Integrability in Grassmann and other Geometries

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General construction

Let $\mathbf{Gr}(d,n)$ be the Grassmannian of d-dimensional linear subspaces of an n-dimensional vector space V^n. A submanifold $X \subset \mathbf{Gr}(d,n)$ gives rise to a differential system $\Sigma(X)$ governing d-dimensional submanifolds of V^n with Gauss image in X. Since d-dimensional submanifolds of V^n are parametrised by $n-d$ functions of d variables, we will assume that the codimension of X in $\mathbf{Gr}(d,n)$ also equals $n-d$: in this case $\Sigma(X)$ will be a determined system of $n-d$ first-order PDEs for $n-d$ unknown functions of d independent variables.

Main question: When is $\Sigma(X)$ integrable?

Based on:

B. Doubrov, E.V. Ferapontov, B. Kruglikov, V.S. Novikov, On the integrability in Grassmann geometries: integrable systems associated with fourfolds in $\mathbf{Gr}(3,5)$, *arXiv:1503.02274.*

Systems associated with fourfolds $X \subset \mathbf{Gr}(3,5)$

Introducing in V^5 coordinates x^1, x^2, x^3, u, v one can parametrise three-dimensional submanifolds of V^5 in the form $u = u(x^1, x^2, x^3)$, $v = v(x^1, x^2, x^3)$. The corresponding system $\Sigma(X)$ reduces to a pair of first-order PDEs for u and v (with $u_i = \partial u/\partial x^i,\ v_i = \partial v/\partial x^i$)

$$F(u_1, u_2, u_3, v_1, v_2, v_3) = 0, \quad G(u_1, u_2, u_3, v_1, v_2, v_3) = 0. \tag{1}$$

Here the Grassmannian $\mathbf{Gr}(3,5)$ is identified with the space of 2×3 matrices,

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix},$$

and equations (1) specify a fourfold $X \subset \mathbf{Gr}(3,5)$.

Non-degeneracy: horizontal conformal structure

We will assume that the characteristic variety of (1) is non-degenerate. This conic curve is given by the following dispersion relation

$$\det\left[\sum_{i=1}^{3}\lambda^i\begin{pmatrix}F_{u_i} & F_{v_i}\\ G_{u_i} & G_{v_i}\end{pmatrix}\right]=0$$

on the projective parameter $[\lambda]=[\lambda^1:\lambda^2:\lambda^3]\in\mathbb{P}^2$. Equivalently, this is written $\lambda\, g^\sharp\lambda^t=0$, where $g^\sharp$ is a symmetric 3×3 matrix with the entries

$$g^{ij}=\tfrac{1}{2}(F_{u_i}G_{v_j}+F_{u_j}G_{v_i}-F_{v_i}G_{u_j}-F_{v_j}G_{u_i}).$$

Non-degeneracy is equivalent to $\det[g^\sharp]\neq 0$, and then the inverse matrix g_{ij} yields the canonical conformal structure $g=g_{ij}dx^idx^j$ on any solution.

The hyperbolicity condition adapted below is that this metric has Lorentzian signature (or one should use complexification).

Examples: Bäcklund transformations

The following 2-component system in 3D

$$\{v_y - \frac{1}{2}v_x^2 - u_x = 0, \quad v_t - \frac{1}{3}v_x^3 - v_x u_x - u_y = 0\}$$

defines a Bäcklund transformation between the dKP and mdKP equations:

$$u_{xt} - u_x u_{xx} - u_{yy} = 0 \qquad \leftrightsquigarrow \qquad v_{xt} - (v_y - \frac{1}{2}v_x^2)v_{xx} - v_{yy} = 0.$$

. . . . .

Let $a_1 + a_2 + a_3 = 0$ and $\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 = 0$ be constants. The system

$$\{a_1\tilde{a}_2 u_x v_y - a_2\tilde{a}_1 u_y v_x = 0, \quad a_1\tilde{a}_3 u_x v_t - a_3\tilde{a}_1 u_t v_x = 0\}$$

defines a Bäcklund transformation between the Veronese web equations

$$a_1 u_x u_{yt} + a_2 u_y u_{xt} + a_3 u_t u_{xy} = 0 \quad \leftrightsquigarrow \quad \tilde{a}_1 v_x v_{yt} + \tilde{a}_2 v_y v_{xt} + \tilde{a}_3 v_t v_{xy} = 0.$$

Equivalence group $\mathbf{SL}(5)$

The linear action of $\mathbf{SL}(5)$ on the variables x^1, x^2, x^3, u, v naturally extends to $\mathbf{Gr}(3,5)$, identified with 2×3 matrices U of partial derivatives u_i, v_i:

$$U \to (AU + B)(CU + D)^{-1};$$

note that the extended action is no longer linear. These transformations preserve the class of equations (1), indeed, first-order derivatives transform through first-order derivatives only. Moreover, they preserve the integrability. Two $\mathbf{SL}(5)$-related equations should be regarded as ‘the same’.

