Tau functions and anomalous dimensions

Twistors based on the Joukowski transformation

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Credits: Andrea Ferrari, (and historically, R.S.Ward, N.M.J.Woodhouse, ...) following Gromov, Kazakov and Volin plus many others.

Work in progress!

The Joukowski transformation

A non-Hausdorff twistor correspondence

- The Joukowski transformation $\lambda \in \mathbb{CP}^1 \xrightarrow{(z,\rho)} \mathbb{CP}^1 \ni u$

$$u = \frac{\rho}{2}\left(\lambda + \frac{1}{\lambda}\right) + iz\,.$$

- Maps unit circle $|x| = 1$ to the slit $[-\rho, \rho] + iz$.
- Family of $2:1$ maps branching at $u = \pm\rho + iz$.
- For $\rho + iz \in U \subset \mathbb{C}$ that contains $\rho = 0$, can connect both pre-images for $u \in U$ but not for $u \notin U$.
- Image is *non-Hausdorff twistor space* $\mathbb{T}(U)$: two Riemann spheres $\mathbb{CP}^1_0, \mathbb{CP}^1_\infty$ glued over U.

$$\lambda = 0 \to u = \infty \in \mathbb{CP}^1_0\,, \qquad \lambda = \infty \to u = \infty \in \mathbb{CP}^1_\infty\,.$$

Correspondence:

$$\begin{array}{ccc} & U \times \mathbb{CP}^1 \ni (z,\rho,\lambda) & \\ \swarrow & & \searrow \\ (z,\rho) \in U & & \mathbb{T}(U) \ni u = \frac{\rho}{2}\left(\lambda + \frac{1}{\lambda}\right) + iz \end{array}$$

Three applications of the non-Hausdorff twistor space

1. Mason & Woodhouse 1986 (following Ward 1983)

$$\left\{\begin{array}{l}\text{Solutions to Ernst}\\ \text{equations on } U\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Certain rank 2 holomorphic}\\ \text{vector bundles } E \to \mathbb{T}(U)\end{array}\right\}$$

2. A consequence of Mason & Woodhouse (1996)

$$\left\{\begin{array}{l}y(\rho),\ \text{Painlevé III}\\ \text{transcendants}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Certain } \partial_u \text{ invariant rank}\\ \text{2 bundles } E \to \mathbb{T}(U)\end{array}\right\}$$

3. Quantum spectral curve of Gromov, Kazakov & Volin 2013

$$\left\{\begin{array}{l}\text{Certain \textit{anomalous di-}}\\ \textit{mensions}\ \gamma(\rho) \text{ for } N = 4\\ \text{Super Yang-Mills}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Certain } u \to u + i \text{ invari-}\\ \text{ant rank 2, 4 or } 4|4 \text{ bundles}\\ E \to \mathbb{T}(U)\end{array}\right\}$$

Ongoing work with Andrea Ferrari.

Stationary axisymmetric self-dual Yang-Mills

Yang's form of Self-dual Yang-Mills on $\mathbb{R}^4$ with coords $(v, y) = (z + it, \rho\, \mathbf{e}^{i\theta})$, metric $\mathrm{d}s^2 = \mathrm{d}v\mathrm{d}\bar{v} + \mathrm{d}y\mathrm{d}\bar{y}$ is:

$$\frac{\partial}{\partial \bar{v}} \left(J^{-1} \frac{\partial J}{\partial v} \right) + \frac{\partial}{\partial \bar{y}} \left(J^{-1} \frac{\partial J}{\partial y} \right)$$

where $J = J(v, w, \bar{y}, \bar{y})$ is a Hermitian matrix function.
t and θ-independence $\rightsquigarrow$

$$\partial_\rho(\rho J^{-1} \partial_\rho J) + \partial_z(\rho J^{-1} \partial_z J) = 0\,.$$

The reduced vacuum equations

Ward's reduction (1983)

Metric:

$$\mathrm{d}s^2 = \pm \mathbf{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}z^2) \pm J_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$k = k(\rho, z)\,, \quad J_{ij} = J_{ij}(\rho, z).$$

$\partial/\partial x^i$, $i = 1, \ldots n-2$ are 2-surface orthogonal Killing vectors (take $n = 4$ hereon).

The Vacuum field equations

$$\rho^2 = \det J$$

$$4i\frac{\partial k}{\partial w} = -\rho\,\mathrm{tr}\,((J^{-1}\partial_w J)^2) - \frac{1}{\rho}\,, \quad w = z + i\rho.$$

$$\partial_\rho(\rho J^{-1}\partial_\rho J) + \partial_z(\rho J^{-1}\partial_z J) = 0$$

Tau function: Solution controlled by k, the *tau function*.

