Mean field games with (nonlocal and) local coupling

P. Cardaliaguet

(Paris-Dauphine)

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Mean Field Games (MFG) study

- Optimal control problems = each agent controls his state in order to minimize a cost which depends on the other agents' positions
- with infinitely many agents = having individually a negligible influence on the global system (Ref : Aumann ('64), Schmeidler ('73), Hildenbrand ('74), Mas-Colell ('84), ...)

Early references :

- Early work by Lasry-Lions (2006) and Huang-Caines-Malhamé (2006)
- Similar models in the economic literature : heterogeneous agent models (Aiyagari ('94), Bewley ('86), Krusell-Smith ('98),...)

A class of $N-$player games

- Fix $N \in \mathbb{N}$, $N \geq 2$ the number of players.

- Fix $i \in \{1, \ldots, N\}$. Player i want to minimize over her control (α_t^i) the quantity

$$J^{N,i}(\alpha^i, (\alpha^j)_{j\neq i}) := \mathbb{E}\left[\int_0^T L(X_t^i, \alpha_t^i) + F^N(X_t^i, m_{\boldsymbol{X}_t}^{N,i})\, dt + G^N(X_T^i, m_{\boldsymbol{X}_t}^{N,i})\right]$$

where $\boldsymbol{X}_t = (X_t^1, \ldots, X_t^N)$ and, for any j,

$$dX_t^j = \alpha_t^j dt + \sqrt{2} dB_t^i, \qquad X_0^j = x_0^j, \qquad \text{and} \qquad m_{\boldsymbol{X}_t}^{N,i} = \frac{1}{N-1}\sum_{j\neq i} \delta_{X_t^j},$$

the (B^i) being independent B.M and (x_0^j) are i.i.d. initial conditions.

- "Good" notion of solution : Nash equilibria.
We say that $\bar{\alpha} = (\bar{\alpha}^1, \ldots, \bar{\alpha}^N)$ is a Nash Equilibrium if

$$J^{N,i}(\bar{\alpha}^i, (\bar{\alpha})_{j\neq i}) \leq J^{N,i}(\alpha^i, (\bar{\alpha})_{j\neq i}) \qquad \forall \alpha^i,\ \forall i.$$

The Nash system

When players play closed-loop controls : $\bar{\alpha}^i = \bar{\alpha}^i(t, X_t^1, \dots, X_t^N)$, the value function $v^{N,i} = v^{N,i}(t, x^1, \dots, x^N)$ of player i associated with a Nash equilibrium satisfies the Nash system :

$$(Nash) \quad \left\{ \begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \displaystyle\sum_{j=1}^N \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) + H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x})) \\ \qquad\qquad + \displaystyle\sum_{j \neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = F^N(x_i, m_{\boldsymbol{x}}^{N,i}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times (\mathbb{R}^d)^N, \ i \in \{1, \dots, N\} \\ v^{N,i}(T, \boldsymbol{x}) = G^N(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } (\mathbb{R}^d)^N, \ i \in \{1, \dots, N\} \end{array} \right.$$

where

- N is the number of players,
- $\boldsymbol{x} = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ is the state variable, $T \geq 0$ is the time horizon,
- for $i \in \{1, \dots, N\}$, $v^{N,i}(t, \boldsymbol{x})$ is the value function of Player i,
- $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Hamiltonian of the system :

$$H(x,p) = \sup_{\alpha \in \mathbb{R}^d} -p \cdot \alpha - L(x, \alpha),$$

- $F^N, G^N : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ are the coupling functions.

The system of optimal trajectories.

We denote by $\boldsymbol{X}_t^N = (X_{1,t}^N, \dots, X_{N,t}^N)$ the “optimal trajectories" of the $N-$player game : they solve the system of N coupled stochastic differential equations (SDE) :

$$dX_{i,t}^N = -D_pH(X_{i,t}, Dv^{N,i}(t, \boldsymbol{X}_t^N))dt + \sqrt{2}dB_t^i, \qquad t \in [0,T],\ i \in \{1, \dots, N\},$$

where

- $v^{N,i}$ is the solution to the Nash system,
- the $(B_t^i)_{t\in[0,T]}$ are $d-$dimensional independent Brownian motions.