In fact, this equivalence group is maximal possible, because it can be considered as the group of transformations preserving the charactericstics consisting of Segre cones rank$(dU) = 1$, which is internal for the problem.

The method of hydrodynamic reductions

Applies to quasilinear equations

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0.$$

Consists of seeking N-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^N).$$

The phases $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x,$$

(called hydrodynamic reductions). Copatability conditions: $\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$.

Definition. A 2+1 quasilinear system is said to be integrable if, for any N, it possesses infinitely many N-component reductions parametrized by N arbitrary functions of one variable.

Hydrodynamic reductions: 2×2 systems

First we represent system (1) in evolutionary form,

$$u_t = f(u_x, u_y, v_x, v_y), \quad v_t = g(u_x, u_y, v_x, v_y). \tag{2}$$

Next, we bring it into quasilinear form by choosing first-order derivatives of u and v as the new dependent variables, and writing out all possible consistency conditions among them. Applying the method of hydrodynamic reductions, one can write down the integrability conditions in symbolic form,

$$d^3 f = R(df, dg, d^2 f, d^2 g), \quad d^3 g = S(df, dg, d^2 f, d^2 g),$$

40 equations altogether (in involution!).

Theorem 1. The parameter space of non-degenerate integrable systems (1) is 30-dimensional.

Moduli of integrable 2×2 systems

Consider the action of the equivalence group $\mathbf{SL}(5)$ on the parameter space of integrable equations (2). The group acts transitively on the Grassmanian, and the stabilizer of a point $o \in \mathbf{Gr}(3,5)$ is the parabolic subgroup $P_o = S(\mathbf{GL}(3) \times \mathbf{GL}(2)) \ltimes (\mathbb{R}^3 \otimes \mathbb{R}^2)$ acting on $T_o\mathbf{Gr}(3,5) = \mathbb{R}^3 \otimes \mathbb{R}^2$. It induces the action on the codimension 2 submanifolds of the latter space.

These are 2×2 systems: $\mathbb{R}^4(a,b,c,d) \to \mathbb{R}^2(f,g)$, where $a = u_x, b = u_y$, $c = v_x, d = v_y, f = u_t, g = v_t$. The algebra $\mathfrak{sl}(5)$ acts transitively on $J^0(\mathbb{R}^4, \mathbb{R}^2) = \mathbb{R}^4 \times \mathbb{R}^2 = \mathbb{R}^6$, and it prolongs to the higher order jet spaces.

There is an open orbit on $J^1(\mathbb{R}^4, \mathbb{R}^2)$ (nondegenerate 4-planes), and the action on $J^2(\mathbb{R}^4, \mathbb{R}^2)$ has no stabilizer at a generic point, so the orbits are 24-dimensional.

The action of $\mathbf{SL}(5)$ on the parameter space $\mathcal{M}^{30} \subset J^2(\mathbb{R}^4, \mathbb{R}^2)$ is algebraic.

Theorem 1^+. The moduli space of non-degenerate integrable systems (1) is a 6-dimensional rational variety $\mathcal{M}^{30}/\mathbf{SL}(5) = \bar{\mathcal{M}}^6$.

Integrability: Dispersionless Lax pairs

System

$$v_y - \frac{1}{2}v_x^2 - u_x = 0, \quad v_t - \frac{1}{3}v_x^3 - v_x u_x - u_y = 0,$$

possesses the Lax pair

$$S_y = S_x^2 + v_x S_x, \quad S_t = \frac{4}{3}S_x^3 + 2v_x S_x^2 + (u_x + v_x^2)S_x,$$

i.e. the compatibility condition is satisfied modulo the equation.

Theorem 2. Every non-degenerate integrable system (1) possesses a dispersionless Lax pair of the type

$$S_y = P(S_x, u_i, v_i), \quad S_t = Q(S_x, u_i, v_i).$$

In the general case parametrization of solutions is given by via generalized hypergeometric functions: Odesskii-Sokolov construction.

