Integrability: Historic Overview

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Plan

- Origins of Integrability - classical dynamics

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- Integrability and XIX-century algebraic geometry

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- Integrability and quantum theory

Part I: Origins of Integrability - classical dynamics

On January 1, 1600 a teacher of mathematics and astronomy from Graz Johannes Kepler set off to Prague by invitation of Tycho Brahe, the imperial astronomer of the Holy Roman Emperor Rudolf II.


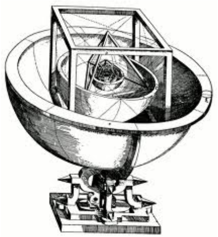

Figure: Kepler (1571-1630) and his "Mysterium Cosmographicum" (1596)

"Forerunner of the Cosmological Essays, Which Contains the Secret of the Universe; on the Marvelous Proportion of the Celestial Spheres, and on the True and Particular Causes of the Number, Magnitude, and Periodic Motions of the Heavens; Established by Means of the Five Regular Geometric Solids"

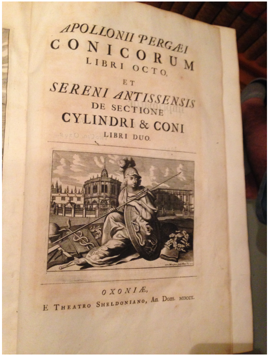

Figure: Apollonius's "Conics" (First Latin edition: Bononiae, 1566)

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The orbits of the planets are ellipses with one of the foci at the Sun

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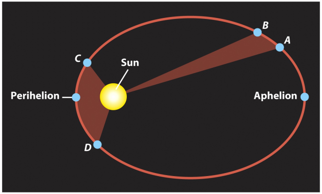
Second Kepler's Law (1602):
The sectorial velocity remains constant along the orbit



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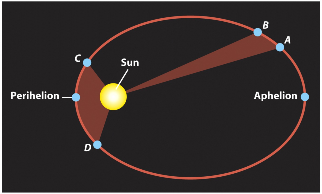
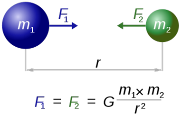


Third Kepler's Law (1619):
The square of the periods are proportional to the cube of the major semi-axes of the orbits

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}.$$

The breakthrough: Isaac Newton

Isaac Newton: Kepler's laws imply the Universal Gravity Law:



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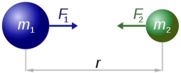

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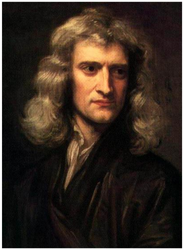

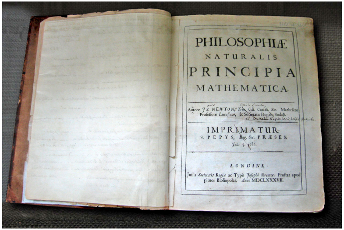

Figure: Isaac Newton (1642-1727) and his own copy of "Principia" (1687)

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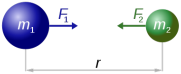

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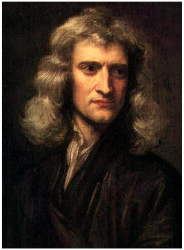

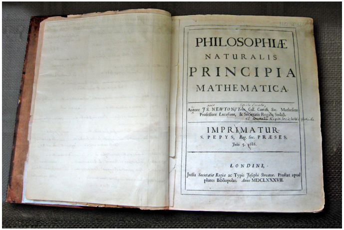

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The Mathematical Physics was born...

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Very lucky...

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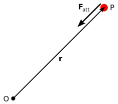

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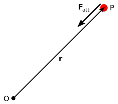

Bertrand (1873): If a central force system has all bounded orbits closed, then either

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or

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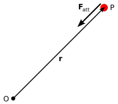

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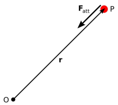

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Albert Einstein: "God may be sophisticated, but not malicious."

