Eynard–Orantin recursion for simple singularities of type A

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W-algebra of an isolated singularity

Frobenius structure

$W \in \mathcal{O}_{\mathbb{C}^{n+1},0}, \quad \{\phi_i\}_{i=1}^{N} \subset H := \mathcal{O}_{\mathbb{C}^{n+1},0}/(\partial_{x_0}W, \ldots, \partial_{x_n}W)$

Miniversal deformation

$$F(x,t) = W(x) + \sum_{i=1}^{N} t_i \phi_i(x), \quad t \in B, \quad (x,t) \in X.$$

The domain of F is appropriately chosen

$$\begin{array}{ccccc} C & \subset & X & \longrightarrow & \mathbb{C}^{n+1} \times \mathbb{C}^{N} \\ & & \pi\downarrow & & \downarrow \mathrm{pr}_2 \\ & & B & \longrightarrow & \mathbb{C}^{N} \end{array}$$

Relative critical set of F

$$\mathcal{O}_C := \mathcal{O}_X/(\partial_{x_0}F, \ldots, \partial_{x_n}F)$$

Kodaira–Spencer isomorphism

$$\mathcal{T}_B \to \pi_* \mathcal{O}_C, \quad \partial_{t_i} \mapsto \partial_{t_i} F \ \mathrm{mod}(\partial_{x_0} F, \ldots, \partial_{x_n} F)$$

Let us fix a primitive form in the sense of K. Saito

$$\omega = g(x,t) dx_0 \wedge \cdots \wedge dx_n \in \Omega^{n+1}_{X/B}(X)$$

$\mathcal{T}_B$ inherits a residue pairing

$$(\psi_1(x,t), \psi_2(x,t)) = \mathrm{Res}_C \frac{\psi_1(x,t)\psi_2(x,t)}{F_{x_0} \cdots F_{x_n}} g(x,t)^2 dx$$

and a multiplication $\bullet$

$\Rightarrow$ Frobenius structure on B (Hertling, Saito–Takahashi).

$$T^*B \cong TB \cong B \times T_0 B \cong B \times H.$$

The total ancestor potential

Higher genus reconstr. of Givental $\Rightarrow$ family of CohFTs

$$^t\Lambda_{g,r} : H^{\otimes r} \to H^*(\overline{\mathcal{M}}_{g,r}, \mathbb{C}), \quad t \in B_{\mathsf{ss}}$$

CohFT correlators

$$\langle v_1\psi_1^{k_1}, \ldots, v_r\psi_r^{k_r}\rangle_{g,n}(t) := \int_{\overline{\mathcal{M}}_{g,r}} {}^t\Lambda_{g,r}(v_1, \ldots, v_r)\, \psi_1^{k_1} \cdots \psi_r^{k_r}.$$

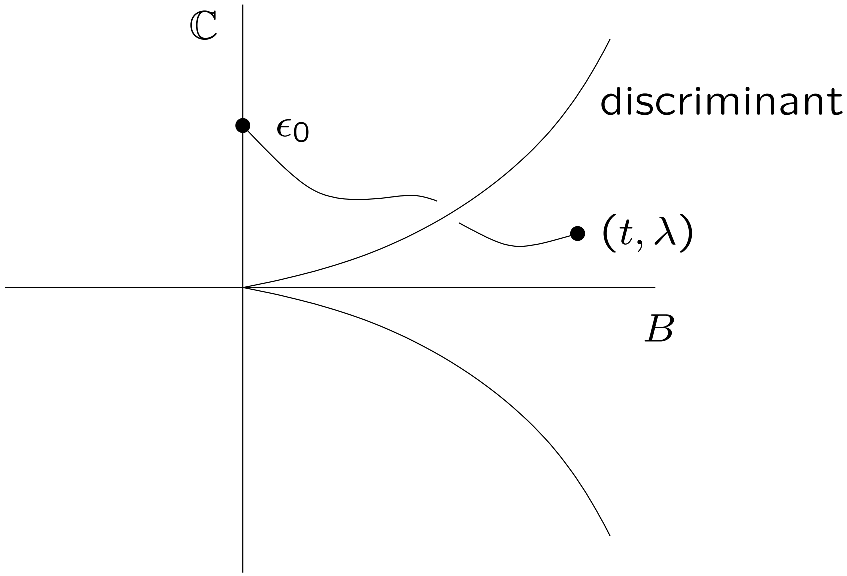
Generating function for the correlators

$$\mathcal{A}_t(\hbar, \mathbf{t}) = \exp\left(\sum_{g,r=0}^{\infty} \frac{\hbar^{g-1}}{r!}\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_r)\rangle_{g,r}(t)\right),$$