> We are interested in the behavior, as $N \to +\infty$, of the $(v^{N,i})$ and of the $(X_{i,\cdot}^N)$.

Two regimes

- $F^N = F(x,m)$ and $G^N = G(x,m)$ are smoothing (nonlocal coupling)
- $G^N = G(x)$ and $F^N = F^N(x, m(\cdot)dx) \to F(x, m(x))$ for smooth measures (local coupling)

Main difficulty : no estimates.

Expected limit : the Mean Field Game.

- **“Probabilistic formulation"**
  Each “small player" wants to minimize over her control (α_t) the cost

$$J(\alpha,(m_t)) := \mathbb{E}\left[\int_0^T L(X_t,\alpha_t) + F(X_t^i,m_t)\, dt + G(X_T)\right]$$

where $dX_t = \alpha_t dt + dB_t,\ X_0 = x_0,$ and (m_t) is the mean field :

$$\mathcal{L}(X_t) = m_t \qquad \forall t \in [0,T] \qquad \text{(Nash equilibrium condition)}.$$

- **“The PDE formulation"**
  Find (u,m) solving the system

$$(MFG) \qquad \left\{\begin{array}{lll} (i) & -\partial_t u - \Delta u + H(x,Du) = F(x,m_t) & \text{in } [0,T]\times\mathbb{R}^d \\ (ii) & \partial_t m - \Delta m - \mathrm{div}(mD_pH(x,Du)) = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ (iii) & u(T,x) = G(x) & \text{in } \mathbb{R}^d \\ (iv) & m(0,\cdot) = m_0 := \mathcal{L}(x_0) & \text{in } \mathbb{R}^d \end{array}\right.$$

where $H(x,p) := \sup_\alpha -p\cdot\alpha - L(x,\alpha)$.

Some results on MFG

For the **MFG equilibrium system :**

$$(MFG) \quad \left\{ \begin{array}{lll} (i) & -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & m(0,\cdot) = m_0, \; u(T,x) = G(x) & \text{in } \mathbb{R}^d \end{array}\right.$$

- **Existence of solutions :** holds under general conditions (Lasry-Lions)
- **Uniqueness** cannot be expected in general,
  - but holds under a monotonicity conditions on F (Lasry-Lions) :

$$\int_{\mathbb{T}^d} (F(x,m) - F(x,m'))d(m-m')(x) \geq 0 \qquad \forall m, m'.$$

- **The mean field limit** (for smoothing coupling function F).

  — from the MFG system to the $N-$player differential games
  Many contributions (Huang-Caines-Malahmé, Carmona-Delarue, Kolokoltsov, ...)

  — from Nash equilibria of $N-$player differential games to the MFG system.
  - LQ differential games (Bardi, Bardi-Priuli)
  - Open loop NE (Lasry-Lions, Fischer, Lacker),
  - Closed loop NE (C.-Delarue-Lasry-Lions).

- Let $v^{N,i}$ be the solution to the Nash system

$$\left\{\begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) + H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x})) \\ \qquad + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = F^N(x_i, m_{\boldsymbol{x}}^{N,i}) \text{ in } [0,T]\times(\mathbb{R}^d)^N \\ v^{N,i}(T,\boldsymbol{x}) = G^N(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } (\mathbb{R}^d)^N \end{array}\right.$$

- Because of the symmetry, the $v^{N,i}$ can be written as $v^{N,i}(t,\boldsymbol{x}) = U^N(t, x_i, m_{\boldsymbol{x}}^{N,i})$.
- Following Lasry-Lions, the expected limit U of the (U^N) should satisfy the master equation.

$$\left\{\begin{array}{l} -\partial_t U - \Delta_x U + H(x, D_x U) - \int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U\right]\, dm(y) \\ \qquad + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U)\, dm(y) = F(x,m) \qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \end{array}\right.$$

Some results on the master equation : Lasry-Lions ('13), Buckdahn-Li-Peng-Rainer ('14), Gangbo-Swiech ('14), Chassagneux-Crisan-Delarue ('15), Bessi ('15), Lacker-Webster ('15), Ahuja ('16),...

