$\mathbb{Z}_N$ Graded Discrete Lax Pairs and Discrete Integrable Systems[1]

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[1]Joint work with Pavlos Xenitidis: arXiv:1411.6059 [nlin.SI]

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Integrable discretisations of “soliton equations”.

MKdV, SG, PKdV, Schwarzian KdV, Boussinesq and modified Boussinesq, etc.

Bianchi permutability (nonlinear superposition) of Bäcklund transformations leads directly to fully discrete equations.

Starting from the PDE, use Darboux transformations. Gives a discrete Lax pair for the discrete system.

Starting from a discrete Lax pair, we may derive the corresponding discrete systems.

May of may not have any relation to a continuous system.

Discrete Lax Pair: Square Lattice: discrete coordinates (m, n).

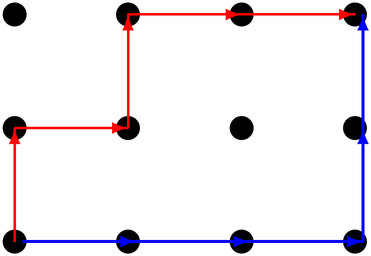
$$\left.\begin{array}{l} \Psi_{m+1,n} = L_{m,n}\,\Psi_{m,n} \\ \Psi_{m,n+1} = M_{m,n}\,\Psi_{m,n} \end{array}\right\} \quad \Rightarrow \quad L_{m,n+1}M_{m,n} \;=\; M_{m+1,n}L_{m,n}.$$

can be pictured as

$$\begin{tikzcd}
(m,n+1) \arrow[r, "L_{m,n+1}"] & (m+1,n+1) \\
(m,n) \arrow[u, "M_{m,n}"] \arrow[r, "L_{m,n}"'] & (m+1,n) \arrow[u, "M_{m+1,n}"']
\end{tikzcd}$$

Commutativity around the quadrilateral.

Path independent evaluation of $\Psi_{m,n}$:



Compatibility $\quad L_{m,n+1}M_{m,n} = M_{m+1,n}L_{m,n}$

implies that components of L and M satisfy

a discrete dynamical system.

Start with the $N \times N$ matrix

$$\Omega = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Definition (Level k matrix)

An $N \times N$ matrix A of the form

$$A = \operatorname{diag}\left(a^{(0)}, \dots, a^{(N-1)}\right) \Omega^k$$

will be said to have level k, written $lev(A) = k$.

Ω is cyclic: $\Omega^N = \mathrm{I}_N$ (level 0) and

$$lev(AB) = lev(BA) = lev(A) + lev(B) \ (\mathrm{mod} N).$$

With

$$\begin{aligned} U_{m,n} &= \operatorname{diag}\left(u_{m,n}^{(0)}, \dots, u_{m,n}^{(N-1)}\right) \Omega^{k_1}, \\ V_{m,n} &= \operatorname{diag}\left(v_{m,n}^{(0)}, \dots, v_{m,n}^{(N-1)}\right) \Omega^{k_2}, \end{aligned}$$

consider the Lax pair

$$\begin{aligned} \Psi_{m+1,n} &= L_{m,n}\Psi_{m,n} \equiv \left(U_{m,n} + \lambda\,\Omega^{\ell_1}\right)\Psi_{m,n}, \quad k_1 \neq \ell_1, \\ \Psi_{m,n+1} &= M_{m,n}\Psi_{m,n} \equiv \left(V_{m,n} + \lambda\,\Omega^{\ell_2}\right)\Psi_{m,n}, \quad k_2 \neq \ell_2, \end{aligned}$$

with compatibility condition $L_{m,n+1}M_{m,n} = M_{m+1,n}L_{m,n}$.

Equating powers of λ:

$$\begin{aligned} U_{m,n+1}V_{m,n} &= V_{m+1,n}U_{m,n}, \\ U_{m,n+1}\Omega^{\ell_2} - \Omega^{\ell_2}U_{m,n} &= V_{m+1,n}\Omega^{\ell_1} - \Omega^{\ell_1}V_{m,n}. \end{aligned}$$

and we find $k_1 + \ell_2 \equiv k_2 + \ell_1 (\mathrm{mod} N)$.