where

$$\mathbf{t}(z) = \sum_{k=0}^{\infty}\sum_{i=1}^{N} t_{k,i}\phi_i z^k.$$

Period vectors



Milnor fibration with fibers

$$X_{t,\lambda} := \{(x,t) \in X \mid F(x,t) = \lambda\}.$$

$$\begin{aligned}\mathfrak{h} &:= H^n(X_{0,\epsilon_0}, \mathbb{C}) \\ \mathfrak{h}^* &:= H_n(X_{0,\epsilon_0}, \mathbb{C}) \supset Q := H_n(X_{0,\epsilon_0}, \mathbb{Z}) \supset \text{van. cycles } \Delta\end{aligned}$$

Given $k \in \mathbb{Z}$ and $\alpha \in \mathfrak{h}^*$ we define

$$I_\alpha^{(k)}(t,\lambda) := -(2\pi)^{-\ell} d_t (\partial_\lambda)^{k+\ell} \int_{\alpha_{t,\lambda}} d^{-1}\omega \quad \in \quad T_t^* B \cong H,$$

where $\alpha_{t,\lambda} \in H_n(X_{t,\lambda}; \mathbb{C})$ and $n =: 2\ell$.

Lemma 1. *There exists a linear isom.* $\Psi : \mathfrak{h}^* \to H$, *s.t.,*

$$I_\alpha^{(-k)}(0,\lambda) = e^{\rho\,\partial_\lambda \partial_k} \left(\frac{\lambda^{\theta+k-\frac{1}{2}}}{\Gamma(\theta+k+\frac{1}{2})} \right) \Psi(\alpha),$$

where $\rho : H \to H$ *is multiplication by* W *(nilp. operator) and* θ *is diagonaliz. and skew-symmetric w.r.t. (,).*

Intersection pairing

$$(\alpha|\beta) := (-1)^\ell \alpha \circ \beta = (I_\alpha^{(0)}(0,\lambda), (\lambda-\rho) I_\beta^{(0)}(0,\lambda))$$

W-algebra

Heisenberg Lie algebra (int. pairing) $H[s, s^{-1}] \oplus \mathbb{C}$

$$[f_1(s), f_2(s)] = \mathrm{Res}_{s=0}(f_1'(s)|f_2(s))ds.$$

Fock space representation

$$\mathcal{F} := \mathbb{C}[\![x_1]\!][x_2, x_3, \ldots], \quad x_k = (x_{k,1}, \ldots, x_{k,N}).$$

$$\phi_i s^{-k} \mapsto x_{k,i}, \quad \phi_j s^k \mapsto \sum_{i=1}^{N} k(\phi_i|\phi_j)\partial/\partial x_{k,i}.$$

Screening operators

$$Q_\alpha = \mathrm{Res}_{z=0}\left(e^{\sum_{n<0} \alpha_n \frac{\zeta^n}{n}} e^{\sum_{n>0} \alpha_n \frac{\zeta^n}{n}}\right) \frac{d\zeta}{\zeta}, \quad \alpha \in \Delta.$$

$$\mathcal{W}_\Delta = \{v \in \mathcal{F} \mid Q_\alpha(v) = 0 \ \forall \alpha \in \Delta\}.$$