Outline

1. The master equation for nonlocal couplings
2. The convergence results for nonlocal couplings
3. The convergence result for a local coupling

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Derivatives in the space of measures

We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed for the Monge-Kantorovich distance

$$\mathbf{d}_1(m, m') = \sup_{\phi} \int_{\mathbb{T}^d} \phi(y)\, d(m - m')(y),$$

where the supremum is taken over all Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$ with a Lipschitz constant bounded by 1.

Derivatives

A map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y) d(m' - m)(y) ds.$$

We set

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y).$$

Standing assumptions

- $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth, globally Lipschitz continuous, with :

$$0 < D^2_{pp}H(x,p) \leq CI_d \qquad \text{for } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

- the maps $F, G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are monotone : for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} (F(x,m) - F(x,m'))d(m-m')(x) \geq 0, \int_{\mathbb{T}^d} (G(x,m) - G(x,m'))d(m-m')(x) \geq 0$$

- the maps F, G are $\mathcal{C}^1$: there exists $n \geq 2$ and $\alpha \in (0,1)$ such that

$$\sup_{m \in \mathcal{P}(\mathbb{T}^d)} \left(\|F(\cdot,m)\|_{n+\alpha} + \left\| \frac{\delta F(\cdot,m,\cdot)}{\delta m} \right\|_{(n+\alpha,n+\alpha)} \right) + \mathrm{Lip}_n\left(\frac{\delta F}{\delta m}\right) \; < \; \infty.$$

and the same for G.

Example. If F is of the form :

$$F(x,m) = \int_{\mathbb{R}^d} f(z, (\rho \star m)(z))\rho(x-z)dz,$$

where

- $\star$ denotes the usual convolution product (in $\mathbb{R}^d$),
- $f = f(x,r)$ is a smooth map, nondecreasing w.r.t. to r,
- ρ is a smooth, even function with compact support.

Then F satisfies our conditions.

Indeed, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} (F(x,m) - F(x,m'))d(m-m')(x)$$
$$= \int_{\mathbb{T}^d} \left[f(y, \rho\star m(y)) - f(y, \rho\star m'(y))\right] \left(\rho\star m(y) - \rho\star m'(y)\right) dy \geq 0,$$

since ρ is even and f is nondecreasing with respect to the second variable. So F is monotone.

The master equation

It is the backward equation

$$(\mathbf{M}) \quad \left\{ \begin{array}{l} -\partial_t U(t,x,m) - \Delta_x U(t,x,m) + H(x, D_x U(t,x,m)) - F(x,m) \\ \qquad - \displaystyle\int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U \right] (t,x,m,y) \, dm(y) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x,m) \in [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(T,x,m) = G(x,m), \qquad \text{for } (x,m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \end{array} \right.$$

Theorem 1 (Chassagneux-Crisan-Delarue)

Under the previous assumptions, the master equation (**M**) has a unique classical solution.

Remark : In C.-Delarue-Lasry-Lions, we extend the result to the master equation with common noise.

Idea of proof

- The proof of Theorem 1 relies on the method of characteristics in infinite dimension.

- Given $(t_0, m_0) \in [0, T) \times \mathcal{P}(\mathbb{T}^d)$, let $(u, m) = (u(t, x), m(t, x))$ be the solution of the MFG system :

$$(MFG) \qquad \left\{ \begin{array}{l} -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) \text{ in } [t_0, T] \times \mathbb{T}^d \\ \partial_t m - \Delta m - \mathrm{div}(mD_pH(x, Du)) = 0 \text{ in } [t_0, T] \times \mathbb{T}^d \\ u(T, x) = G(x, m(T)), \;\; m(t_0, \cdot) = m_0 \text{ in } \mathbb{T}^d \end{array} \right.$$

- Under our monotonicity assumptions on F and G, the (MFG) system is well-posed. (Lasry-Lions, 2007)
- We *define* U by

$$U(t_0, \cdot, m_0) := u(t_0, \cdot)$$

- One easily check that U is formally a solution to (**M**).
- Difficult part : show that U is smooth enough to justify the computation.