Luck continues: Two-fixed centre problem

Leonhard Euler (1760): Integrability of two-fixed centre problem

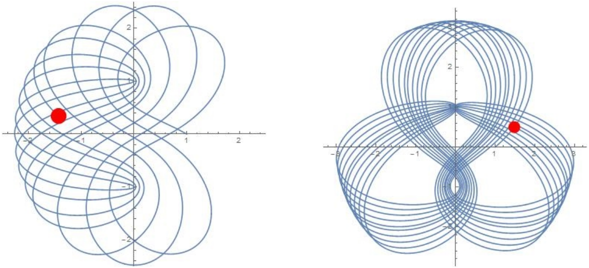

Figure: Orbits of two-fixed centre problem (produced by R. Sakamoto)

Hamiltonian formalism

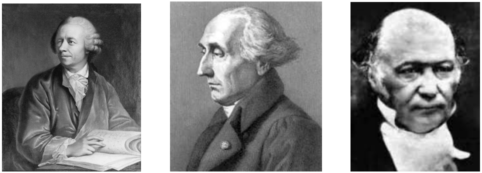

Figure: Euler (1707-1787), Lagrange (1736-1813) and W.R. Hamilton (1805-1865)

William Rowan Hamilton (1837): Euler-Lagrange equations of mechanics can be re-written as **Hamiltonian systems**

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \;\; \dot{q}_i = \frac{\partial H}{\partial p_i},$$

where $i = 1, ..., n$ and $H = H(p, q)$ is called **Hamiltonian** of the system.

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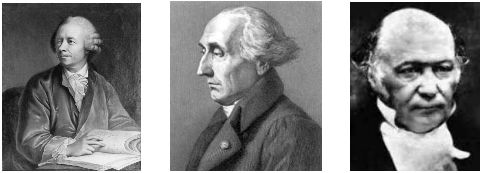

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Define **Poisson bracket** of two functions F and G on the phase space $\mathbf{R}^{2n}$ as

$$\{F, G\} := \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right).$$

The equations of motion can be re-written then in an elegant form

$$\dot{F} = \{H, F\}.$$

Poisson algebra

Poisson bracket has the following properties:

$$\{F, G\} = -\{G, F\}$$

$$\{c_1 F_1 + c_2 F_2, G\} = c_1\{F_1, G\} + c_2\{F_2, G\}$$

$$\{FG, H\} = \{F, H\}G + F\{G, H\}$$

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0 \quad \textbf{(Jacobi identity)}.$$

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Corollary. The integrals of a Hamiltonian system also form a Lie algebra with respect to the Poisson bracket.

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Emmy Noether (1918): there is a correspondence between integrals and symmetries:

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Example. For a central force system in $\mathbf{R}^3$ we have

$$Angular\ momentum\ M = p \times q \leftrightarrow Rotational\ symmetry.$$

Poisson algebra of components of M is nothing but the Lie algebra $so(3)$:

$$\{M_1, M_2\} = M_3,\ \{M_2, M_3\} = M_1,\ \{M_3, M_1\} = M_2.$$

Isotropic harmonic oscillator

$$H = \frac{1}{2}|p|^2 + \frac{1}{2}\omega^2|q|^2, \; p, q \in \mathbb{R}^n$$

has the integrals $M_{ij} = p_i q_j - p_j q_i, \; i < j$, corresponding to rotational symmetry, and additional integrals

$$N_{ij} = p_i p_j + q_i q_j, \; i \leq j,$$

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Indeed, assume for simplicity that $\omega = 1$, then the equations of motion are $\dot{p} = -q, \ \ \dot{q} = p$, or for $z = p + iq \in \mathbf{C}^n$

$$\dot{z} = iz.$$

Its solutions are circles $z = z_0 e^{it}$. The orbits are their projections on q-space, which are ellipses.

Hidden symmetry of Kepler system

Kepler system has the Hamiltonian

$$H = \frac{1}{2}|p|^2 - \frac{k}{|q|}, \quad p, q \in \mathbf{R}^3.$$

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Components of M and L form together a Lie algebra isomorphic to $so(4)$, which is the full symmetry of Kepler system.

Liouville integrability

We say that a Hamiltonian system in $\mathbb{R}^{2n}$ is **integrable in Liouville sense** (or, simply, **integrable**) if it has n independent integrals $F_1, \ldots, F_n$ **in involution**:

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Liouville-Arnold theorem. Assume that a Hamiltonian system in $\mathbf{R}^{2n}$ has n independent integrals $F_1 = H, F_2, \ldots, F_n$ in involution and consider a level set

$$M_c = \{x \in \mathbf{R}^{2n} : F_j(x) = c_j, j = 1, \ldots, n\}.$$

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3. In a neighbourhood of such M_c there is a canonical change of variables $(p, q) \to (I, \phi \mod 2\pi)$ (**action-angle variables**), such that in the new coordinates the Hamiltonian $H = H(I)$. The flow is linear in angle variables:

$$\phi = \omega(I)t + \phi_0, \quad \omega_j(I) = \frac{\partial H}{\partial I_j}(I),$$

so the orbits are winding lines on the corresponding torus.