Geometry 'on solutions': Einstein-Weyl geometry

Given conformal structure g^{ij}, introduce the covector ω,

$$\omega_k = 2g_{kj}\mathcal{D}_{x^s}(g^{js}) + \mathcal{D}_{x^k}(\ln\det g_{ij}), \tag{3}$$

and the symmetric Weyl connection $\mathbb{D}$ such that $\mathbb{D}_k g_{ij} = \omega_k g_{ij}$.

Theorem 3. System (1) is integrable if and only if on every solution the triple $\mathbb{D}, g, \omega$ satisfies the Einstein-Weyl equations,

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad Ric_{(ij)} = \Lambda g_{ij}.$$

Here $Ric_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$, and Λ is some function.

Einstein-Weyl geometry is integrable (Cartan, Hitchin). Thus, solutions to integrable equations carry integrable geometry.

Deviation: Scalar second order PDE

Consider a PDE of Hirotha type

$$F(u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0. \tag{4}$$

with associated conformal symmetric bivector $g^\sharp$, $g^{ij} = \frac{\partial F}{\partial u_{ij}}$. The symbol of PDE is non-degenerate if the characteristic varity is a nondegenerate conic curve in $\mathbb{P}^2$, i.e. $\det g^\sharp \neq 0$. Then $(g_{ij}) = (g^{ij})^{-1}$ defines a conformal structure on any solution $g = g_{ij}dx^i dx^j$. Formula (3) defines the covector ω and so Weyl structure.

The following was proved in E.Ferapontov & BK, *Dispersionless integrable systems in 3D and Einstein-Weyl geometry*, Journal of Differential Geometry **97**, (2014):

Theorem 3*. PDE (4) is integrable iff $(\mathbb{D}, g, \omega)$ satisfies the EW equations,

$$\mathbb{D}g = \omega \otimes g, \quad Ric_{\mathbb{D}}^{\text{sym}} = \Lambda \cdot g.$$

This is also true for general 2nd order PDE (D.Calderbank & BK: work in progress).

Geometry ‘on equation’: $\mathrm{GL}(2)$ geometry

The tangent bundle to the Grassmannian $\mathbf{Gr}(3,5)$ carries canonical generalised conformal structure defined by the family of Segre cones $du_i dv_j - du_j dv_i = 0$. Given a fourfold $X \subset \mathbf{Gr}(3,5)$, the intersection of its tangent space $\mathrm{T}X$ with the Segre cone is a two-dimensional rational cone of degree three; its projectivisation is a rational normal curve of degree three (twisted cubic). This is known as a $\mathbf{GL}(2)$ structure on X. It was demonstrated by Bryant that every four-dimensional $\mathbf{GL}(2)$ structure defines on X a canonical affine connection (with torsion).

Theorem 4. System (1) is integrable if and only if X possesses infinitely many holonomic 3-folds. This is equivalent to the condition that the curvature R and the covariant derivative ∇T of the torsion T of the Bryant connection are certain invariant quadratic expressions in T,

$$R = f(T^2), \quad \nabla T = g(T^2).$$

$\mathfrak{sl}(2)$ decomposition into irreps for $\mathrm{GL}(2)$-structures:

$$\tau \otimes \tau^* \otimes \Lambda^2 \tau^* = 2V_0 \oplus 4V_2 \oplus 5V_4 \oplus 4V_6 \oplus 2V_8 \oplus V_{10}.$$

$$R = R_{(0)} + R_{(2)} + R_{(4)} + R_{(6)}, \quad \nabla T = \nabla T_{(4)} + \nabla T_{(6)} + \nabla T_{(8)} + \nabla T_{(10)},$$

$$(T^2)^k_{lij} = T^k_{la} T^a_{ij}, \quad (T^2_\alpha)^k_{lij} = T^k_{la} \Omega^{ab} T^c_{b[i} \Omega_{cj]}, \quad (T^2_\beta)^k_{lij} = T^k_{[ja} \Omega^{ab} T^c_{bl} \Omega_{ci]},$$
$$(T^2_\gamma)^k_{lij} = T^k_{[ia} \Omega^{ab} T^c_{bj]} \Omega_{cl}, \quad (T^2_\delta)^k_{lij} = \Omega^{ka} T^b_{al} \Omega_{bc} T^c_{ij};$$

$$R_{(0)} = 0, \;\; R_{(4)} = 0, \;\; \nabla T_{(4)} = 0, \;\; \nabla T_{(8)} = 0, \;\; \nabla T_{(10)} = -28\, T^2_{\alpha\,(10)},$$
$$R_{(2)} = \tfrac{44}{3}\, T^2_{\alpha\,(2)} + 2\, T^2_{\beta\,(2)} - \tfrac{40}{3}\, T^2_{\gamma\,(2)} - 2\, T^2_{\delta\,(2)},$$
$$R_{(6)} = -24\, T^2_{\alpha\,(6)} - 30\, T^2_{\beta\,(6)} - 60\, T^2_{\gamma\,(6)} - 24\, T^2_{\delta\,(6)},$$
$$\nabla T_{(6)} = -8\, T^2_{\alpha\,(6)} - 8\, T^2_{\beta\,(6)} - 16\, T^2_{\gamma\,(6)} - 4\, T^2_{\delta\,(6)}.$$