Twisted representation

Heisenberg Lie algebra (res.pairing) $H[z,z^{-1}]\oplus\mathbb{C}((\hbar^{1/2}))$

$$[f_1(z),f_2(z)] = \hbar\ \mathrm{Res}_{z=0}(f_1(-z),f_2(z))dz.$$

Ancestor Fock space representation

$$\mathcal{A}_t \in \mathbb{C}_\hbar[\![t_0,t_1,\dots]\!]$$

defined by

$$\phi_i z^k \mapsto -\hbar\,\partial_{q_{k,i}}, \qquad \phi^i(-z)^{-k-1} \mapsto q_{k,i},$$

and the *dilaton shift*

$$t_{k,i} = q_{k,i} + \delta_{k,1} a_i, \qquad \sum a_i \phi_1 = 1.$$

We want to define state-field correspondence

$$\mathcal{F} \ni v \mapsto \text{diff. oper. acting on anc. Fock space}$$

W-constraints

Generating fields

$$Y_{t,\lambda}(\alpha s^{-1}) = \hbar^{-1/2} \sum_{k\in\mathbb{Z}} I_\alpha^{(k+1)}(t,\lambda)\,(-z)^k$$

OPE axiom holds

$$Y_{t,\lambda}(\alpha s^{-k-1}v) = \mathrm{Res}_{\lambda'=\lambda}\Big(Y_{t,\lambda'}(\alpha s^{-k-1})Y_{t,\lambda}(v)\Big)\frac{d\lambda'}{(\lambda'-\lambda)^{k+1}}$$

Theorem 2. *(Bakalov-M) If $w \in \mathcal{W}_\Delta$, then*

$$\mathrm{Res}_{\lambda=\infty}\Big(\lambda^k\, Y_{t,\lambda}(w)\mathcal{A}_t(\hbar;\mathbf{t})\Big) = 0$$

for all $k \geq 0$.

Problem: It is very hard to construct states in the W-algebra. Only for ADE singularities the structure is well understood.

The Eynard–Orantin recursion

Spectral curve

(1) $x : \Sigma \to \mathbb{P}^1$ branched covering, s.t., if $u_i \in \mathbb{C}$ $(1 \leq i \leq N)$ is a finite branch point, then $x^{-1}(u_i)$ has a unique ramif. point p_i and p_i is a double ramif. point.

(2) $y : \Sigma \to \mathbb{P}^1$ meromorphic function, s.t., x and y generate the field $K(C)$ and $dy(p_i) \neq 0$ for all i.

(3) Fix a symplectic basis $\{\mathcal{A}_i, \mathcal{B}_i\}_{i=1}^{g} \subset H_1(\Sigma; \mathbb{Z})$.

The recursion defines symmetric forms (called correlator forms)

$$\omega_{g,n}(q_1,\ldots,q_n) = f_{g,n}(q_1,\ldots,q_n)dq_1\cdots dq_n,$$

where $f_{g,n} \in K(\Sigma^n)$ with poles only at the ramif. points.

$\omega_{g,n}$ is called *stable* if $2g-2+n>0$.

Riemann's 2nd fundamental form $B(q_1,q_2)$: unique symmetric quadratic differential with an order 2 pole (no residue) along the diagonal $q_1 = q_2$, s.t.,

$$B(q_1,q_2) = \frac{dt(q_1)dt(q_2)}{(t(q_1)-t(q_2))^2} + \cdots, \qquad \oint_{q_1\in\mathcal{A}_i} B(q_1,q_2) = 0.$$

Initial condition:

$$\omega_{0,2}(q_1, q_2) := B(q_1, q_2)$$

Recursion:

$$\omega_{g,n+1}(q, q_{n+1}) = \sum_{i=1}^{N} \text{Res}_{p=p_i} \frac{\frac{1}{2}\int_p^{\tau_i(p)} B(p', q_{n+1})}{(y(p) - y(\tau_i(p)))dx(p)}$$

$$(\omega_{g-1,n+2}(p, \tau_i(p), q) + \sum \omega_{g',n'+1}(p, q_{I'})\omega_{g'',n''+1}(\tau_i(p), q_{I''})),$$

where τ_i is the local deck transformation near p_i, $q = (q_1, \dots, q_n)$, the sum is over all splittings

$$g = g' + g'', \quad \{1, \dots, n\} = I' \sqcup I'',$$

and all unstable correlators, except for $\omega_{0,2}$, are 0.