Liouville tori

Figure: Joseph Liouville (1809-1882), Vladimir I. Arnold (1937-2010) and Liouville torus in a central force system

Examples of integrable systems

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2 (Euler). Two-fixed centre problem on the plane with

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{m_1}{r_1} - \frac{m_2}{r_2},$$

where

$$r_1 = \sqrt{(q_1 + c)^2 + q_2^2},\ r_2 = \sqrt{(q_1 - c)^2 + q_2^2}$$

are the distances from the centres, has an additional integral

$$F = \frac{1}{2}(p_1 q_2 - p_2 q_1)^2 + \frac{1}{2}c^2 p_1^2 + c q_1 \left(\frac{m_1}{r_1} - \frac{m_2}{r_2} \right).$$

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3. Harmonic oscillator with

$$H = \frac{1}{2}|p|^2 + \frac{1}{2}\sum_{i=1}^{n} \omega_i^2 q_i^2$$

has n commuting independent integrals

$$F_i = \frac{1}{2}p_i^2 + \frac{1}{2}\omega_i^2 q_i^2,\ \ i = 1, \ldots, n.$$

Integrability in rigid body dynamics

Euler: motion of rigid body fixed at centre of mass

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Alfred Clebsch (1871): special case of rigid body motion in infinite fluid

Sofia Kowalevskaya (1888): a special asymmetric top, "Prix Bordin" (1888), arguably the most complicated integrable system of XIX century.

Figure: Alfred Clebsch (1833-1872) and Sofia Kowalevskaya (1850-1891)

Paradise lost: Poincare and chaos

Poincare (1892-99): non-integrability and chaos in 3-body problem in celestial mechanics

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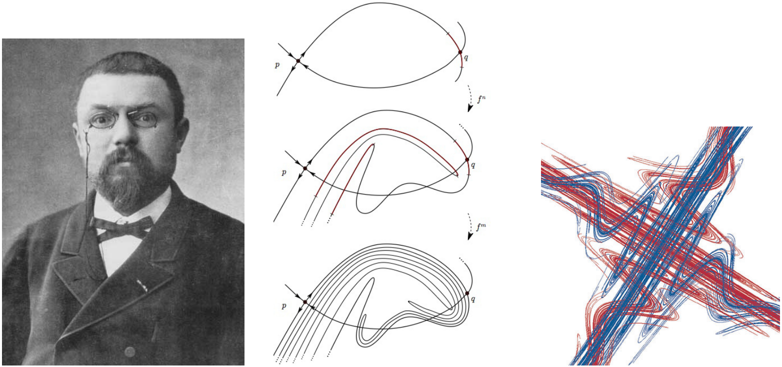


Figure: Henri Poincaré (1854-1912) and homoclinic tangles

Paradise lost: Poincare and chaos

Poincare (1892-99): non-integrability and chaos in 3-body problem in celestial mechanics

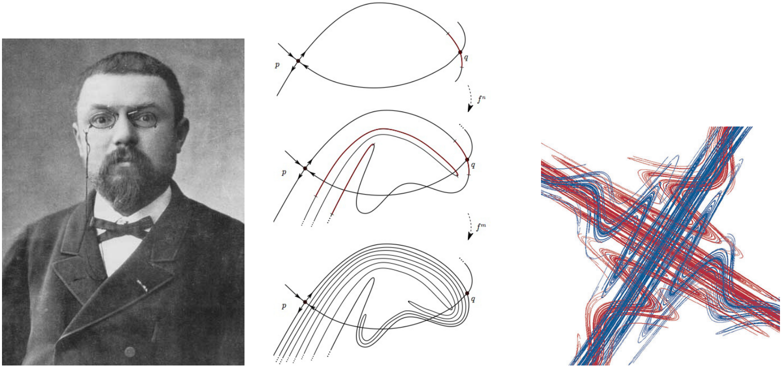
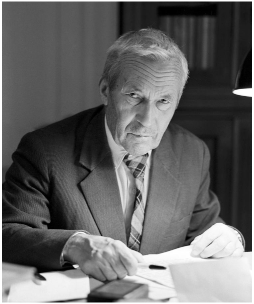


Figure: Henri Poincaré (1854-1912) and homoclinic tangles

Much of this came as a result of correcting the mistake in his early 1887 work...

