# Bi-Hamiltonian structures of KdV type

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# Bi-Hamiltonian structures of KdV-type

It was observed (Olver and Rosenau, 1996) that many PDEs admit a bi-Hamiltonian structure which is indeed defined by a trio of mutually compatible Hamiltonian operators.

Examples: the scalar case

$$P_1 = \partial_x, \qquad Q_1 = 2u\partial_x + u_x, \quad R_3 = \partial_x^3.$$

Poisson pencil of KdV hierarchy (Magri (1978)):

$$\Pi_\lambda = Q_1 + \epsilon^2 R_3 - \lambda P_1 = 2u\partial_x + u_x - \lambda\partial_x + \epsilon^2\partial_x^3$$

Poisson pencil of Camassa–Holm hierarchy:

$$\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_3) = 2u\partial_x + u_x - \lambda(\partial_x + \epsilon^2\partial_x^3).$$

Examples: the 2-component case

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 2u\partial_x + u_x & v\partial_x \\ \partial_x v & -2\partial_x \end{pmatrix},$$
$$R_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix}$$

- $\Pi_\lambda = Q_1 + \epsilon^2 R_3 - \lambda P_1$ AKNS (or two-boson) hierarchy;
- $\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_3)$ two-component Camassa-Holm hierarchy.

We say the pencils of the type of $\Pi_\lambda$ (or $\tilde{\Pi}_\lambda$) to be bi-Hamiltonian structures of KdV-type.

# Classification of bi-Hamiltonian structures of KdV type

The problem: classify compatible trios of Hamiltonian operators $P_1$, $Q_1$, $R_n$ where $P_1$ and $Q_1$ are homogeneous first-order Hamiltonian operators (Dubrovin and Novikov, 1983)

$$P_1 = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x, \qquad Q_1 = h^{ij}\partial_x + \Xi^{ij}_k u^k_x,$$

and $R_n$ is a homogeneous Hamiltonian operator

$$R_n = \sum_{l=0}^{n} A^{ij}_{n,l}(u, u_x, \dots, u_{(l)})\partial_x^{(n-l)}$$

of degree $n > 1$ (Dubrovin and Novikov 1984), where $A^{ij}_{n,l}$ are homogeneous polynomials of degree $l$ in $u_x, \dots, u_{(l)}$, $x$-derivative has degree 1.
Homogeneous operators are form-invariant with respect to point transformations $\tilde{u}^i = \tilde{u}^i(u^j)$.

# A strategy for the classification

The above pencils can be thought as a deformation of a Poisson pencil of hydrodynamic type.

Due to the general theory of deformations the only interesting cases are $n = 2$ and $n = 3$. In the remaining case the deformations can always be eliminated by Miura type transformations (Liu and Zhang, 2005).

Our strategy: knowing the normal forms of $R_2$ and $R_3$ we find all possible compatible first-order Poisson pencils of hydrodynamic type $P_1 - \lambda Q_1$. This yields bi-Hamiltonian structures of KdV type with $n = 2$ (or $n = 3$).

# Homogeneous Hamiltonian operators, degree 2

Second-order operators $R_2$ have been completely described in the non degenerate case $\det(\ell^{ij}) \neq 0$ (Potemin 1987, 1991, 1997; Doyle 1993):

$$R_2 = \partial_x \ell^{ij} \partial_x,$$

where $\ell_{ij} = T_{ijk} u^k + T^0_{ij}$, and $T_{ijk}$, $T^0_{ij}$ are constant and completely skew-symmetric, without further conditions.

When $m = 2$ there is only one homogeneous second-order Hamiltonian operator (up to point transformations):

$$R_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2.$$

# Homogeneous Hamiltonian operators, degree 3

Third-order operators $R_3$ have been classified ($\det(\ell^{ij}) \neq 0$) in the $m$-component case with $m = 1$ (in this case the operator can be reduced to $\partial_x^3$ by a point transformation (Potemin 1987, 1991, 1997; Doyle 1993) and $m = 2, 3, 4$ (Ferapontov, Pavlov, V. 2014, 2016).

$$R_3 = \partial_x \left( \ell^{ij} \partial_x + c^{ij}_k u^k_x \right) \partial_x,$$

where, introducing $c_{ijk} = \ell_{iq}\ell_{jp}c^{pq}_k$, the following conditions must be fulfilled:

$$c_{nkm} = \frac{1}{3}(\ell_{nm,k} - \ell_{nk,m}),$$

$$\ell_{mn,k} + \ell_{nk,m} + \ell_{km,n} = 0,$$

$$c_{mnk,l} = -\ell^{pq} c_{pml} c_{qnk}.$$

# The geometry of third-order operators

Projective-geometric interpretation: $g_{ij}$ is the Monge form of a **quadratic line complex**, $c_{ijk}$ is the corresponding **tangential line complex**. A quadratic line complex is a subvariety of the Plücker's variety of all lines of $\mathbb{P}^m(\mathbb{C})$.