Example $N = 4$

Matrix L with $(k_1, \ell_1) = (1, 2)$

$$L_{m,n} = \begin{pmatrix} 0 & u^{(0)}_{m,n} & \lambda & 0 \\ 0 & 0 & u^{(1)}_{m,n} & \lambda \\ \lambda & 0 & 0 & u^{(2)}_{m,n} \\ u^{(3)}_{m,n} & \lambda & 0 & 0 \end{pmatrix}$$

The Level Structure of matrices L and M is labelled

$$(k_1, \ell_1; k_2, \ell_2).$$

with

$$\ell_2 - k_2 \equiv \ell_1 - k_1 \ (\mathrm{mod} N)$$

The compatibility conditions

$$
\begin{aligned}
U_{m,n+1}V_{m,n} &= V_{m+1,n}U_{m,n}, \\
U_{m,n+1}\Omega^{\ell_2} - \Omega^{\ell_2}U_{m,n} &= V_{m+1,n}\Omega^{\ell_1} - \Omega^{\ell_1}V_{m,n},
\end{aligned}
$$

are explicitly written as

$$
\begin{aligned}
u^{(i)}_{m,n+1}v^{(i+k_1)}_{m,n} &= v^{(i)}_{m+1,n}u^{(i+k_2)}_{m,n}, \\
u^{(i)}_{m,n+1} - u^{(i+\ell_2)}_{m,n} &= v^{(i)}_{m+1,n} - v^{(i+\ell_1)}_{m,n},
\end{aligned}
$$

which can be solved to give:

$$
\begin{aligned}
u^{(i)}_{m,n+1} &= \frac{u^{(i+\ell_2)}_{m,n} - v^{(i+\ell_1)}_{m,n}}{u^{(i+k_2)}_{m,n} - v^{(i+k_1)}_{m,n}}\, u^{(i+k_2)}_{m,n}, \\
v^{(i)}_{m+1,n} &= \frac{u^{(i+\ell_2)}_{m,n} - v^{(i+\ell_1)}_{m,n}}{u^{(i+k_2)}_{m,n} - v^{(i+k_1)}_{m,n}}\, v^{(i+\ell_1)}_{m,n}.
\end{aligned}
$$

Equivalent Lax Pairs

1. Switching m and n, so
$$L \leftrightarrow M \quad \text{and} \quad (k_1, \ell_1) \leftrightarrow (k_2, \ell_2).$$
2. $(k_i, \ell_i) \mapsto (N - k_i, N - \ell_i)$, so
$$(u^{(i)}_{m,n}\,,\, v^{(i)}_{m,n}) \mapsto (u^{(N-i)}_{m,n}\,,\, v^{(N-i)}_{m,n}).$$

The coprime case satisfies

$$(N, \ell_1 - k_1) = (N, \ell_2 - k_2) = 1.$$

We then have

$$|L| = a - (-\lambda)^N, \quad \text{where} \quad a = \prod_{j=0}^{N-1} u^j \quad \text{and} \quad \Delta_n a = 0.$$
$$|M| = b - (-\lambda)^N, \quad \text{where} \quad b = \prod_{j=0}^{N-1} v^j \quad \text{and} \quad \Delta_m b = 0.$$

Subdivision of the coprime case $(N, \ell_1 - k_1) = (N, \ell_2 - k_2) = 1$.

We may reduce to the submanifold

$$\prod_{j=0}^{N-1} u_{m,n}^{(j)} = a, \quad \prod_{j=0}^{N-1} v_{m,n}^{(j)} = b \quad \text{(constants)}.$$

The generic subcase: $ab \neq 0$.

The above relations allow us to express one function from each set in terms of the remaining ones.

The degenerate subcase: $a \neq 0$, $b = 0$.

We can eliminate one of the $u^{(j)}$ and set (wlog) $v^{(N-1)} = 0$.

The degenerate case $a = b = 0$ is empty.

The non-coprime case: $(N, \ell_1 - k_1) = (N, \ell_2 - k_2) = p > 1$.
Determinant factorises:

$(N, k_1, \ell_1) = (6, 1, 3) \quad (p = 2)$

$$|L| = -(\lambda^3 + u_{m,n}^{(0)} u_{m,n}^{(2)} u_{m,n}^{(4)})(\lambda^3 + u_{m,n}^{(1)} u_{m,n}^{(3)} u_{m,n}^{(5)}).$$

$(N, k_1, \ell_1) = (6, 1, 4) \quad (p = 3)$

$$|L| = (\lambda^2 - u_{m,n}^{(0)} u_{m,n}^{(3)})(\lambda^2 - u_{m,n}^{(1)} u_{m,n}^{(4)})(\lambda^2 - u_{m,n}^{(2)} u_{m,n}^{(5)}).$$

Can transform to block matrix form, corresponding to a coupling of smaller coprime systems.