The Lax pair

- Let $U \subset H = \{(z, \rho) \in \mathbb{R}^2 | \, \rho \geq 0\}$ and set $w = \rho + iz$.
- For $(w, \lambda) \in U \times \mathbb{CP}^1$ define

$$
\begin{aligned}
L &:= \partial_w + \frac{\lambda(\lambda - 1)}{2\rho(\lambda + 1)} \partial_\lambda + \frac{1}{\lambda + 1} J^{-1} \partial_w J \,, \\
\tilde{L} &:= \partial_{\bar{w}} - \frac{\lambda(\lambda + 1)}{2\rho(\lambda - 1)} \partial_\lambda - \frac{1}{\lambda - 1} J^{-1} \partial_{\bar{w}} J
\end{aligned}
$$

- Then

$$
[L, \tilde{L}] = 0 \qquad \Leftrightarrow \qquad \partial_\rho(\rho J^{-1} \partial_\rho J) + \partial_z(\rho J^{-1} \partial_z J) = 0
$$

Twistor space

For a given $U \subset H$ define the reduced twistor space to be

$$\mathbb{T}(U) = U \times \mathbb{CP}^1 / \{l, \tilde{l}\}$$

where $\{l, \tilde{l}\}$ is the distribution spanned by

$$l = \partial_w + \frac{\lambda(\lambda - 1)}{2\rho(\lambda + 1)}\partial_\lambda \,, \qquad \tilde{l} = \partial_{\bar{w}} - \frac{\lambda(\lambda + 1)}{2\rho(\lambda - 1)}\partial_\lambda \,.$$

Points of $\mathbb{T}(U)$ are the leaves of $\{l, \tilde{l}\}$ given by constant

$$u = \frac{\rho}{2}(\lambda + \frac{1}{\lambda}) + iz$$

$\mathbb{T}(U)$ is a non-Hausdorff Riemann surface with affine holomorphic coordinate u so $u : \mathbb{T}(U) \to \mathbb{CP}^1$.

Reduced Ward correspondence,

Theorem (M. & Woodhouse 1986)

Solutions to the stationary axisymmetric SDYM equations on U are in 1:1 correspondence with holomorphic vector bundles $E \to \mathbb{T}(U)$ such that p^*E is trivial over each $\mathbb{CP}^1$ in $U \times \mathbb{CP}^1$. Suppose further

- *$\bar{E}$ = pull-back of E^* by complex conjugation $u \to \bar{u}$*
- *interchange of sheets of $\mathbb{T}(U) \to \mathbb{CP}^1$ sends $E \to E^*$,*

then J is real and symmetric and is a solution of the reduced vacuum equations.

(Triviality condition is generic for small enough U.)

Assume solution analytic (or meromorphic) at $\rho = 0$.

- $\mathbb{T}(U) = \mathbb{CP}^1_0$ glued to $\mathbb{CP}^1_\infty$ over U.
- There is a canonical normalization of twistor data:

$$E|_{\mathbb{CP}^1_0} \cong E|_{\mathbb{CP}^1_\infty} \cong \mathcal{O}(p) \oplus \mathcal{O}(q), \qquad (\mathcal{O}(p) = \mathbb{C}\text{-line bundle, } c_1 = p)$$

- Need $P(u) : U \to \mathrm{SL}(2,\mathbb{C})$ to patch $E|_{\mathbb{CP}^1_0}$ to $E|_{\mathbb{CP}^1_\infty}$.
- J is obtained from *Riemann-Hilbert problem* in λ-plane

$$G_0(z,\rho,\lambda) = \begin{pmatrix} \frac{\rho^p}{\lambda^p} & 0 \\ 0 & \frac{\rho^q}{\lambda^q} \end{pmatrix} P(u) \begin{pmatrix} (-\rho\lambda)^p & 0 \\ 0 & (-\rho\lambda)^q \end{pmatrix} G_\infty(z,\rho,\lambda),$$

$u = iz + \frac{\rho}{2}(\frac{1}{\lambda} + \lambda)$, G_0 holomorphic on $|\lambda| \leq 1$, G_∞ on $|\lambda| \geq 1$.

- Finally

$$J(z,\rho) = G_0(z,\rho,0) G_\infty(z,\rho,\infty)^{-1}.$$

P can be retrieved from finite order ρ-expansion of J at $\rho = 0$.