Differential-geometric interpretation: $c^i_{jk} = g^{is}c_{sjk}$ is a **flat metric connection** with torsion of the **first Cartan type**.

# Example of Monge metric in the case $m = 3$

$$
\begin{aligned}
g_{11} &= -[R_{12}(u^2)^2 + R_{13}(u^3)^2 + 2B_{12}u^2u^3 + 2H_{12}u^2 + 2H_{13}u^3 + D_1],\\
g_{22} &= -[R_{12}(u^1)^2 + R_{23}(u^3)^2 + 2B_{22}u^1u^3 + 2H_{21}u^1 + 2H_{23}u^3 + D_2],\\
g_{33} &= -[R_{23}(u^2)^2 + R_{13}(u^1)^2 + 2B_{32}u^1u^2 + 2H_{31}u^1 + 2H_{32}u^2 + D_3],\\
g_{12} &= R_{12}u^1u^2 + B_{12}u^1u^3 + B_{22}u^2u^3 - B_{32}(u^3)^2 + H_{12}u^1 + H_{21}u^2 + (E_2 - E_1)u^3 + F_{12},\\
g_{13} &= R_{13}u^1u^3 + B_{12}u^1u^2 - B_{22}(u^2)^2 + B_{32}u^2u^3 + H_{13}u^1 + H_{31}u^3 + (E_1 - E_3)u^2 + F_{13},\\
g_{23} &= R_{23}u^2u^3 - B_{12}(u^1)^2 + B_{22}u^1u^2 + B_{32}u^1u^3 + H_{23}u^2 + H_{32}u^3 + (E_3 - E_2)u^1 + F_{23},
\end{aligned}
$$

# Classification results for operators of degree 3

$\mathbf{m = 2}$: three normal forms of homogeneous third-order Hamiltonian operators up to **point transformations** (Ferapontov, Pavlov, V, JGP 2014)

$$R_3^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x^3, \qquad R_3^{(2)} = \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{u^1} \\ \frac{1}{u^1}\partial_x & \frac{u^2}{(u^1)^2}\partial_x + \partial_x \frac{u^2}{(u^1)^2} \end{pmatrix} \partial_x,$$

$$R_3^{(3)} = \partial_x \begin{pmatrix} \partial_x & \partial_x \frac{u^2}{u^1} \\ \frac{u^2}{u^1}\partial_x & \frac{(u^2)^2+1}{2(u^1)^2}\partial_x + \partial_x \frac{(u^2)^2+1}{2(u^1)^2} \end{pmatrix} \partial_x.$$

# Classification results for operators of degree 3

$\mathbf{m=3}$: six normal forms of homogeneous third-order Hamiltonian operators up to **reciprocal transformations of projective type** (Ferapontov, Pavlov, V, JGP 2014)

$$g^{(1)} = \begin{pmatrix} (u^2)^2+c & -u^1u^2-u^3 & 2u^2 \\ -u^1u^2-u^3 & (u^1)^2+c(u^3)^2 & -cu^2u^3-u^1 \\ 2u^2 & -cu^2u^3-u^1 & c(u^2)^2+1 \end{pmatrix},$$

$$g^{(2)} = \begin{pmatrix} (u^2)^2+1 & -u^1u^2-u^3 & 2u^2 \\ -u^1u^2-u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix},$$

$$g^{(3)} = \begin{pmatrix} (u^2)^2+1 & -u^1u^2 & 0 \\ -u^1u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{m=4}$: 38 normal forms, Ferapontov, Pavlov, V. 2016 (IMRN).

# Results: trios $P_1$, $Q_1$, $R_2$

$\mathbf{m} = \mathbf{1}$: nothing new, KdV and Camassa-Holm hierarchies.

We focus on the $\mathbf{m} =$ **2-component** case.
In what follows $c_i$ are constants, Levi-Civita conditions:

$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s$$
$$\Gamma^{ij}_k + \Gamma^{ji}_k = \partial_k g^{ij}$$

Theorem: $P_1$ is compatible with $R_2$ if and only if

$$g^{11} = c_1 u^1 + c_2, \tag{1a}$$
$$g^{12} = \frac{1}{2}c_3 u^1 + \frac{1}{2}c_1 u^2 + c_5 \tag{1b}$$
$$g^{22} = c_3 u^2 + c_4. \tag{1c}$$

The above metric is flat for every value of the parameters. Any $Q_1$ with a metric $h^{ij}$ of the above form makes a trio $P_1$, $Q_1$, $R_2$.