A glimpse of hope: KAM-theory



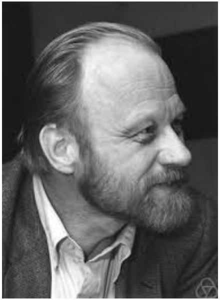


Figure: A.N. Kolmogorov (1903-1987), V.I. Arnold (1937-2010) and J. Moser (1928-1999)

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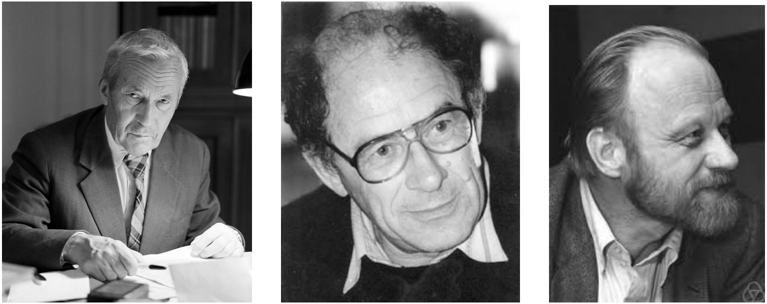

Figure: A.N. Kolmogorov (1903-1987), V.I. Arnold (1937-2010) and J. Moser (1928-1999)

KAM-theorem (1954-63): most of Liouville's tori survive under a small perturbation of integrable system

$$H = H(I) + \varepsilon H_1(I, \varphi).$$

Renaissance of Integrability: soliton theory (1967-)

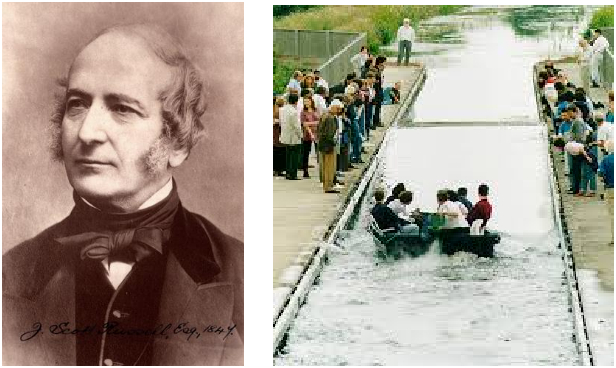

Figure: John Scott Russell (1808-82) and his soliton re-created in 1995

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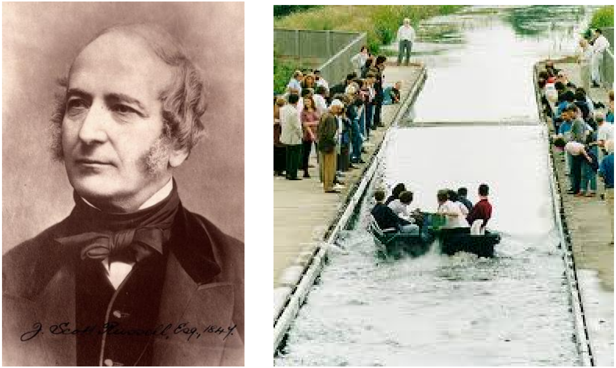
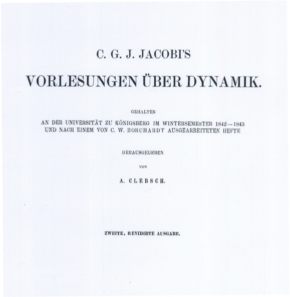


Figure: John Scott Russell (1808-82) and his soliton re-created in 1995

More in Mark Ablowitz's lecture

Part II: Integrability and XIX-century algebraic geometry

Carl Gustav Jacobi (1843): a famous lecture course on Dynamics in Königsberg, which were later edited by Clebsch and published in 1866.



C. G. J. JACOBI'S

VORLESUNGEN ÜBER DYNAMIK.

GEHALTEN

AN DER UNIVERSITÄT ZU KÖNIGSBERG IM WINTERSEMESTER 1842—1843
UND NACH EINEM VON C. W. BORCHARDT AUSGEARBEITETEN HEFTE

HERAUSGEGEBEN

VON

A. CLEBSCH.

ZWEITE, REVIDIRTE AUSGABE.