The general equations

$$\begin{aligned} u^{(i)}_{m,n+1} v^{(i+k_1)}_{m,n} &= v^{(i)}_{m+1,n} u^{(i+k_2)}_{m,n}, \\ u^{(i)}_{m,n+1} - u^{(i+\ell_2)}_{m,n} &= v^{(i)}_{m+1,n} - v^{(i+\ell_1)}_{m,n}, \end{aligned}$$

can be reduced by introducing potentials.

The first holds identically if we set **Quotient Potential**

$$u^{(i)}_{m,n} = \alpha \frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}}, \quad v^{(i)}_{m,n} = \beta \frac{\phi^{(i)}_{m,n+1}}{\phi^{(i+k_2)}_{m,n}}, \quad i \in \mathbb{Z}_N.$$

The second holds identically if we set **Additive Potential**

$$u^{(i)}_{m,n} = \chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}, \quad v^{(i)}_{m,n} = \chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}, \quad i \in \mathbb{Z}_N.$$

Quotient Potential: we set ($i \in \mathbb{Z}_N$ throughout)

$$u_{m,n}^{(i)} = \alpha \frac{\phi_{m+1,n}^{(i)}}{\phi_{m,n}^{(i+k_1)}}, \quad v_{m,n}^{(i)} = \beta \frac{\phi_{m,n+1}^{(i)}}{\phi_{m,n}^{(i+k_2)}}.$$

The second equation then take the form

$$\alpha \left(\frac{\phi_{m+1,n+1}^{(i)}}{\phi_{m,n+1}^{(i+k_1)}} - \frac{\phi_{m+1,n}^{(i+\ell_2)}}{\phi_{m,n}^{(i+k_1+\ell_2)}} \right) = \beta \left(\frac{\phi_{m+1,n+1}^{(i)}}{\phi_{m+1,n}^{(i+k_2)}} - \frac{\phi_{m,n+1}^{(i+\ell_1)}}{\phi_{m,n}^{(i+k_2+\ell_1)}} \right).$$

The solved form is written as

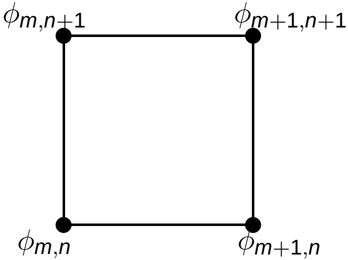
$$\phi_{m+1,n+1}^{(i)} = \frac{\phi_{m,n+1}^{(i+k_1)} \phi_{m+1,n}^{(i+k_2)}}{\phi_{m,n}^{(i+k_1+\ell_2)}} \left(\frac{\alpha\, \phi_{m+1,n}^{(i+\ell_2)} - \beta\, \phi_{m,n+1}^{(i+\ell_1)}}{\alpha\, \phi_{m+1,n}^{(i+k_2)} - \beta\, \phi_{m,n+1}^{(i+k_1)}} \right).$$

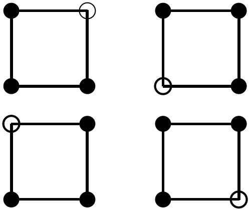
$\prod_{i=0}^{N-1} \phi_{m,n}^{(i)} = 1$ Reduces number of components.

The equation

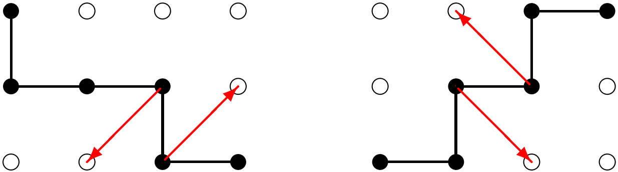
$$\alpha\left(\frac{\phi^{(i)}_{m+1,n+1}}{\phi^{(i+k_1)}_{m,n+1}} - \frac{\phi^{(i+\ell_2)}_{m+1,n}}{\phi^{(i+k_1+\ell_2)}_{m,n}}\right) = \beta\left(\frac{\phi^{(i)}_{m+1,n+1}}{\phi^{(i+k_2)}_{m+1,n}} - \frac{\phi^{(i+\ell_1)}_{m,n+1}}{\phi^{(i+k_2+\ell_1)}_{m,n}}\right).$$

can be solve for each vertex of the square:





Initial values given on a staircase:



The Lax Pair in potential form

$$\begin{aligned}\Psi_{m+1,n} &= \left(\alpha\phi_{m+1,n}\Omega^{k_1}\phi_{m,n}^{-1} + \lambda\Omega^{\ell_1}\right)\Psi_{m,n},\\ \Psi_{m,n+1} &= \left(\beta\phi_{m,n+1}\Omega^{k_2}\phi_{m,n}^{-1} + \lambda\Omega^{\ell_2}\right)\Psi_{m,n},\end{aligned}$$

where

$$\phi_{m,n} := \mathrm{diag}\left(\phi_{m,n}^{(0)}, \cdots, \phi_{m,n}^{(N-1)}\right).$$

Equivalence relation: $\widetilde{\Psi}_{m,n} = \alpha^{-m}\beta^{-n}\lambda^{-m-n}\phi_{m,n}^{-1}\Psi_{m,n}$.

gives

$$(\phi^{(i)}; k_1, \ell_1, \alpha; k_2, \ell_2, \beta) \leftrightarrow (\widetilde{\phi}^{(i)}; \ell_1, k_1, \tilde{\alpha}; \ell_2, k_2, \tilde{\beta})$$

where

$$\phi_{m,n}^{(i)}\widetilde{\phi}_{m,n}^{(i)} = 1, \quad \alpha\,\tilde{\alpha} = 1, \quad \beta\,\tilde{\beta} = 1, \quad \lambda \mapsto \lambda^{-1}$$

Additive Potential: we set ($i \in \mathbb{Z}_N$ throughout)

$$u^{(i)}_{m,n} = \chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}, \quad v^{(i)}_{m,n} = \chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}.$$

The first equation then take the form

$$\begin{aligned}\left(\chi^{(i)}_{m+1,n+1} - \chi^{(i+\ell_1)}_{m,n+1}\right)\left(\chi^{(i+k_1)}_{m,n+1} - \chi^{(i+k_1+\ell_2)}_{m,n}\right)\\ = \left(\chi^{(i)}_{m+1,n+1} - \chi^{(i+\ell_2)}_{m+1,n}\right)\left(\chi^{(i+k_2)}_{m+1,n} - \chi^{(i+k_2+\ell_1)}_{m,n}\right).\end{aligned}$$

The solved form is written as

$$\chi^{(i)}_{m+1,n+1} = \frac{\chi^{(i+k_1)}_{m,n+1}\chi^{(i+\ell_1)}_{m,n+1} - \chi^{(i+k_2)}_{m+1,n}\chi^{(i+\ell_2)}_{m+1,n} - \chi^{(i+k_1+\ell_2)}_{m,n}(\chi^{(i+\ell_1)}_{m,n+1} - \chi^{(i+\ell_2)}_{m+1,n})}{\chi^{(i+k_1)}_{m,n+1} - \chi^{(i+k_2)}_{m+1,n}}.$$

These are Bäcklund related to the quotient potential equations.

In $2D$ we have a unified description of several well known examples.

For quotient potentials we can set $\phi^{(0)}_{m,n}\phi^{(1)}_{m,n} = 1$.

Level structure $(0,1;0,1)$

$$\alpha\left(\phi_{m,n}\phi_{m,n+1} - \phi_{m+1,n}\phi_{m+1,n+1}\right) \\ -\beta\left(\phi_{m,n}\phi_{m+1,n} - \phi_{m,n+1}\phi_{m+1,n+1}\right) = 0,$$

where $\phi_{m,n} = \phi^{(0)}_{m,n} = 1/\phi^{(1)}_{m,n}$. (Discrete MKdV equation.)

Level structure $(0,1;1,0)$

$$\alpha\left(\phi_{m,n}\phi_{m+1,n+1} - \phi_{m+1,n}\phi_{m,n+1}\right) \\ -\beta\left(\phi_{m,n}\phi_{m+1,n}\phi_{m,n+1}\phi_{m+1,n+1} - 1\right) = 0,$$

where $\phi_{m,n} = \phi^{(0)}_{m,n} = 1/\phi^{(1)}_{m,n}$.

(Hirota's discrete sine-Gordon equation.)

For additive potentials we have the first integrals

$$\prod_{i=0}^{N-1}\left(\chi^{(i)}_{m+1,n}-\chi^{(i+\ell_1)}_{m,n}\right) = \alpha^N, \quad \prod_{i=0}^{N-1}\left(\chi^{(i)}_{m,n+1}-\chi^{(i+\ell_2)}_{m,n}\right) = \beta^N.$$

Level structure $(0,1;0,1)$

Using the first integrals to replace either $\chi^{(0)}$ or $\chi^{(1)}$:

$$\left(\chi_{m+1,n+1}-\chi_{m,n}\right)\left(\chi_{m+1,n}-\chi_{m,n+1}\right) = \alpha^2-\beta^2\,,$$

which is the discrete potential KdV.