Tau functions for Painlevé III transcendants

- Stationary axisymmetric SDYM with ∂_z symmetry reduces to Painlevé III [M.& Woodhouse (1997)].

$$y'' = \frac{y'^2}{y} - \frac{y'}{\rho} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}$$

- ∂_z symmetry implies exponential dependence for $P(u)$.
- Explicit Tau function solutions follow from combinations of Ward Ansatze, to solve Riemann-Hillbert problem:

$$G_0(\rho, \lambda) = \begin{pmatrix} \lambda^{-L}\mathbf{e}^{\theta u} & \mathbf{e}^{u} \\ 0 & \lambda^{L}\mathbf{e}^{-\theta u} \end{pmatrix} G_\infty(\rho, \lambda)$$

- Tau functions obtained as determinants of $L \times L$ Matrices

$$\tau(\rho) = \det(A_{ij}), \qquad A_{ij} = \psi^{i+j-2} \quad \psi^k = (\rho\partial_\rho)^k(c_1 I_\nu(\rho) + c_2 I_{-\nu}),$$

where $I_{\pm\nu}(\rho)$ are Bessel functions cf [Okamoto, Masuda 2007].

Integrability of N=4 Super Yang-Mills (MSYM)

- **Fields:** Yang-Mills connection A, 6 scalars Φ_i, and Fermions.
- Action $\int_M \mathrm{Tr}(F_A^2 + [\Phi_i, \Phi_j]^2 + \mathrm{Fermions})$
- Conformal field theory.
- QFT controlled by strings in $\mathrm{AdS}^5 \times S^5$
- Strings are minimal surfaces $Y : \Sigma \to AdS^5 \times S^5$.
- Integrable with Lax pair

$$L = d + \frac{1-\lambda}{1+\lambda} j + \frac{1+\lambda}{1-\lambda} \bar{j}$$

  where $j \in \Omega^{1,0}_\Sigma \otimes su(2,2|4)$ and L has $\mathbb{Z}_4$ symmetry.
- This all needs to be quantized $\rightsquigarrow$ integrable spin chains ...

What can you calculate?

Anomalous dimensions

- The *cusp anomalous dimension* is a Wilson loop

$$\gamma(\theta, \phi, \rho) = \left\langle \mathrm{tr}\ \mathrm{Pexp} \int_{\Gamma_{\theta,\phi}} (A + \mathrm{d}s\, n^i \Phi_i) \right\rangle$$

  with $|n| = 1$, $\Gamma =$ two arcs of circles in $\mathbb{M}$ with angle θ, and $n \cdot n' = \cos\phi$.
- QFT calculation $\rightsquigarrow$ Feynman integrals, multi-zeta values, polylogs ...
- Calculate using *quantum spectral curve* [Gromov,Kazakov, Volin 2013. . .].
- Near BPS to $O((\theta - \phi)^2)$, QSC becomes

$$G_0(\rho, \lambda) = \begin{pmatrix} \lambda^{-2L} \mathbf{e}^{\theta u} & \sinh 2\pi u \\ 0 & \lambda^{2L} \mathbf{e}^{-\theta u} \end{pmatrix} G_\infty(\rho, \lambda)$$

- $\gamma = \tau$-function with $\psi^k = I^\theta_k$ generalized Bessel functions $\sinh(2\pi\rho(\lambda + \frac{1}{\lambda}) \exp(\theta\rho(\lambda - \frac{1}{\lambda}) = \sum_\nu I^\theta_k \lambda^k$.

Full Quantum Spectral Curve

More generally

- $G_0^a(\rho, \lambda)$ on $|\lambda| < 1 + \epsilon$, $a = 1, \ldots, 4$,
- $G_\infty^a(\rho, \lambda)$ on $|\lambda| > 1 - \epsilon$
- Instead of P have $\mu_{ab} = -\mu_{ba}$ near $|\lambda| = 1$.
- Near $|\lambda| = 1$ we have

$$G_0^a = \mu^{ab} G_\infty^b$$

- In near BPS cases $\mu_{ab} = \mu_{ab}(u)$ periodic in $u \to u + i$.
- In general it satisfies its own Riemann Hilbert problem in terms of G_0^a and G_∞^a and asymptotics as $\lambda \to 0, \infty$

$$\mu^{ab} - \tilde{\mu}^{ab} = G_0^{[a} G_\infty^{b]}$$

Gives all anomalous dimensions for $N = 4$ SYM.

Summary

- Non-Hausdorff reduced twistor space underlies geometry of Ernst equations, Painlevé III, and Quantum spectral curve.
- Near BPS anomalous dimensions arise from adaptations of twistor correspondences and satisfy reductions of SDYM equations.

Further projects:

- Near BPS is close to classical minimal surface problem for string in AdS^5 where ρ is radius of AdS^5.
- Away from BPS, system becomes more complicated and coupled, although still geometric.
  - Is this some underlying geometric structure on $\mathbb{T}(U)$?
  - Do these give stationary axisymmetric SDYM solutions periodic in z?
  - Any relation to $N = 4$ SYM twistor action?

Thank you!