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Convergence of the Nash system

We consider the solution $(v^{N,i})$ of the Nash system :

$$(Nash) \qquad \left\{\begin{array}{l} -\partial_t v^{N,i} - \sum_j \Delta_{x_j} v^{N,i} + H(x_i, D_{x_i} v^{N,i}) \\ \qquad\qquad + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}) \cdot D_{x_j} v^{N,i} = F(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } [0,T]\times \mathbb{T}^{Nd} \\ v^{N,i}(T,\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } \mathbb{T}^{Nd} \end{array}\right.$$

where we have set, for $\boldsymbol{x} = (x_1, \dots, x_N) \in (\mathbb{T}^d)^N$, $m_{\boldsymbol{x}}^{N,i} = \frac{1}{N-1}\sum_{j\neq i} \delta_{x_j}$.

Theorem 2 (C.-Delarue-Lasry-Lions)

Let $(v^{N,i})$ be the solution to the Nash system and U be the classical solution to the master equation (**M**). Then, for any $N \in \mathbb{N}^*$ and any $\boldsymbol{x} \in (\mathbb{T}^d)^N$,

$$\left| v^{N,i}(t_0, \boldsymbol{x}) - U(t_0, x_i, m_{\boldsymbol{x}}^N) \right| \leq CN^{-1}.$$

where $m_{\boldsymbol{x}}^N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$.

Convergence of the optimal trajectories

Let $t_0 \in [0, T)$, $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and let (Z_i) be an i.i.d family of random variables of law m_0. Let also (B^i) and W be independent B.M. and independent of the (Z_i).

We consider

- the optimal trajectories $(\boldsymbol{X}_t^N = (X_{1,t}^N, \dots, X_{N,t}^N))_{t\in[t_0,T]}$ of the Nash system :

$$\begin{cases} dX_{i,t}^N = -D_pH(X_{i,t}^N, D_{x_i}v^{N,i}(t, \boldsymbol{X}_t^N))dt + \sqrt{2}dB_t^i, & t \in [t_0, T] \\ X_{i,t_0}^N = Z_i \end{cases}$$

- and the solution $(\boldsymbol{Y}_t^N = (Y_{1,t}^N, \dots, Y_{N,t}^N))_{t\in[t_0,T]}$ of stochastic differential equation of McKean-Vlasov type :

$$\begin{cases} dY_{i,t}^N = -D_pH\left(Y_{i,t}^N, D_xU(t, Y_{i,t}^N, \mathcal{L}(Y_{i,t}^N)\right) dt + \sqrt{2}dB_t^i, \\ Y_{i,t_0}^N = Z_i. \end{cases}$$

Theorem 3 (C.-Delarue-Lasry-Lions)

For any $N \geq 1$ and any $i \in \{1, \ldots, N\}$, we have

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\left|X_{i,t}^N - Y_{i,t}^N\right|\right] \leq CN^{-\frac{1}{d+8}}$$

for some constant $C > 0$ independent of t_0, m_0 and N.

As the $(Y_{i,t}^N)$ are independent, the above result shows the propagation of chaos.

Key ingredient of proofs

Let U be the solution of the master equation.