Level structure $(1,0;1,0)$

Using the first integrals to eliminate one of the variables:

$$\alpha^2\left(\chi_{m,n}-\chi_{m,n+1}\right)\left(\chi_{m+1,n}-\chi_{m+1,n+1}\right) \\ -\beta^2\left(\chi_{m,n}-\chi_{m+1,n}\right)\left(\chi_{m,n+1}-\chi_{m+1,n+1}\right)=0.$$

which is the Schwarzian KdV equation

The 2D degenerate case

$$u^{(0)}_{m,n}u^{(1)}_{m,n} = a, \quad v^{(1)}_{m,n} = 0,$$

gives Hirota's KdV equation:

$$\frac{a}{u_{m+1,n+1}} + u_{m,n+1} = u_{m+1,n} + \frac{a}{u_{m,n}}.$$

In higher dimensions, we derive a new generalisation of this, involving 2N points.

In $3D$ and above our scheme gives either generalisations of well known $2D$ examples or new families of integrable systems.

For quotient potentials we can set $\prod_{i=0}^{2} \phi_{m,n}^{(i)} = 1$.
We use the following substitution:

$$\left(\phi_{m,n}^{(0)}, \phi_{m,n}^{(1)}, \phi_{m,n}^{(2)}\right) \mapsto \left(\frac{1}{\phi_{m,n}^{(0)}}, \phi_{m,n}^{(1)}, \frac{\phi_{m,n}^{(0)}}{\phi_{m,n}^{(1)}}\right).$$

Two Equivalence Relations for the quotient potential:

1. $(k_i, \ell_i) \mapsto (N - k_i, N - \ell_i)$,
2. $(k_i, \ell_i) \leftrightarrow (\ell_i, k_i)$.

In $3D$ we therefore have the following inequivalent cases:

1. Level structure $(0, 1; 0, 1)$ (modified Boussinesq),
2. Level structure $(0, 1; 1, 2)$ (a new integrable system),
3. Level structure $(0, 1; 2, 0)$ (a new integrable system),
4. Level structure $(1, 2; 1, 2)$ (a new integrable system).

The case $(0, 1; 2, 0)$ is specific to $N = 3$, since

$$2 + 1 \equiv 0 + 0 \ (mod 3).$$

Level structure $(0,1;0,1)$

$$\phi^{(0)}_{m+1,n+1} = \frac{\alpha\,\phi^{(0)}_{m,n+1} - \beta\,\phi^{(0)}_{m+1,n}}{\alpha\,\phi^{(1)}_{m+1,n} - \beta\,\phi^{(1)}_{m,n+1}}\,\phi^{(1)}_{m,n},$$

$$\phi^{(1)}_{m+1,n+1} = \frac{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1} - \beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}{\alpha\,\phi^{(1)}_{m+1,n} - \beta\,\phi^{(1)}_{m,n+1}}\,\frac{\phi^{(1)}_{m,n}}{\phi^{(0)}_{m,n}}.$$

This equation is related to the modified Boussinesq equation.

Special case of nonlinear superposition of 2D Toda lattice, related to modified Lax equations. (Fordy-Gibbons 1980)

Rediscovered by (Nijhoff, et al, 1992) in the context of discrete integrable systems.

Level structure $(1, 2; 1, 2)$

$$\begin{aligned}
\phi^{(0)}_{m+1,n+1} &= \frac{\alpha\,\phi^{(1)}_{m+1,n} - \beta\phi^{(1)}_{m,n+1}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1} - \beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\,\frac{1}{\phi^{(0)}_{m,n}},\\
\phi^{(1)}_{m+1,n+1} &= \frac{\alpha\,\phi^{(0)}_{m,n+1} - \beta\phi^{(0)}_{m+1,n}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1} - \beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\,\frac{1}{\phi^{(1)}_{m,n}}.
\end{aligned}$$

This is a new integrable system.