Figure: Carl Gustav Jacob Jacobi (1804-51) and second edition of his "Lectures on Dynamics"

To find the geodesics on an ellipsoid Jacobi introduced a "remarkable substitution" -*Elliptic Coordinates*.

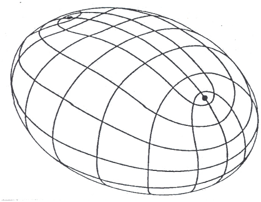


Figure: Elliptic coordinates on an ellipsoid

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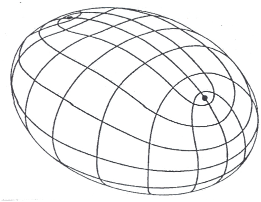

Figure: Elliptic coordinates on an ellipsoid

The elliptic coordinates $u_1, \dots, u_n$ are the roots of

$$\frac{x_1^2}{a_1 + u} + \frac{x_2^2}{a_2 + u} + \quad \dots + \frac{x_n^2}{a_n + u} = 1$$

and correspond to the confocal quadrics extensively studied in XIX-th century (e.g. in Salmon and Fiedler "Analytische Geometrie des Raumes" (1863-65)).

Abel map and Jacobi inversion problem

Jacobi showed that for $n = 2$ the elliptic coordinates u_1, u_2 satisfy

$$\dot{\xi}_1 = \frac{\dot{u}_1}{\sqrt{R(u_1)}} + \frac{\dot{u}_2}{\sqrt{R(u_2)}} = 1,$$

$$\dot{\xi}_2 = \frac{u_1 \dot{u}_1}{\sqrt{R(u_1)}} + \frac{u_2 \dot{u}_2}{\sqrt{R(u_2)}} = 0,$$

where $R(z)$ is some polynomial of degree 5, and

$$\xi_1 = \int^{u_1} \frac{dz}{\sqrt{R(z)}} + \int^{u_2} \frac{dz}{\sqrt{R(z)}},$$

$$\xi_2 = \int^{u_1} \frac{zdz}{\sqrt{R(z)}} + \int^{u_2} \frac{zdz}{\sqrt{R(z)}}$$

is the Abel map of $S^2\Gamma \to J(\Gamma)$ for hyperelliptic curve Γ given by

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This was the origin of the classical Jacobi inversion problem, which is one of the most fundamental in classical algebraic geometry.

In December 28, 1838 Jacobi wrote to his colleague Friedrich Bessel:

The day before yesterday I reduced the geodesic line of an ellipsoid with three unequal axes to quadratures. The formulas are the simplest in the world, Abelian integrals, transforming into the known elliptical ones if two axes are made equal.

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Riemann's solution and θ-functions

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In full generality, the solution was found by Bernhard Riemann, who introduced the classical **Riemann θ-function**

$$\theta(z, B) = \sum_{m \in \mathbb{Z}^g} \exp 2\pi i (m^t z + m^t B m),$$

where B is Riemann matrix of b-periods.

Hyperelliptic case: Klein's sigma-functions

Figure: Karl Weierstrass (1815-97), Bernhard Riemann (1826-66) and Felix Klein (1849-1925)

Hyperelliptic case: Klein's sigma-functions



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In 1925 Felix Klein complained:

When I was a student, abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics and each of us was ambitious to make progress in this field. And now? The younger generation hardly knows abelian functions.

Modern development

The situation changed from 1974 when θ-functions became common tool in the theory of integrable PDEs (S.P. Novikov, Dubrovin, Its, Matveev, Krichever).

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Another very important development was

- ADHM construction of instantons (Atiyah, Drinfeld, Hitchin, Manin, 1976)

Part III: Integrability in classical differential geometry

Line congruences in $\mathbb{R}^3$

Line congruence is a 2-parameter family of straight lines in $\mathbb{R}^3$. For a general line congruence there are exactly 2 **focal surfaces**:

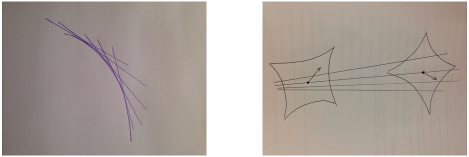

Figure: Focal curve (envelope) in the plane and 2 focal surfaces in space

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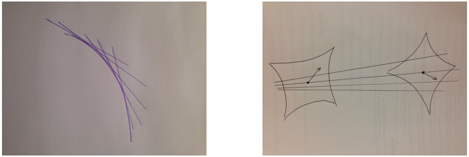

