Stochastic variational principles on Lie groups

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General idea:

To derive variational methods for certain deterministic equations of motion corresponding to dissipative systems (that cannot be treated by such methods in a classical setting), ode's or pde's (in the infinite dimensional case) by deforming stochastically the underlying Lagrangian paths and interpreting velocities in a generalized sense. Example: Navier-Stokes, MHD.

In this case Lagrangian is the classical one, but computed over stochastic processes (inspired by Feynman path integral approach to QM). Equations are not perturbed.

Also to derive sde's (spde's in the infinite dimensional case) by stochastic variational principles. Here the Lagrangian is randomly perturbed.

Lie group structures

G Lie group, left (right) translations smooth, with

$< >$ left (right) invariant metric

∇ left (right) invariant connection, torsion free

e identity element, $\mathcal{G} \simeq T_e G$ Lie algebra;

For $g_1 \in G$, $T_{g_2} L_{g_1} : T_{g_2} G \to T_{g_1 g_2} G$ tangent map (derivative of L_{g_1} at g_2)

Semi-direct products

U vector space, U^* dual; suppose that G has a left representation on $U \to$ naturally induce left representations of G and $\mathcal{G}$ on U and U^*; denote

$\diamond : U \times U^* \to T_e^* G$

$< a \diamond \alpha, v >_{T_e G} := - < v\alpha, a >_U = < \alpha, va >_U$, for $a \in U$, $\alpha \in U^*, v \in T_e G$

Let g be of the form

$$\begin{cases} dg(t) = T_e L_{g_(t)} \left(\sum_{i=1}^k H_i dW^i(t) + u(t)dt \right), \\ g(0) = e. \end{cases} \tag{1}$$

($dW^i(t)$ Itô integral) or, equivalently

$$\begin{cases} dg(t) = T_e L_{g(t)} \left(\sum_{i=1}^k H_i \circ dW^i(t) - \frac{1}{2} \nabla_{H_i} H_i dt + u(t)dt \right), \\ g(0) = e. \end{cases} \tag{2}$$

($\circ dW^i(t)$ Stratonovich integral) with $H_i \in T_e G$, $1 \le i \le k$. Then

$$\frac{D^\nabla g(t)}{dt} = T_e L_{g(t)} u(t)$$

(∇**-generalized derivative**)

$$d^\nabla g(t) = \sum_{i=1}^k T_e L_{g(t)} H_i dW^i(t)$$

∇**-stochastic differential with respect to the martingale part**

Stochastic variational principles
Action functional

$$
\begin{aligned}
A(g^1(\cdot), g^2(\cdot)) := & \int_0^T l\left(T_{g^1(t)} L_{g^1(t)^{-1}} \frac{D^\nabla g^1(t)}{dt}, \alpha(t) \right) dt \\
& + \int_0^T \left\langle q\left(T_{g^1(t)} L_{g^1(t)^{-1}} \frac{D^\nabla g^1_\omega(t)}{dt}, \alpha(t) \right), T_{g^1(t)} L_{g^1(t)^{-1}} d^\nabla g^1(t) \right\rangle \\
& - \sum_{i=1}^m \int_0^T \left\langle q\left(T_{g^1_\omega(t)} L_{g^1_\omega(t)^{-1}} \frac{D^\nabla g^1(t)}{dt}, \alpha(t) \right), H_i dW^i(t) \right\rangle
\end{aligned} \tag{3}
$$

Here $l : T_eG \times U^* \to \mathbb{R}$ is the Lagrangian function, $q : T_eG \times U^* \to T_e^*G$, and

$$\alpha(t) = g^2(t)^{-1}\alpha_0, \ \ \alpha_0 \in U^*$$

Notice that if $q = 0$ the action functional is simply a classical action computed on stochastic paths. In this case we define $\alpha(t)$ as the expectation of the above $\alpha(t)$.