Bouchard–Eynard recursion

Differential of the 3rd kind

$$\omega_p(q) := \int_\circ^p B(p', q),$$

where $\circ \neq p_i$ is chosen arbitrary.

For $p \in \Sigma$ and $\emptyset \neq J \subset \tau(p) := x^{-1}(x(p)) \setminus \{p\}$ define

$$K_J(p,q) = -\frac{\omega_p(q)}{\prod_{p' \in J} ((y(p) - y(p')dx(p))}, \quad q \in \Sigma.$$

Remark 3. *We have to assume that y separates the points in $x^{-1}(u_i)$ for all finite branch points $u_i \in \mathbb{C}$.*

Fix $q_i \in \Sigma$, $1 \leq i \leq n$ and $J \subset \tau(p)$.

$C^m_{g,n}$ = the topological type of a genus-g Riemann surface with m boundaries and n non-ordered marked points.

J-**partition of** $(C_{g,n}, q_1, \ldots, q_n)$ is the data

$\{(C^{m_i}_{g_i,n_i}, J_i, I_i)\}^s_{i=1}, \quad J \cup \{p\} = \sqcup_i J_i, \quad \{q_1, \ldots, q_n\} = \sqcup_i I_i,$ s.t.,

each boundary is labeled by a point from J_i,

each marked point is labeled by a point from I_i,

$$\bigsqcup_{i=1}^{s} (C^{m_i}_{g_i,n_i}, J_i, I_i) \bigcup (C^m_{0,0}, J \cup p) = (C_{g,n}, q_1, \ldots, q_n),$$

where we glue boundaries, so that their labels match.

Define

$$\Omega_J^{(g)}(p, q_1, \ldots, q_n) = \sum \prod_{i=1}^{s} \omega_{g_i, m_i+n_i}(J_i, I_i),$$

where the sum is over all J-partitions of $(C_{g,n}, q_1, \ldots, q_n)$.

Bouchard–Eynard recursion:

$$\omega_{g,n+1}(q, q_{n+1}) = \frac{1}{2\pi\sqrt{-1}} \oint_{p \in \Gamma} \sum_J K_J(p, q_{n+1}) \Omega_J^{(g)}(p, q),$$

where $q = (q_1, \ldots, q_n) \in \Sigma^n$ and Γ is a union of closed loops around the ramification points p_i.

EO recursion in singularity theory

Main idea: Use the BE recursion to find states in the W-algebra.

Local EO recursion

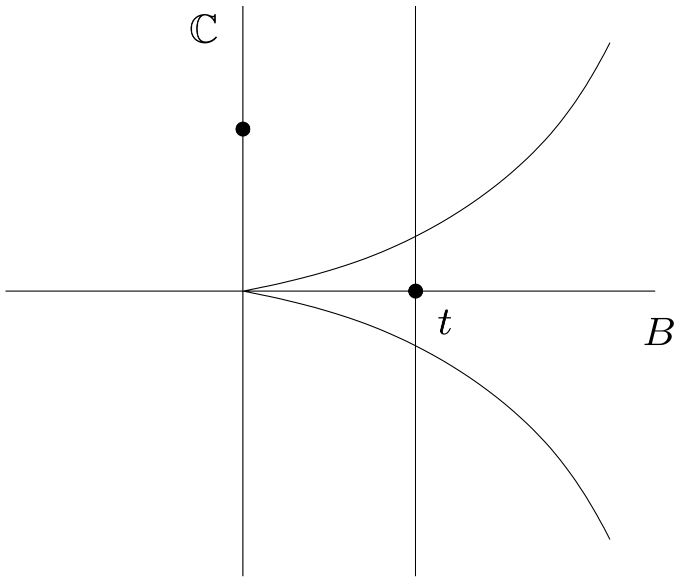
D-BOSS (arXiv: 1211.5273), M (arXiv: 1211.5847)