- For $N \geq 2$ and $i \in \{1, \ldots, N\}$ we set

$$u^{N,i}(t,\boldsymbol{x}) = U(t, x_i, m_{\boldsymbol{x}}^{N,i}) \quad \text{where } \boldsymbol{x} = (x_1, \ldots, x_N) \in (\mathbb{T}^d)^N, \; m_{\boldsymbol{x}}^{N,i} = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}.$$

- Then one can compute the derivatives of $u^{N,i}$ in terms of those for U : e.g.,

$$D_{x_j} u^{N,i}(t,\boldsymbol{x}) = \frac{1}{N-1} D_m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j) \qquad (j \neq i),$$

- The $u^{N,i}$ are “almost" solution to the Nash system : for any $i \in \{1, \ldots, N\}$,

$$\begin{cases} -\partial_t u^{N,i} - \sum_j \Delta_{x_j} u^{N,i} + H(x_i, D_{x_i} u^{N,i}) \\ \qquad + \sum_{j \neq i} D_{x_j} u^{N,i}(t,\boldsymbol{x}) \cdot D_p H(x_j, D_{x_j} u^{N,j}(t,\boldsymbol{x})) = F(x_i, m_{\boldsymbol{x}}^{N,i}) + r^{N,i}(t,\boldsymbol{x}) \\ \qquad\qquad\qquad\qquad \text{in } (0,T) \times \mathbb{T}^{Nd} \\ u^{N,i}(T,\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } \mathbb{T}^{Nd} \end{cases}$$

where $\|r^{N,i}\|_\infty \leq \dfrac{C}{N}$.

- Let $(\boldsymbol{X}_t^N = (X_{1,t}^N, \dots, X_{N,t}^N))_{t\in[t_0,T]}$ be the optimal trajectories of the Nash system :

$$\begin{cases} dX_{i,t}^N = -D_pH(X_{i,t}^N, D_{x_i}v^{N,i}(t, \boldsymbol{X}_t^N))dt + \sqrt{2}dB_t^i, & t \in [t_0, T] \\ X_{i,t_0}^N = Z_i \end{cases}$$

and $(\tilde{\boldsymbol{Y}}_t^N = (\tilde{Y}_{1,t}^N, \dots, \tilde{Y}_{N,t}^N))_{t\in[t_0,T]}$ be the solution to

$$\begin{cases} d\tilde{Y}_{i,t}^N = -D_pH(\tilde{Y}_{i,t}^N, D_{x_i}u^{N,i}(t, \tilde{\boldsymbol{Y}}_t^N))dt + \sqrt{2}dB_t^i & t \in [t_0, T] \\ \tilde{Y}_{i,t_0}^N = Z_i \end{cases}$$

- Note that $\tilde{Y}_{i,t}^N$ and $Y_{i,t}^N$ are close (classical mean field limit).
- Using the equation satisfied by the $(v^{N,i}(t, \boldsymbol{X}_t^N))$ and the $(u^{N,i}(t, \boldsymbol{X}_t^N))$, one can show that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} |\tilde{Y}_{i,t} - X_{i,t}| + \sup_{t\in[t_0,T]} \left|u^{N,i}(t, \boldsymbol{X}_t) - v^{N,i}(t, \tilde{\boldsymbol{Y}}_t)\right|\right] \leq CN^{-1},$$

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The Nash system for a singular coupling

We now study the limit of the Nash system (with no common noise) :

$$(Nash) \qquad \begin{cases} -\partial_t v^{N,i} - \sum_j \Delta_{x_j} v^{N,i} + H(x_i, D_{x_i} v^{N,i}) \\ \qquad + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}) \cdot D_{x_j} v^{N,i} = F^N(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } [0,T]\times \mathbb{T}^{Nd} \\ v^{N,i}(T, \boldsymbol{x}) = G(x_i) \qquad \text{in } \mathbb{T}^{Nd} \end{cases}$$

when the coupling becomes singular (local) :

$$F^N(x, m(\cdot)dx) \to f(x, m(x)) \qquad \text{as } N \to +\infty$$

for any smooth density $m(\cdot)dx$. Namely :

$$F^N(x,m) = [f(\cdot, m \star \xi^{\varepsilon_N})] \star \xi^{\varepsilon_N}$$

where $\xi^\varepsilon(x) = \varepsilon^{-d}\xi(x/\varepsilon)$, ξ is a “nice kernel" and $f : \mathbb{T}^d \times [0,+\infty) \to \mathbb{R}$ is smooth, Lipschitz continuous and increasing in the second variable.