# Results: trios $P_1$, $Q_1$, $R_3^{(1)}$

Theorem: $P_1$ is a Hamiltonian operator compatible with $R_3^{(1)}$ if and only if

$$\begin{aligned} g^{11} &= c_1 u^1 + c_2 u^2 + c_3, \\ g^{12} &= c_4 u^1 + c_1 u^2 + c_5 \\ g^{22} &= c_6 u^1 + c_4 u^2 + c_7 \end{aligned} \tag{2}$$

together with the Levi-Civita conditions

$$c_1 c_4 - c_2 c_6 = 0, \quad c_3 c_4 - c_7 c_2 = 0, \quad c_3 c_6 - c_1 c_7 = 0. \tag{3}$$

The above conditions imply the flatness of $g$.
There is a 5 parameter family of mutually commuting pairs $P_1$, $Q_1$ that commute with $R_3^{(1)}$.

## Results: trios $P_1$, $Q_1$, $R_3^{(2)}$

Theorem: $P_1$ is a Hamiltonian operator compatible with $R_3^{(2)}$ if and only if

$$g^{11} = c_1 u^1 + c_2 u^2, \tag{4a}$$

$$g^{12} = c_4 u^1 + \frac{c_3}{u^1} + \frac{c_2 (u^2)^2}{2u^1}, \tag{4b}$$

$$g^{22} = 2c_4 u^2 + \frac{c_6}{u^1} - \frac{c_1 (u^2)^2}{u^1} + c_5, \tag{4c}$$

together with the Levi-Civita conditions

$$c_2 c_6 + 2c_1 c_3 = 0, \quad c_2 c_5 = 0, \quad c_1 c_5 = 0. \tag{5}$$

The above conditions imply the flatness of $g$.
There exists a 4 parameter family of mutually commuting pairs $P_1$, $Q_1$ that commute with $R_3^{(2)}$.

# Results: trios $P_1$, $Q_1$, $R_3^{(3)}$

Theorem: $P_1$ is a Hamiltonian operator compatible with $R_3^{(3)}$ if and only if

$$g^{11} = c_1 u^1 + c_2 u^2 + c_3, \tag{6a}$$

$$g^{12} = c_4 u^1 - \frac{c_2}{2u^1} + \frac{c_3 u^2}{u^1} + \frac{c_2 (u^2)^2}{2u^1}, \tag{6b}$$

$$g^{22} = 2c_4 u^2 + \frac{c_1}{u^1} + \frac{c_5 u^2}{u^1} - \frac{c_1 (u^2)^2}{u^1} + c_6, \tag{6c}$$

together with the Levi-Civita conditions

$$c_2 c_5 + 2c_1 c_3 = 0, \quad c_2 c_6 - 2c_3 c_4 = 0, \quad c_1 c_6 + c_4 c_5 = 0, \tag{7}$$

The above conditions imply the flatness of $g$.
There exists a 4 parameter family of mutually commuting pairs $P_1$, $Q_1$ that commute with $R_3^{(3)}$.

# Known examples with $R_2$

- The Kaup–Broer system (Kupershmidt 1985):

$$\begin{cases} u^1_t = ((u^1)^2/2 + u^2 + \beta u^1_x)_x, \\ u^2_t = (u^1 u^2 + \alpha u^1_{xx} - \beta u^2_x)_x, \end{cases} \tag{8}$$

- In De Sole, Kac, Turhan 2014, a six-parameter family of pairwise compatible Hamiltonian operators defined by the cohomology spaces of curves is considered. A subset of these operator belongs to our class, with $R_2$.

# Known examples with $R_3^{(1)}$

- A version of the Dispersive Water Waves system (Antonowicz-Fordy, 1989):

$$u_t^1 = \frac{1}{4}u_{xxx}^2 + \frac{1}{2}u^2u_x^1 + u^1u_x^2,$$
$$u_t^2 = u_x^1 + \frac{3}{2}u^2u_x^2$$

- Coupled Harry-Dym hierarchy (Antonowicz-Fordy, 1988):

$$u_1^1 = \left(\frac{1}{4(u^2)^{1/2}}\right)_{xxx} - \alpha\left(\frac{1}{(u^2)^{1/2}}\right)_x$$
$$u_t^2 = u^1\left(\frac{1}{(u^2)^{1/2}}\right)_x + \frac{u_x^1}{2(u^2)^{1/2}}$$

# New example with $R_3^{(2)}$

Two identical copies of the metric which solves the compatibility problem with $R_3^{(2)}$, $g$ and $h$.
Metric $g$ of $P_1$ parametrized by $c_i$.
Metric $h$ of $Q_1$ parametrized by $d_i$.