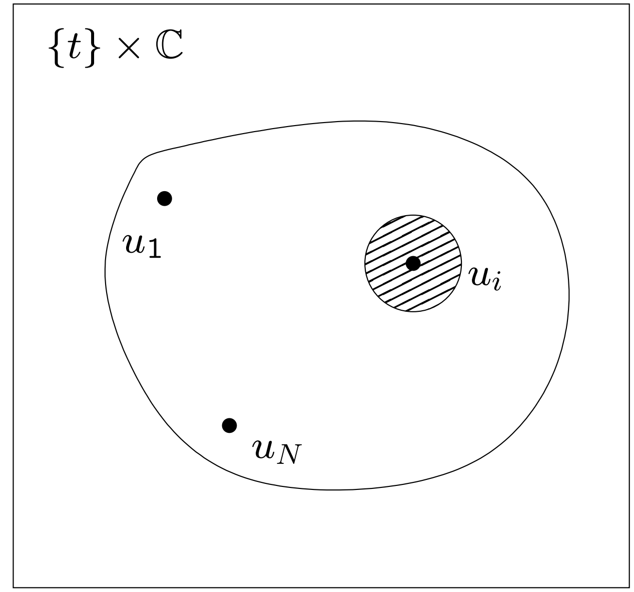
Given

$$c^1, \ldots, c^r \in H, \quad (t, \lambda) \in B \times \mathbb{C} \setminus \text{discriminant}$$

define $\Omega^{(g)}_{c^1,\ldots,c^r}(t, \lambda; \mathbf{t})$ by

$$Y_{t,\lambda}(c^1 \cdots c^r)\, \mathcal{A}_t(\hbar; \mathbf{t}) =: \left(\sum_{g=0}^{\infty} \hbar^{g-\frac{r}{2}} \Omega^{(g)}_{c^1,\ldots,c^r}(t, \lambda; \mathbf{t}) \right) \mathcal{A}_t(\hbar; \mathbf{t})$$





$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle \phi_a \psi^m, \mathbf{t}, \ldots, \mathbf{t} \rangle_{g,n+1} =$$

$$\frac{1}{4} \sum_{i=1}^{N} \mathrm{Res}_{\lambda=u_i} \frac{(I_{\beta_i}^{(m+1)}(t,\lambda), \phi_a)}{(I_{\beta_i}^{(-1)}(t,\lambda), 1)} \;\Omega^{(g)}_{\beta_i,\beta_i}(t,\lambda;\mathbf{t})\; d\lambda$$

Global recursion for A_N-singularity

For A_N singularity we have

$$F(x,t) = x^{N+1} + t_1 x^{N-1} + \cdots + t_N$$

with vanishing cycles

$$\Delta = \{\chi_i - \chi_j \mid 1 \le i \ne j \le N+1\}$$

Theorem 4. *(M) The RHS of the local recursion is equivalent to*

$$-\frac{1}{2\pi\sqrt{-1}} \oint_C \sum_{i=1}^{N+1} \sum_J \frac{(I^{(m+1)}_{\chi_i}(t,\lambda), \phi_a)}{\prod_{j\in J}(I^{(-1)}_{\chi_i-\chi_j}(t,\lambda), 1)} \ \Omega^{(g)}_{\chi_i\chi_{j_1}\cdots\chi_{j_r}}(t,\lambda;\mathbf{t})\ d\lambda$$

where we sum $\forall\ J = \{j_1, \ldots, j_r\} \subset \{1, 2, \ldots, N+1\} \setminus \{i\}$.

We get differential operators

$$\partial_{t_{m,a}} + \frac{1}{2\pi\sqrt{-1}} \oint_C \sum_{i=1}^{N+1} \sum_{J} {}^{*}\ \hbar^{(r-1)/2} Y_{t,\lambda}(\chi_i \chi_{j_1} \cdots \chi_{j_r})\ d\lambda$$

annihilating the total ancestor potential.

Theorem 5. *(Lewanski-M) Under the dilaton shift the above operator turns into*

$$\frac{1}{2\pi\sqrt{-1}}\ \oint_C \lambda^m Y_{t,\lambda}(e_{N+2-a}),$$

where e_r is the rth elementary symmetric polynomial in χ_i, $1 \le i \le N+1$.

It is easy to check that $e_r \in \mathcal{W}_\Delta$, $2 \le r \le N+1 \Rightarrow$ the EO recursion $\Leftrightarrow$ the W-constraints.

Thank you!