Domain of the action: the collection of all G-valued semimartingales defined for $t \in [0, T]$.

Admissible variations: For every $\varepsilon \in [0, 1)$ and every process $v \in C^1([0, 1]; T_eG)$ a.s., let $e_{\varepsilon,v}(\cdot) \in C^1([0, T]; G)$ be the unique solution of the (random) time-dependent ode

$$\begin{cases} \frac{d}{dt} e_{\varepsilon,v}(t) = \varepsilon T_e L_{e_{\varepsilon,v(t)}} \dot{v}(t), \\ e_{\varepsilon,v}(0) = e, \end{cases} \tag{4}$$

Then (g^1, g^2) is **critical** for A if for every such variation,

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} A\big(g^1(\cdot)e_{\varepsilon,v(\cdot)}, g^2(\cdot)e_{\varepsilon,v(\cdot)}\big) = 0$$

Let
$dg^j(t) = T_eL_{g^j(t)}\left(\sum_{i=1}^{m_i} H_i^j \circ dW^{j,i}(t) + u(t)dt\right), \quad g^j(0) = e, \;\; j = 1, 2.$
Assume $\sum_i \nabla_{H_iH_i} = 0$.

Theorem. (g^1, g^2) is a critical point of A if and only if the process $u(t)$ coupled with $\alpha(t)$ satisfies the following semidirect (stochastic) product Euler-Poincaré equation for stochastic particle paths:

$$\left\{\begin{array}{l} d\left(\frac{\delta l}{\delta u}(u(t),\alpha(t))\right) = \sum_{i=1}^{m_1} \mathrm{ad}^*_{H_i} q(u(t),\alpha(t))dW^{1,i}(t) \\ +\mathrm{ad}^*_{u(t)}(\frac{\delta l}{\delta u}(u(t),\alpha(t)))dt + (\frac{\delta l}{\delta \alpha}(u(t),\alpha(t))) \diamond \alpha(t)dt \\ +K(\frac{\delta l}{\delta u}(u(t)))dt, \\ d\alpha(t) = -\sum_{i=1}^{m_2} H_i^2\alpha(t)dW^{2,i}(t) - u^2(t)\alpha(t)dt + \frac{1}{2}\sum_{i=1}^{m_2} H_i^2(H_i^2\alpha(t))dt \end{array}\right. \tag{5}$$

where $\frac{\delta l}{\delta u} \in T_e^*G$ and $\frac{\delta l}{\delta \alpha} \in U$ denote the partial functional derivatives of l.

Who is K?

The linear operator $K : T_e^*G \to T_e^*G$ is defined for all $\mu \in T_e^*G$ and $v \in T_eG$ by

$$\langle K(\mu), v\rangle = -\left\langle \mu, \frac{1}{2}\sum_{i=1}^{m_1}\left(\nabla_{\mathrm{ad}_v H_i^1} H_i^1 + \nabla_{H_i^1}(\mathrm{ad}_v H_i^1)\right)\right\rangle .$$

BUT, in the right invariant case:

If ∇ is the Levi-Civita connection and $\nabla_{H_i} H_i = 0$ for all i. Then

$$K^*(u) = -\frac{1}{2}\sum_i (\nabla_{H_i}\nabla_{H_i} u + R(u, H_i)H_i)$$

where R is the Riemannian curvature tensor. If, in addition, H_i is an o.n. basis of $\mathcal{G}$, identifying T_e^*G with T_eG, we have $K(u) = -\frac{1}{2}(\Delta u + \text{Ricci } u)$.

Right invariant case:

Just change the tangent map for $T_e R_g$ and the sign of the terms that do not involve U (and U^*).

Proof: long but essentially use of Itô calculus.

Infinite dimensional groups

The group of homeomorphisms on the torus, $M = \mathbb{T}^3$:

$G := G^s = \{g : M \to M \text{ bijective } : g, g^{-1} \in H^s\}$
H^s Sobolev maps of order $s > 1 + \frac{3}{2}$

G^s is an open subset of H^s so a smooth Hilbert manifold and a group under composition.