Fix $t_0 \in [0, T]$ and m_0 a smooth positive density and set

$$w^{N,i}(t_0, x_i, m_0) := \int_{(\mathbb{T}^d)^{N-1}} v^{N,i}(t_0, \boldsymbol{x}) \prod_{j \neq i} m_0(dx_j) \qquad \text{where } \boldsymbol{x} = (x_1, \ldots, x_N).$$

Theorem 4 (C., 2017)

Assume that $\varepsilon_N = \ln(N)^{-\beta}$ for some $\beta \in (0, (6d(2d+15))^{-1})$. Then

$$\left\| w^{N,i}(t_0, \cdot, m_0) - u(t_0, \cdot) \right\|_{L^1(m_0)} \leq A \ln(N)^{-B}$$

where (u, m) solves the MFG system with local interactions :

$$\left\{ \begin{array}{l} -\partial_t u - \Delta u + H(x, Du) = f(x, m(t,x)) \qquad \text{in } [t_0, T] \times \mathbb{T}^d, \\ \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du)) = 0 \qquad \text{in } [t_0, T] \times \mathbb{T}^d, \\ u(T, x) = G(x), \;\; m(t_0, \cdot) = m_0 \qquad \text{in } \mathbb{T}^d \end{array} \right.$$

Moreover, the optimal trajectories converge :

$$\mathbb{E}\left[\sup_{t \in [t_0, T]} \left| Y^N_{i,t} - X^N_{i,t} \right| \right] \leq A \ln(N)^{-B}.$$

- **Main issue :** No master equation associated with the local coupling.

- Arguments of proof : estimate of (lack) of regularity of the solution U^N associated with the master equation

$$\left\{\begin{array}{l} -\partial_t U^N - \Delta_x U^N + H(x, D_x U^N) - \int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U^N\right]\ dm(y) \\ \qquad + \int_{\mathbb{R}^d} D_m U^N \cdot D_p H(y, D_x U^N)\ dm(y) = F^N(x,m) \qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \\ U^N(T,x,m) = G(x) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \end{array}\right.$$

The open-loop Nash system with singular coupling

When players play with "open-loop" controls, the Nash system reads :

$$\left\{\begin{array}{l} -\partial_t v^{N,i} - \Delta v^{N,i} + H(x_i, Dv^{N,i}) = \displaystyle\int_{(\mathbb{T}^d)^{N-1}} F^N(x_i, m_{\boldsymbol{x}}^{N,i})\Pi_{j\neq i} m^{N,j}(t,x_j)dx_j \text{ in } [t_0,T]\times\mathbb{T}^d \\ \partial_t m^{N,i} - \Delta m^{N,i} - \mathrm{div}(m^{N,i} D_pH(x_i, Dv^{N,i})) = 0 \qquad \text{in } [t_0,T]\times\mathbb{T}^d \\ m^{N,i}(t_0,\cdot) = \mathcal{L}(Z_i), \qquad v^{N,i}(T,x_i) = G(x_i) \qquad \text{in } \mathbb{T}^d \end{array}\right.$$

Theorem 5 (C., 2017)

Assume $\varepsilon_N = N^{-\beta}$ where $\beta \in (0, (6d(2d+15))^{-1})$ and $(v^{N,i})$ is a symmetric solution. Then

$$\|w^{N,i}(t_0,\cdot,m_0) - u(t_0,\cdot)\|_{L^1(m_0)} \leq AN^{-B}$$

where (u,m) solves the MFG system with local interactions.
Moreover, the optimal trajectories converge :

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\left|Y^N_{i,t} - X^N_{i,t}\right|\right] \leq AN^{-B}.$$

- Convergence in the case of nonlocal smooth coupling : Lasry-Lions, Fischer, Lacker (compactness arguments).
- The singular behavior of F^N makes the compactness argument difficult (no rate).
- Proof without master equation (!)
- Key idea : reproduce the Lasry-Lions monotony argument at the level of $v^{N,i} - u$.