The reduction

$$\phi^{(0)}_{m,n} = \phi^{(1)}_{m,n} = \frac{-1}{2^{1/3}u_{m,n}}, \quad \beta = -\alpha.$$

leads to a discrete Tzitzeica equation (Mikhailov and Xenitidis):

$$u_{m,n}u_{m+1,n+1}\left(u_{m+1,n} + u_{m,n+1}\right) + 1 = 0.$$

Additive Potential   Level structure $(0,1;0,1)$

$$\chi^{(i)}_{m+1,n+1} = \frac{(\chi^{(i)}_{m+1,n}-\chi^{(i+1)}_{m,n})\chi^{(i+1)}_{m+1,n}-(\chi^{(i)}_{m,n+1}-\chi^{(i+1)}_{m,n})\chi^{(i+1)}_{m,n+1}}{\chi^{(i)}_{m+1,n}-\chi^{(i)}_{m,n+1}},$$

$$\chi^{(i+1)}_{m+1,n+1} = \chi^{(i)}_{m,n}+\frac{1}{\chi^{(i+1)}_{m+1,n}-\chi^{(i+1)}_{m,n+1}}\left(\frac{\alpha^3}{\chi^{(i)}_{m+1,n}-\chi^{(i+1)}_{m,n}} - \frac{\beta^3}{\chi^{(i)}_{m,n+1}-\chi^{(i+1)}_{m,n}}\right).$$

This is a two component system with fixed $i=0$ (or 1 or 2).

This is a new integrable system, which can be decoupled to a nine point scalar equation (the discrete potential Boussinesq equation).

The Non-Coprime Case has $(N, \ell_i - k_i) = p > 1$.

For $N = pq,\ \ell_i - k_i = pr,\ (q, r) = 1$ the variables are grouped together

$$\mathbf{u}_i = (u^{(i)}, u^{(i+p)}, \dots, u^{(i+p(q-1))}),\ i = 0, \dots, p-1.$$

and

$$\mathbf{v}_i = (v^{(i)}, v^{(i+p)}, \dots, v^{(i+p(q-1))}),\ i = 0, \dots, p-1.$$

The permutation matrix which re-orders them like this, can be used to put L and M in block form:

$$p \times p \quad \text{matrices of} \quad q \times q \quad \text{blocks.}$$

For $N = 6$ with $p = 3$, $q = 2$, $r = 1$ there are several compatible level structures.

The choices $(1, 4)$ and $(2, 5)$ give the following forms for L:

$$L = \begin{pmatrix} 0 & L_{03}^{(0,1)} & 0 \\ 0 & 0 & L_{14}^{(0,1)} \\ L_{25}^{(1,0)} & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & L_{03}^{(0,1)} \\ L_{14}^{(1,0)} & 0 & 0 \\ 0 & L_{25}^{(1,0)} & 0 \end{pmatrix},$$

where $L_{ab}^{(k,\ell)}$ is the 2×2 Lax matrix of level structure (k, ℓ) and depending on variables $u_{m,n}^{(a)}$ and $u_{m,n}^{(b)}$.

For example

$$L_{03}^{(0,1)} = \begin{pmatrix} u_{m,n}^{(0)} & \lambda \\ \lambda & u_{m,n}^{(3)} \end{pmatrix}, \quad L_{25}^{(1,0)} = \begin{pmatrix} \lambda & u_{m,n}^{(2)} \\ u_{m,n}^{(5)} & \lambda \end{pmatrix}.$$

Similarly for M (but depending upon $v_{m,n}^{(a)}, v_{m,n}^{(b)}$).

The choice $(2,5;2,5)$ leads to the system

$$\begin{aligned}
L_{03}^{(0,1)}(\mathbf{u}_{m,n+1})M_{25}^{(1,0)}(\mathbf{v}_{m,n}) &= M_{03}^{(0,1)}(\mathbf{v}_{m+1,n})L_{25}^{(1,0)}(\mathbf{u}_{m,n}),\\
L_{14}^{(1,0)}(\mathbf{u}_{m,n+1})M_{03}^{(0,1)}(\mathbf{v}_{m,n}) &= M_{14}^{(1,0)}(\mathbf{v}_{m+1,n})L_{03}^{(0,1)}(\mathbf{u}_{m,n}),\\
L_{25}^{(1,0)}(\mathbf{u}_{m,n+1})M_{14}^{(1,0)}(\mathbf{v}_{m,n}) &= M_{25}^{(1,0)}(\mathbf{v}_{m+1,n})L_{14}^{(1,0)}(\mathbf{u}_{m,n}),
\end{aligned}$$