Linearisable systems

Systems of Monge-Ampère type are linear combinations of minors of the 2×3 matrix U:

$$a^{ij}(u_i v_j - u_j v_i) + b^i u_i + c^i v_i + m = 0,$$
$$\alpha^{ij}(u_i v_j - u_j v_i) + \beta^i u_i + \gamma^i v_i + \mu = 0.$$

Proposition. For non-deg system (1), the following conditions are equivalent:

(a) System is linearisable by a transformation from the equivalence group $\mathbf{SL}(5)$.

(b) System belongs to the Monge-Ampère class.

(c) System is invariant under an 8-dimensional subgroup of $\mathbf{SL}(5)$.

(d) The principal symbol conformal structure is flat on every solution.

Linearly degenerate systems: definition

The definition is inductive. Start with a 2D system,

$$F(u_x, u_t, v_x, v_t) = 0, \quad G(u_x, u_t, v_x, v_t) = 0.$$

Writing it in evolutionary form, $u_t = f(u_x, v_x),\ v_t = g(u_x, v_x)$, differentiating by x and setting $u_x = a, v_x = b$, we obtain a 2-component system of hydrodynamic type, $a_t = f(a, b)_x,\ b_t = g(a, b)_y$. The system is said to be linearly degenerate if the corresponding characteristic speeds λ^i are constant in the direction of the associated eigenvectors ξ_i: $L_{\xi_i} \lambda^i = 0$.

In 3D, system (1) is said to be linearly degenerate if every its travelling wave reduction to 2D is linearly degenerate in the above sense.

The parameter space of 3D linear degenerate systems $\mathcal{M}^{30}$ is 30-dimensional.

Linearly degenerate integrable systems: moduli

The integrability constraint gives a 22-dimensional sub-variety $\mathcal{M}^{22} \subset \mathcal{M}^{30}$. The equivalence group $\mathbf{SL}(5)$ acts on it with 4D stabilizer at generic point. Thus the generic orbits have dimension $24 - 4 = 20$ and the quotient $\mathcal{M}^{22}/\mathbf{SL}(5) = \bar{\mathcal{M}}^2$ is 2-dimensional. The general stratum of this moduli space is given by the equation:

$$u_x v_y = \alpha u_y v_x, \quad u_x v_t = \beta u_t v_x.$$

Proposition. (a) System (1) is linearly degenerate iff there exists a unique symmetric connection on X in which the associated $\mathbf{GL}(2)$ structure is parallel.

(b) System (1) is linearly degenerate and integrable iff there exists a unique flat symmetric connection on X in which the associated $\mathbf{GL}(2)$ structure is parallel.

(c) Such fourfolds X with integrable $\Sigma(X)$ come from the Chasles construction

$$\mathbb{P}^4 \dashrightarrow \mathbf{Gr}(3,5), \quad [\xi] \mapsto \langle \xi, A\xi \rangle^{\perp}, \quad A \in \mathbf{SL}(5).$$

Integrability in 4D: Moduli

Consider 4D systems,

$$F(u_i, v_i) = 0, \quad G(u_i, v_i) = 0. \tag{5}$$

Theorem 5. The parameter space of non-degenerate integrable systems in 4D is 36-dimensional. Any such system is necessarily linearly degenerate. Furthermore, the following conditions are equivalent:

(a) System is integrable by the method of hydrodynamic reductions.

(b) The principal symbol conformal structure g is anti-self-dual on every solution.

No explicit description yet. Similarly to the 3D case we obtain

Theorem 5^{+}. The moduli space of 4D non-degenerate integrable systems (1) is a 1-dimensional rational variety $\mathcal{M}^{36}/\mathbf{SL}(6) = \bar{\mathcal{M}}^1$.

Particular integrable examples are provided by systems of Monge-Ampère type.

Monge-Ampère systems in higher dimensions

Any such system is specified by a pair of differential d-forms in a $(d+2)$-dimensional vector space V with coordinates $x^1, \ldots, x^d, u, v$. Utilising the isomorphism between Λ^d and Λ^2, we can reduce the theory of normal forms of Monge-Ampère systems to the classification of pencils of skew-symmetric 2-forms.