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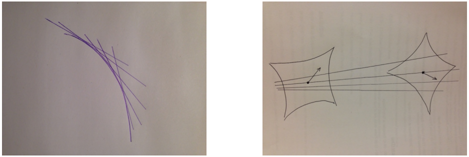

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Lie (1879), Bäcklund (1883): converse to this statement and one-parameter generalization

Bäcklund transform and sine-Gordon equation

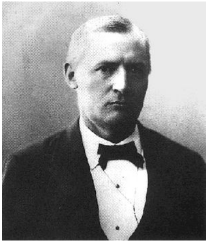

Figure: Luigi Bianchi (1856-1928), Sophus Lie (1842-99) and Albert Victor Bäcklund (1845-1922)

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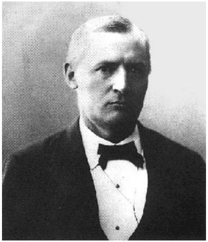

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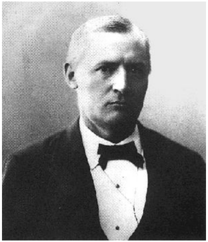

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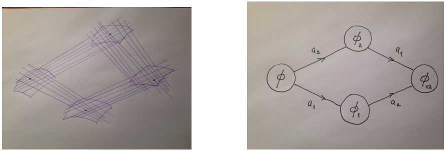
Lie-Bäcklund transform:

$$\psi_x = \phi_x + 2a\sin\left(\frac{\phi+\psi}{2}\right), \quad \psi_y = -\phi_y + \frac{2}{a}\sin\left(\frac{\psi-\phi}{2}\right).$$

Bianchi permutability

Bianchi: Lie-Bäcklund transforms commute:



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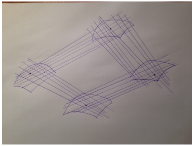


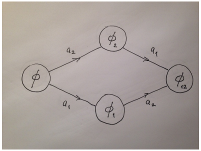


Figure: Bianchi permutability diagram

Corresponding solutions of sine-Gordon equation satisfy the relation

$$\phi_{12} = \phi + 4\tan^{-1}\left(\frac{a_2 - a_1}{a_2 + a_1}\tan\frac{\phi_2 - \phi_1}{4}\right),$$

which is an example of integrable **purely discrete** 2D equation.

Menelaus theorem and discrete SKP equation

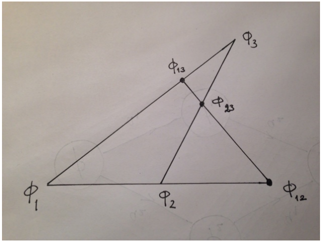
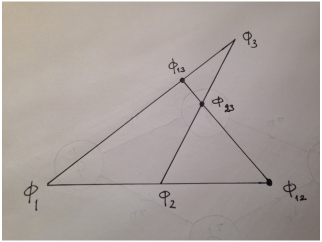


Figure: Menelaus of Alexandria (70-140AD) and his theorem

Menelaus theorem: $\phi_{13}, \phi_{23}, \phi_{12}$ lie on a straight line iff

$$\frac{(\phi_1 - \phi_{12})}{(\phi_{12} - \phi_2)} \frac{(\phi_2 - \phi_{23})}{(\phi_{23} - \phi_3)} \frac{(\phi_3 - \phi_{13})}{(\phi_{13} - \phi_1)} = -1.$$



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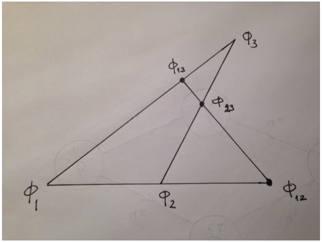

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Adler, Bobenko, Suris (2010): classification of integrable discrete equations of this (octahedral) type

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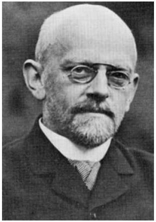