$T_e G^s = \mathcal{G}^s = H^s$ vector fields on M. Not quite a Lie algebra (the bracket loses one derivative).

$d\mu$ the Riemannian volume. Weak L^2 metric:

$$< U_g, V_g > := \int_M < U_g(\theta), V_g(\theta) >_{g(\theta)} d\mu(\theta)$$

for $U_g, V_g \in T_g G^s = \{U : M \to TM \text{ in } H^s, U(\theta) \in T_{g(\theta)} M\}$
Not right-invariant.

Connection:

$$(\nabla_X Y)(g) := \frac{\partial}{\partial t}(Y(g_t) \circ g_t^{-1}) \circ g + \nabla_{X \circ g^{-1}} Y \circ g^{-1}) \circ g$$

where g_t is such that $g_0 = g$, $\frac{d}{dt}|_{t=0} g_t = X(g)$.

We cannot apply the general theorem directly but we can go through the proof and repeat it for specific cases.

Take the constant vector fields $H_i = \sqrt{2\nu}e_i$, e_i canonical bases of $\mathbb{R}^3$; $TM = M \times \mathbb{R}^3$ and consider

$$\begin{cases} dg(t,\theta) = \sqrt{2\nu}dW(t) + u(t, g(t,\theta))dt, \\ g(0,\theta) = \theta. \end{cases} \tag{6}$$

with $u \in C[0,T]; \mathcal{G}^s(M))$, s large enough to define a flow of G^s diffeomorphisms. U^* can be a space of functions or differential forms on M. Action of G^s on U^*:

$$\alpha(t) := \alpha_0 g(t)^{-1} = (g(t)^{-1})^* \alpha_0$$

for $\alpha_0 \in U^*$ is the pullback by $g(t)^{-1}$.

$\alpha_0 : M \to \mathbb{R}$ smooth function:

$$d\alpha(t,\theta) = -\sqrt{2\nu}\nabla\alpha(t,\theta)\cdot dW(t) - u(t,\theta)\cdot\nabla\alpha(t,\theta)dt + \nu\Delta\alpha(t,\theta)dt$$

$\alpha_0 = \rho_0(\theta)d\theta$ a density form: $\alpha(t,\theta) = \rho(t,\theta)d\theta$ with

$$d\rho(t,\theta) = -\sqrt{2\nu}\nabla\rho(t,\theta)\cdot dW(t) - \nabla\cdot(\rho u)(t,\theta) + \nu\Delta\rho(t,\theta)$$

α_0 a one-form:
$\alpha_0 = \sum_{i=1}^{3} A_{0,i}d\theta_i$:

$$\begin{aligned} dA_i(t,\theta) = &-\sum_{j=1}^{3}\sqrt{2\nu}\partial_j A_i(t,\theta)dW^j(t) \\ &-\sum_{j=1}^{3}\Big(u_j(t,\theta)\partial_j A_i(t,\theta) + A_j(t,\theta)\partial_i u_j(t,\theta)\Big)dt \\ &+\nu\Delta A_i(t,\theta)dt \end{aligned} \tag{7}$$

The compressible Navier-Stokes equation

Let U^* be the space of probability densities on $M = \mathbb{T}^3$.

Consider the Lagrangian $l : \mathcal{G}^s \times U^* \to \mathbb{R}$

$$l(u, \alpha) := \int_M \left(\frac{\rho(\theta)}{2}|u(\theta)|^2 - \rho(\theta)e(\rho(\theta))\right)d\theta,\ \forall u \in \mathcal{G}^s(\mathbb{T}^3), \alpha = \rho(\theta)d\theta \in U^*$$

$e(\rho) =$ fluid's internal energy

$p(\rho) =$ pressure, $\mathbf{d}e = -p\mathbf{d}(\frac{1}{\rho})$, $\alpha = \rho(\theta)d\theta$.