In potential form, with

$$\phi_{m,n}^{(0)} = 1/\phi_{m,n}^{(3)} = \psi_{m,n}^{(0)},\ \phi_{m,n}^{(4)} = 1/\phi_{m,n}^{(1)} = \psi_{m,n}^{(1)},\ \phi_{m,n}^{(2)} = 1/\phi_{m,n}^{(5)} = \psi_{m,n}^{(2)},$$

we obtain the coupled discrete MKdV system

$$\psi_{m+1,n+1}^{(i)} = \left(\frac{\alpha\psi_{m,n+1}^{(i+2)} - \beta\psi_{m+1,n}^{(i+2)}}{\alpha\psi_{m+1,n}^{(i+2)} - \beta\psi_{m,n+1}^{(i+2)}}\right)\psi_{m,n}^{(i+1)}, \quad i \in \mathbb{Z}_3.$$

The choice $(1,4;2,5)$ leads to the system

$$\psi^{(i)}_{m,n}\psi^{(i)}_{m+1,n+1} = \frac{\alpha - \beta\psi^{(i+1)}_{m,n+1}\psi^{(i+2)}_{m+1,n}}{\alpha\psi^{(i+1)}_{m,n+1}\psi^{(i+2)}_{m+1,n} - \beta}, \quad i \in \mathbb{Z}_3,$$

which is a coupled system of Hirota's discrete sine-Gordon equations.

A consistent lattice is such that around each elementary quadrilateral, we have

$$L_{m,n+1}M_{m,n} = M_{m+1,n}L_{m,n}.$$

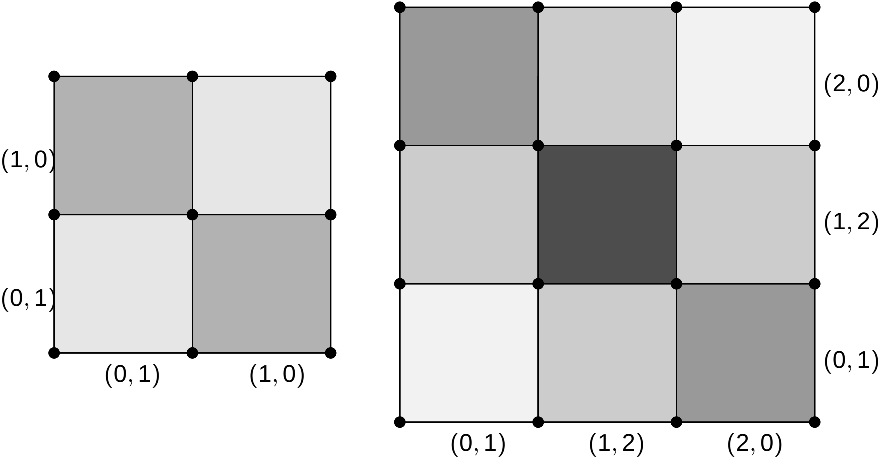
One choice is to take the same L and M around each quadrilateral.

However, we can choose a variety of level structures $(k_1, \ell_1; k_2, \ell_2)$, subject only to

$$\ell_i - k_i \quad \text{being fixed} \quad (\text{mod} N) \quad \text{over the lattice.}$$

Opposite edges carry matrices with exactly the same structure.

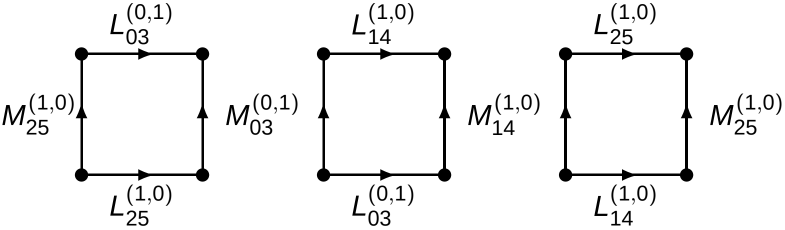
For $N = 2$ and $N = 3$ we can choose:



Non-coprime systems form subsystems.

For the discrete MKdV case we had separate consistency equations:

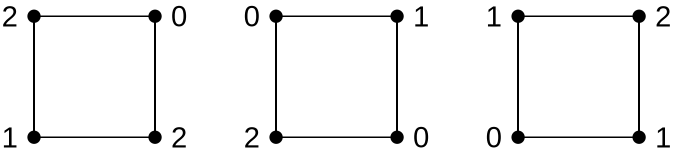
$$L_{03}^{(0,1)}(\mathbf{u}_{m,n+1})M_{25}^{(1,0)}(\mathbf{v}_{m,n}) = M_{03}^{(0,1)}(\mathbf{v}_{m+1,n})L_{25}^{(1,0)}(\mathbf{u}_{m,n})$$



Matching edges can be glued together.