Conclusion

We have established

- The well-posedness of the master equation (with common noise),
- Limit results for the Nash system and the associated optimal trajectories,
- Case of local coupling (for Nash in closed-loop and open-loop form).

Open problems :

- Analysis in more realistic setting (with boundary conditions, non constant diffusion matrices,...)
- Stronger convergence for the solutions $(v^{N,i})$ of the Nash system.
- Convergence in the non-monotone setting.

Common noise

The Nash system with common noise :

$$\left\{\begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^N \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \beta \sum_{j,k=1}^N \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) + H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x})) \\ \qquad + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = F^{N,i}(\boldsymbol{x}) \\ \qquad\qquad\qquad\qquad \text{in } [0,T]\times(\mathbb{R}^d)^N,\ i\in\{1,\dots,N\} \\ v^{N,i}(T,\boldsymbol{x}) = G^{N,i}(\boldsymbol{x}) \qquad \text{in } (\mathbb{R}^d)^N,\ i\in\{1,\dots,N\} \end{array}\right.$$

where $\beta \geq 0$ is the intensity of the noise.

The associated optimal trajectories

$$dX^N_{i,t} = -D_p H(X_{i,t}, Dv^{N,i}(t,\boldsymbol{X}^N_t))dt + \sqrt{2}dB^i_t + \sqrt{2\beta}dW_t, \qquad t\in[0,T],\ i\in\{1,\dots,N\},$$

The master equation with common noise :

$$\left\{\begin{array}{l} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) \\ \qquad -(1+\beta)\displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y\,[D_m U]\; dm(y) + \int_{\mathbb{R}^d} D_m U\cdot D_p H(y, D_x U)\, dm(y) \\ \qquad -2\beta \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_x\,[D_m U]\, dm(y) - \beta\int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right]\; dm\otimes dm = F(x,m) \\ \qquad\qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \end{array}\right.$$

with associated stochastic MFG system :

$$(MFGs)\quad \left\{\begin{array}{l} d_t u_t = \left\{-(1+\beta)\Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta}\,\mathrm{div}(v_t)\right\} dt \\ \qquad\qquad\qquad\qquad + v_t\cdot\sqrt{2\beta}\, dW_t \qquad \text{in } [t_0, T]\times\mathbb{T}^d, \\ d_t m_t = \left[(1+\beta)\Delta m_t + \mathrm{div}\left(m_t D_p H(m_t, Du_t)\right)\right] dt - \sqrt{2\beta}\,\mathrm{div}(m_t dW_t) \\ \qquad\qquad\qquad\qquad \text{in } [t_0, T]\times\mathbb{T}^d \\ m_{t_0} = m_0,\; u_T(x) = G(x, m_T) \qquad \text{in } \mathbb{T}^d. \end{array}\right.$$

where (v_t) is a vector field which ensures (u_t) to be adapted to the filtration $(\mathcal{F}_t)_{t\in[t_0,T]}$ generated by the M.B. $(W_t)_{t\in[0,T]}$.

(actually, $v_t(x) = \displaystyle\int_{\mathbb{T}^d} D_m U(x, m_t, y) dm_t(y)$)

The master equation with common noise :

$$\left\{\begin{array}{l}
-\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) \\
\qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U\right] \, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U) \, dm(y) \\
\qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x \left[D_m U\right] dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right] \, dm \otimes dm = F(x, m) \\
\qquad\qquad\qquad \text{in } [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\
U(T, x, m) = G(x, m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)
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\qquad\qquad\qquad\qquad\qquad \text{in } [t_0, T] \times \mathbb{T}^d \\
m_{t_0} = m_0, \; u_T(x) = G(x, m_T) \qquad \text{in } \mathbb{T}^d.
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