Stochastic force $q : \mathcal{G}^s \times U^* \to \mathbb{R}$

$q(u, \alpha) = \frac{\delta l}{\delta u}(u, \alpha) = u\rho$.

Action:

$$A(g^{\nu_1}, g^{\nu_2}) := \int_0^T \int_M (\frac{1}{2}|w(t,\theta)|^2 \rho(t,\theta) - \rho(t,\theta) e(\rho(t,\theta))) d\theta dt$$
$$+ \int_0^T \int_M < w(t,\theta), dM(t,\theta) > \rho(t,\theta) d\theta$$
$$- \sum_{i=1}^{3} \sqrt{2\nu} \int_0^T \int_M w_i(t,\theta)\rho(t,\theta) d\theta dW^i(t)$$

where $w(t,\cdot) := T_{g^{\nu_1}(t)} R_{g^{\nu_1}(t)^{-1}} (\frac{D^\nabla g^{\nu_1}(t)}{dt}))$,
$dM(t,\cdot) := T_{g^{\nu_1}(t)} R_{g^{\nu_1}(t)^{-1}} (d^\nabla g^{\nu_1}(t))$,
$\rho(t,\theta) d\theta = (g^{\nu_2}(t,\cdot)^{-1})^* \alpha_0$.

Theorem. The semi-martingale (g^{ν_1}, g^{ν_2}) is a critical point of A if and only if (u, ρ) satisfies the following SPDE

$$\begin{cases} du(t) = -\sqrt{2\nu_1}\nabla u \cdot dW(t) - (\sqrt{2\nu_1} - \sqrt{2\nu_2})u\nabla log\rho \cdot dW(t) \\ -u \cdot \nabla udt + \nu_1\Delta udt + 2\nu_1\nabla log\rho \cdot \nabla udt \\ +(\nu_1 - \nu_2)u\frac{\Delta\rho}{\rho}dt + \frac{\nabla p}{\rho}dt \\ d\rho(t) = -\sqrt{2\nu_2}\nabla\rho \cdot dW(t) - \nabla \cdot (u\rho)dt + \nu_2\Delta\rho dt, \end{cases} \tag{8}$$

where $\rho(t,\theta)d\theta := (g^{\nu_2}(t,\cdot)^{-1})^*(\rho_0(\theta)d\theta)$.

In particular, if $\nu_1 = \nu_2 = \nu$,

$$\begin{cases} du(t) = -\sqrt{2\nu}\nabla u \cdot dW(t) - u \cdot \nabla u dt + \nu \Delta u dt \\ +2\nu \nabla log\rho \cdot \nabla u dt + \frac{\nabla p}{\rho} dt \\ d\rho(t) = -\sqrt{2\nu}\nabla \rho \cdot dW(t) - \nabla \cdot (u\rho) dt + \nu \Delta \rho dt. \end{cases} \tag{9}$$

For the deterministic action, $\nu_1 = \nu_2 = \nu$, we obtain:

$$\begin{cases} du(t) = -(u \cdot \nabla u) dt + \nu \Delta u dt + 2\nu \nabla log\tilde{\rho} \cdot \nabla u dt \\ +\frac{\nabla p}{\tilde{\rho}} dt, \\ d\tilde{\rho}(t) = -\nabla \cdot (u\tilde{\rho}) dt + \nu \Delta \tilde{\rho} dt. \end{cases} \tag{10}$$

with $\tilde{\rho}(t,\cdot) d\theta = E[(g(t,\cdot)^{-1})^* \alpha_0]$.