Choosing $c_3 = 0$, $d_3 = 1$, $c_2 = 2$, $c_4 = 1$, $d_4 = 0$, $d_5 = 0$ we obtain the bi-Hamiltonian system

$$\begin{aligned} u^1_{t_2} &= 2u^2u^1_x + u^1u^2_x \\ u^2_{t_2} &= u^1u^1_x + 2u^2u^2_x - \frac{u^1_x u^1_{xx}}{(u^1)^2} + \frac{u^1_{xxx}}{u^1}, \end{aligned}$$

# Another new example with $R_3^{(2)}$

Choosing $c_4 = 0$, $c_1 = -1$, $c_6 = -1$, $c_2 = 0$, $d_2 = 0$, $d_1 = 0$ we obtain the bi-Hamiltonian system

$$
\begin{aligned}
u^1_{t_2} =& \frac{3}{2}\frac{u^2_x}{u^1} - \frac{3}{2}\frac{u^2 u^1_x}{(u^1)^2} - \frac{u^1_{xxx}}{(u^1)^3} + 9\frac{u^1_x u^1_{xx}}{(u^1)^4} - 12\frac{(u^1_x)^3}{(u^1)^5}\\
u^2_{t_2} =& \frac{3}{2}\frac{(1-(u^2)^2)u^1_x}{(u^1)^3} + \frac{3}{2}\frac{u^2 u^2_x}{(u^1)^2} - \frac{30u^2(u^1_x)^3}{(u^1)^6} + 10\frac{u^2_x(u^1_x)^2}{(u^1)^5}\\
&+ 12\frac{u^2_x(u^1)^2_x}{(u^1)^5} + -\frac{3u^2_x u^1_{xx}}{(u^1)^4} - 2\frac{u^2 u^1_{xxx}}{(u^1)^4} - \frac{u^2_{xx}u^1_x}{(u^1)^4}.
\end{aligned}
$$

# New examples with $R_3^{(3)}$

Choosing

$$c_1 = 1, \quad c_2 = -1, \quad d_3 = 1, \quad c_3 = 0, \quad c_4 = 0$$

one easily gets the first non trivial flows of the associated bi-Hamiltonian hierarchy. Too big to be shown.

The multiparametric families of solutions allow for a great variety of bi-Hamiltonian systems.

# Dubrovin and Zhang's perturbative approach

Our pencils can be regarded as deformations of a Poisson pencil of hydrodynamic type. The classification of deformations with respect to the Miura group

$$\tilde{u}^i = f^i(u^1, \ldots, u^n) + \sum_{k\geq 1} \epsilon^k F^i_k(u, u_x, \ldots, u_{(k)}), \qquad (9)$$

has been obtained in recent years in the semisimple case (see Liu and Zhang (2005) and Carlet, Posthuma, Shadrin (2015)).

Deformations are uniquely determined by their dispersionless limit and by $n$ functions of one variable, the central invariants. Deformations with vanishing central invariants can be transformed to their dispersionless limit, and are *trivial*.

# Central invariants of the examples with $R_3^{(2)}$

First example, canonical coordinates:

$$\lambda^1 = (u^1 + u^2)^2, \qquad \lambda^2 = (u^1 - u^2)^2,$$

central invariants:

$$s_1 = -\frac{1}{8\sqrt{\lambda^1}}, \qquad s_2 = \frac{1}{8\sqrt{\lambda^2}}.$$

Second example, canonical coordinates:

$$\lambda^1 = \frac{u^2 + 1}{u^1}, \qquad \lambda^2 = \frac{u^2 - 1}{u^1}$$

central invariants:

$$s_1 = \frac{1}{2}, \qquad s_2 = -\frac{1}{2}.$$

# Central invariants of the example with $R_3^{(3)}$

In the example with $R_3^{(3)}$ (not shown), canonical coordinates:

$$\lambda^1 = -\frac{1}{2}\,\frac{(u^2)^2 - 1}{u^2}, \qquad \lambda^2 = \frac{1}{2}\,\frac{4\,(u^1)^2 - 4\,u^1u^2 + (u^2)^2 - 1}{2u^1 - u^2},$$

central invariants:

$$s_1 = \frac{1}{2}\,\frac{\lambda^1\sqrt{(\lambda^1)^2+1} - (\lambda^1)^2 - 1}{(\lambda^1)^2+1},$$

$$s_2 = -\,\frac{1}{2}\,\frac{\lambda^2\sqrt{(\lambda^2)^2+1} + (\lambda^2)^2 + 1}{(\lambda^2)^2+1}.$$

This means that all the new examples of Poisson pencils obtained in the previous Section are not Miura-trivial.

# Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `http://gdeq.org`.

CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

Thank you!

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