Proposition. In four dimensions, any non-degenerate system of Monge-Ampère type is $\mathbf{SL}(6)$-equivalent to one of the following normal forms:

1. $u_2 - v_1 = 0, \quad u_3 + v_4 = 0,$
2. $u_2 - v_1 = 0, \quad u_3 + v_4 + u_1v_2 - u_2v_1 = 0,$
3. $u_2 - v_1 = 0, \quad u_3v_4 - u_4v_3 - 1 = 0,$
4. $u_2 - v_1 = 0, \quad u_1 + v_2 + u_3v_4 - u_4v_3 = 0.$

All these systems are integrable by the method of hydrodynamic reductions.

All of them are equivalent to various heavenly-type equations.

$\mathbf{GL}(2)$-geometry (= paraconformal geometry)

$\mathbf{GL}(2)$-structure on a manifold M^n is defined by a field of rational normal curves

$$\operatorname{Ker} \omega(\lambda) \subset \mathbb{P}T^*M, \quad \omega(\lambda) = \omega_0 + \lambda\omega_1 + \cdots + \lambda^{n-1}\omega_{n-1}, \qquad (6)$$

where ω_i is a coframe on M. The parameter λ is projective. Dually to (6) a $\mathbf{GL}(2)$-structure is defined via osculating hyperplanes by the field of rational normal curves $\Pi_\omega(\lambda) = \operatorname{Ker}(\omega, \omega_\lambda, \ldots, \omega_{\lambda\ldots\lambda}) \subset \mathbb{P}TM$.

A codimension one submanifold of M is an α-manifold if all its tangent spaces are α-hyperplanes $\Pi_\omega(\lambda) \subset TM$. A $\mathbf{GL}(2)$-structure on M is α-integrable [Krynski] if every α-hyperplane is tangential to some α-manifold.

Such structures arise on the solution spaces of scalar ODEs with vanishing Wünschmann invariants. Conversely, as shown by W.Krynski, every α-integrable $\mathbf{GL}(2)$-structure can be obtained from an ODE of this type. In 4D such structures were studied by R.Bryant in the context of exotic holonomy. In other aspects they were also studied by M.Dunajski, P.Tod, B.Doubrov, P.Nurowski, M.Godlinski.

Integrable system for α-integrable $\mathrm{GL}(2)$-structures

The general α-integrable $\mathbf{GL}(2)$-structure can be brought to the normal form

$$\omega(\lambda) = \sum_{i=1}^{n} \frac{u_i}{\lambda - \frac{u_i}{v_i}} dx^i,$$

where the functions u and v satisfy the system of PDEs with $a_i = \frac{u_i}{v_i},\ b_i = \frac{v_i}{u_i}$:

$$\mathop{\mathfrak{S}}_{(jkl)} (a_i - a_j)(a_k - a_l)\left(\frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} + \frac{2u_{kl} - (a_k + a_l)v_{kl}}{u_k u_l}\right) = 0,$$

$$\mathop{\mathfrak{S}}_{(jkl)} (b_i - b_j)(b_k - b_l)\left(\frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} + \frac{2v_{kl} - (b_k + b_l)u_{kl}}{v_k v_l}\right) = 0.$$

Solutions of this involutive system depend on $2(n-2)$ functions of 3 variables.

Theorem 6. This system possesses a dispersionless Lax representation, and can be viewed as a dispersionless integrable hierarchy. Consequently, the system of vanishing Wünschmann's conditions is an integrable dispersionless system of PDE.

Some open problems

- For $d = 3$, the moduli space of non-degenerate integrable systems $\Sigma(X)$ associated with submanifolds of codimension $n - 3 \geq 2$ in $\mathbf{Gr}(3, n)$ is finite-dimensional. Submanifolds X corresponding to 'generic' integrable systems are not algebraic.

- It would be challenging to classify integrable systems that correspond to *algebraic* fourfolds $X \subset \mathbf{Gr}(3, 5)$. The homology class of any such X can be represented as $a\sigma + b\eta$ where a, b are nonnegative integers, and σ, η are the standard four-dimensional Schubert cycles. Which values of a and b are compatible with the requirement of integrability?

- In higher dimensions $d \geq 4$, any non-degenerate integrable system $\Sigma(X)$ associated with a submanifold of codimension $n - d \geq 2$ in $\mathbf{Gr}(d, n)$ is necessarily linearly degenerate. Submanifolds X corresponding to linearly degenerate integrable systems are rational (generally, singular).