Compressible MHD (magnetohydrodynamical) equation

Let $\alpha_0 := (b_0(\cdot), B_0(\theta) \cdot dS, D_0(\theta)d\theta)$, b_0 a C^2 function , $B_0(\theta) \cdot dS$ an exact two-form on $\mathbb{T}^3$, i.e., there is some one-form $A_0(\theta) \cdot d\theta$ such that

$$B_0(\theta) \cdot dS = d\Big(A_0(\theta) \cdot d\theta\Big) = \sum_{1 \leq j < k \leq 3, i \neq j, i \neq k} \big(curl\ A_0(\theta)\big)_i d\theta_j \wedge d\theta_k,$$

and $D_0(\theta)d^3\theta$ a density. We let U^* denote the vector space of all such triples $(b(\cdot), B(\theta) \cdot dS, D(\theta)d\theta)$.

$I : T_eG^s \times U^* \to \mathbb{R}$ be defined by

$$I(u, b, B, D) = \int (\frac{D(\theta)}{2}|u(\theta)|^2 - D(\theta)e(D(\theta), b(\theta)) - \frac{1}{2}|B(\theta)|^2)d\theta,$$

with $u \in T_eG^s$ the Eulerian velocity of the fluid, $b \in C^2$ the entropy function, $B(\theta) \cdot dS$ an exact differential two-form representing the magnetic field in the fluid, $D(\theta)d\theta$ the mass density, and the function $e(D, b)$ the fluid's internal energy. The pressure $p(D, b)$ and the temperature $T(D, b)$ are related by $\mathbf{d}e = -p\mathbf{d}\left(\frac{1}{D}\right) + T\mathbf{d}b$.

Lagrangian:

$$l(u,b,B,D)=\int(\frac{D(\theta)}{2}|u(\theta)|^2-D(\theta)e(D(\theta),b(\theta))-\frac{1}{2}|B(\theta)|^2)d\theta,$$

In the case of the non-stochastic action, namely

$$A(g^{\nu_1},g^{\nu_2},g^{\nu_3},g^{\nu_4}):=\int_0^T l(w(t),\tilde{B}(t),\tilde{D}(t),\tilde{b}(t))dt,$$

where $w(t,\cdot):=T_{g^{\nu_1}(t)}L_{g^{\nu_1}(t)^{-1}}(\frac{D^{\nabla}g^{\nu_1}(t)}{dt})$,

$\tilde{B}(t,\theta)\cdot\mathbf{d}S:=E[(g^{\nu_3}(t,\cdot)^{-1})^*(B_0(\theta)\cdot\mathbf{d}S)]$,

$\tilde{b}(t,\theta):=E[(g^{\nu_4}(t,\cdot)^{-1}t)^*b_0]$,

$\tilde{D}(t,\cdot)d\theta=E[(g^{\nu_2}(t,\cdot)^{-1})^*(D_0(\theta)d\theta)]$.

Then (g^{ν_1}, g^{ν_2}) is a critical point of A if and only if $(u, \tilde{B}, \tilde{D}, \tilde{b})$ satisfies the following PDE

$$\begin{cases} du(t) = -u \cdot \nabla u dt + \frac{\tilde{B} \times curl\ \tilde{B}}{\tilde{D}} + \nu_1 \Delta u + (\nu_1 - \nu_2) \frac{u \Delta \tilde{D}}{\tilde{D}} \\ \qquad\qquad +2\nu_1 \nabla log \tilde{D} \cdot \nabla u dt + \frac{\nabla p}{\tilde{D}} dt \\ d\tilde{D}(t) = -\nabla \cdot (u\tilde{D}) dt + \nu_2 \Delta \tilde{D} dt \\ d\tilde{b}(t) = -u \cdot \nabla \tilde{b} dt + \nu_4 \Delta \tilde{b} dt \\ d\tilde{B}(t) = curl\ (u \times \tilde{B}) dt + \nu_3 \Delta \tilde{B} dt \\ \nabla \cdot \tilde{B}(t) = 0. \end{cases} \tag{11}$$

Existence of critical paths: the best method seems to be the use of entropy methods.

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