The coupled discrete MKdV system

$$\psi^{(i)}_{m+1,n+1} = \left(\frac{\alpha \psi^{(i+2)}_{m,n+1} - \beta \psi^{(i+2)}_{m+1,n}}{\alpha \psi^{(i+2)}_{m+1,n} - \beta \psi^{(i+2)}_{m,n+1}} \right) \psi^{(i+1)}_{m,n}, \quad i \in \mathbb{Z}_3.$$



Matching edges can be glued together.

We can build 3 different lattices with a single component $\psi^{(i)}_{m,n}$ at each site.

$$\begin{array}{ccccccc}
1 & \text{---} & 2 & \text{---} & 0 & \text{---} & 1 \\
| & & | & & | & & | \\
0 & \text{---} & 1 & \text{---} & 2 & \text{---} & 0 \\
| & & | & & | & & | \\
2 & \text{---} & 0 & \text{---} & 1 & \text{---} & 2 \\
| & & | & & | & & | \\
1 & \text{---} & 2 & \text{---} & 0 & \text{---} & 1
\end{array}$$

This can then be periodically extended in any direction.

The original lattice is a superposition of the 3 single component lattices.

The Initial Value Problem: 3 steps.

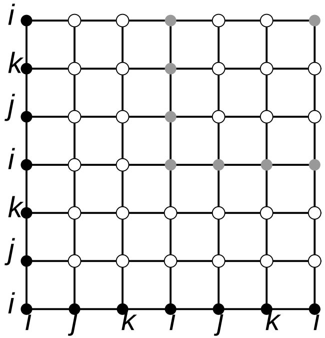
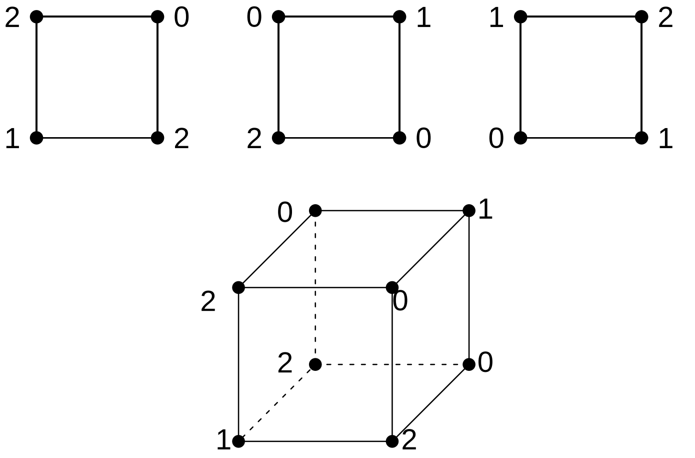
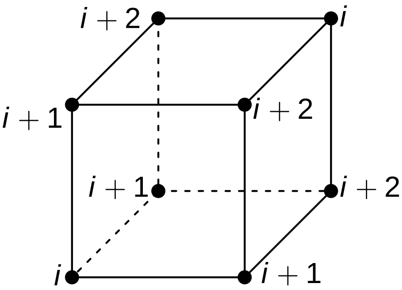


Figure: *Patterns on the lattice and initial value problems with* $(i,j,k) \in \{(0,1,2),(1,2,0),(2,0,1)\}$ *: Every black vertex carries the initial value of the corresponding variable, e.g. the left bottom vertex carries the initial value of* $\psi^{(i)}$*. This initial pattern repeats after three diagonal steps leading to the updated gray vertices.*

3D Consistency. We can build a 3D cube with these faces:



There are 3 such cubes: $i \in \{0, 1, 2\}$



We can place 27 such cubes to form a $3 \times 3 \times 3$ cube, which can be periodically extended.

Each face is one of our *2D* lattices.

Conclusions:

- The general scheme we introduced has led to a unified description of many known discrete integrable systems.
- Each of these is generalised to arbitrary N dimensions.
- Many new systems are included.
- Multicoloured lattices.
- Non-Coprime Case: coupled systems of lower dimensional equations.
- $3D$ Consistency

Further Results:

- Continuous symmetries and master symmetries: classification and hierarchies.
- Nonlocal symmetries and Bäcklund Transformations of the 2D Toda Lattice.
- Nonlinear superposition formula as Discrete Integrable Systems.