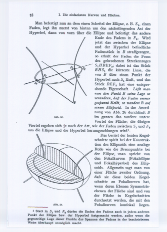

18 I. Die einfachsten Kurven und Flächen.

Man befestigt nun an dem einen Scheitel der Ellipse, z. B. S_1, einen Faden, legt ihn zuerst von hinten um den nächstliegenden Ast der Hyperbel, dann von vorn über die Ellipse und befestigt das andere Ende des Fadens in F_2. Wird jetzt das zwischen der Ellipse und der Hyperbel befindliche Fadenstück in B straffgezogen, so erhält der Faden die Form des gebrochenen Streckenzuges S_1HBEF_2; dabei ist das Stück BHS_1 die kürzeste Linie, die von B über einen Punkt der Hyperbel nach S_1 läuft, und das Stück BEF_2 hat eine entsprechende Eigenschaft. *Läßt man nun den Punkt B seine Lage so verändern, daß der Faden immer gespannt bleibt, so wandert B auf einem Ellipsoid.* In der Anordnung von Abb. 26 durchläuft B im ganzen das vordere untere Viertel der Fläche; die übrigen Viertel ergeben sich je nach der Art, wie der Faden zwischen S_1 und F_2 um die Ellipse und die Hyperbel herumgeschlungen wird[1].

Abb. 26.

Das Gerüst der beiden Kegelschnitte spielt bei der Konstruktion des Ellipsoids eine analoge Rolle wie die Brennpunkte bei der Ellipse, man spricht von den Fokalkurven (Fokalellipse und Fokalhyperbel) des Ellipsoids. Allgemein sagt man von einer Fläche zweiter Ordnung, daß sie diese beiden Kegelschnitte zu Fokalkurven hat, wenn deren Ebenen Symmetrieebenen der Fläche sind und von der Fläche in Kegelschnitten durchsetzt werden, die mit den Fokalkurven konfokal liegen.

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[1] Statt in S_1 und F_2 dürfen die Enden des Fadens auch in jedem anderen Punkt der Ellipse bzw. der Hyperbel festgemacht werden, außer wenn die gegenseitige Lage dieser Punkte das Spannen des Fadens in der beschriebenen Weise überhaupt unmöglich macht.

Figure: David Hilbert (1862-1943) and Staude's construction from Hilbert-Cohn Vossen "Anschauliche Geometrie"

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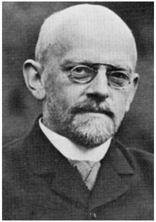

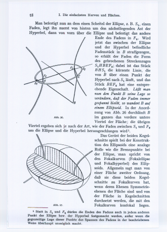

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The last one is based on the integrability of Jacobi's geodesic problem.

Part VI: Integrability and quantum theory

Hamilton-Jacobi equation and separation of variables

W.R. Hamilton (1834) used deep analogy between mechanics and optics to introduce **Hamilton-Jacobi equation**

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial q}, q, t) = 0$$

where $S = S(q, t)$ is the *action* of the corresponding Hamiltonian system.

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An amazing discovery of Jacobi that this PDE is sometimes easier to solve than the original system of ODEs !

Namely, the complete solution can be found as a **sum** (?!)

$$S = Et + W_1(Q_1, \alpha) + W_2(Q_2, \alpha) + ... + W(Q_n, \alpha),$$

where $Q_1, Q_2, ..., Q_n$ are some new coordinates. This method is known as **separation of variables** in the Hamilton-Jacobi equation and still remains arguably the most effective method in the theory of integrable systems.

"Older Quantum Theory"

Bohr, Sommerfeld (1913-15): quantisation conditions

$$\oint p_k dq_k = n_k \hbar,$$

where n_k are integers, $\hbar$ is a Planck constant.

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Schwarzschild, Epstein (1916), Einstein (1917): true only in separation coordinates of Hamilton-Jacobi equation...

Figure: Karl Schwarzschild (1873-1916) and Paul Epstein (1871-1939), who were first to emphasise the role of Jacobi's work in the "Older Quantum Theory."

Schrödinger equation

$$i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(\vec{r}) \right] \Psi(\vec{r},t)$$

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BY E. SCHRÖDINGER

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which is exactly the Hamilton-Jacobi equation with $H = |p|^2 + V(q)$.

The separation of variables in the quantum case looks quite natural:

$$\Psi = \Psi_1(X_1)\Psi_2(X_2)...\Psi_n(X_n).$$

So Jacobi's method is *SEMI-QUANTUM* !

- Spectral theory: existence of commuting operators imply "finite-gap" properties (S.P. Novikov; Lax (1974))

Quantum integrability in modern Mathematics

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