute numerical approximations of transmission eigenvalues for anisotropic media using a finite element method due to MONK-SUN.



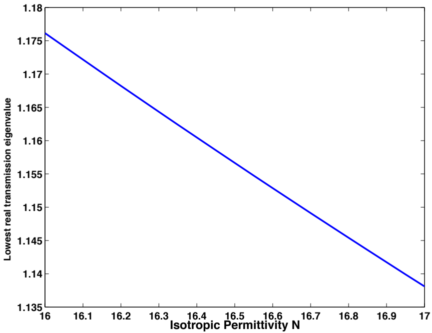
Perturbation of $N = 16I$



Perturbation of $N = 5I$

Numerical Examples (cont)

Using the same finite element code we can compute the transmission eigenvalues for isotropic N then compute the isotropic n discussed previously for any measured transmission eigenvalue

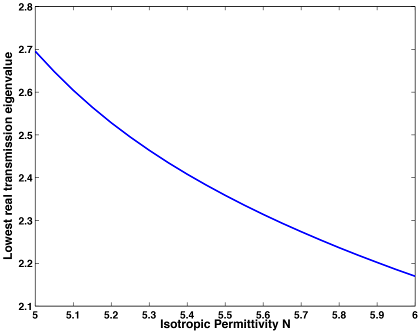


Lowest transmission eigenvalue against N (isotropic)

N	$\lambda_{1,D,N(x)}$	n
diag([15.5, 16, 16.5])	1.163	16.33
diag([15, 16, 17])	1.151	16.65
diag([16, 16, 16.5])	1.161	16.38
diag([16, 16, 17])	1.146	16.77

Numerical Examples (cont)

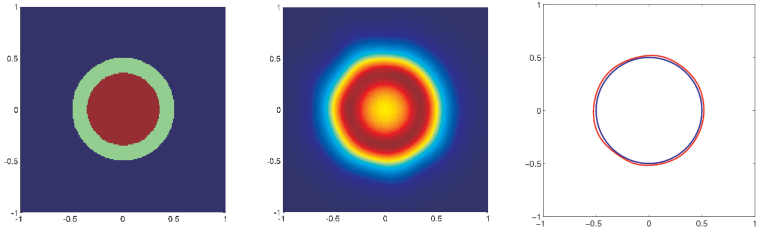
The same procedure can be carried out at lower N as well (the lowest transmission eigenvalue increases and so the calculations become more expensive)



Lowest transmission eigenvalue against N (isotropic)

N	$\lambda_{1,D,N(x)}$	n
$\mathrm{diag}([4.5, 5, 5.5])$	2.442	5.339
$\mathrm{diag}([4, 5, 6])$	2.302	5.631
$\mathrm{diag}([5, 5, 5.5])$	2.410	5.397
$\mathrm{diag}([5, 5, 6])$	2.245	5.778

Numerical Example: Inhomogeneous Isotropic Media



n_e	n_i	k_1	n-exact shape	n-recon. shape
8	8	2.98	8.07	7.61
11	5	3.27	7.05	6.69
22	19	1.76	20.28	18.86
67	61	0.97	64.11	59.42

Example from

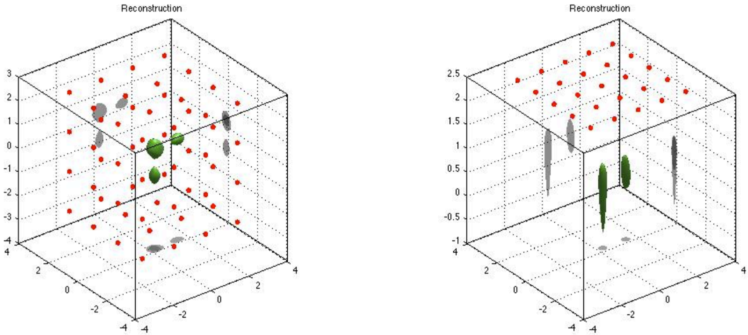
GIORGI-HADDAR (2012) - *Inverse Problems*

Future Directions

- Drawback of these methods is the amount of spatial data needed.
- Possible remedy using time domain data.

Development of qualitative methods in the time domain for Maxwell's equation!

Future Directions



GUO-MONK-COLTON (2014) - *Inverse Problems*

Future Directions

- Drawback of the use of transmission eigenvalues is that it needs data for a range of frequencies and it does not work for media with absorption.
- Possible remedy

Introduce a new eigenvalue problem by modifying the far field equations!

(see P. Monk's talk)

CAKONI-COLTON-MENG-MONK (2016) - *SIAM J. Appl. Math*