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On numerical solutions of parabolic stochastic PDEs given on the whole space

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- Motivation, Nonlinear filtering
- Localisation error for SPDEs
- Finite difference schemes for the Zakai equation
- Truncated schemes
- Accelerated schemes

The talk is based on M. Gerencsér and I. G. (2017)

1. Motivation, Nonlinear filtering

Consider a partially observed system $Z_t = (X_t, Y_t)$ governed by

$$\begin{aligned} dX_t =& b(Z_t)\,dt + \theta(Z_t)\,dW_t + \rho(Z_t)\,dV_t, \quad X_0 = \xi, \\ dY_t =& B(Z_t)\,dt + \,dV_t, \quad Y_0 = \eta, \end{aligned} \tag{1}$$

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Task: Calculate the mean square estimate $\hat{\phi}$ of $\varphi(X_t)$ from observations $(Y_s)_{s\in[0,t]} =: \mathcal{Y}_t$, i.e.,

$$E|\hat{\varphi} - \varphi(X_t)|^2 = \min_f E|f(Y_{[0,t]}) - \varphi(X_t)|^2 .$$

Clearly,

$$\hat{\varphi} = E(\varphi(X_t)|\mathcal{Y}_t) = \int_{\mathbb{R}^d} \varphi(x) P_t(dx) = \int_{\mathbb{R}^d} \varphi(x)\pi_t(x)\, dx,$$

where

$$P_t(dx) := P(X_t \in dx|\mathcal{Y}_t) = \pi_t(x)\, dx.$$

One knows that under suitable conditions

$$\pi_t = \frac{u_t}{\int_{\mathbb{R}^d} u_t(x)\, dx},$$

where u is the solution of the *Zakai equation*:

$$du_t(x) = \mathcal{L}_t u_t(x)\, dt + \mathcal{M}_t^k u_t(x)\, dY_t^k, \quad t \in [0, T], x \in \mathbb{R}^d.$$

Aim: Find numerical solutions to the Zakai equation
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Challenges:

- Zakai equation is in the whole $\mathbb{R}^d$
- It may degenerate (may not be uniformly parabolic)
- Methods of artificial boundary conditions do not work!

2. Localisation error for SPDEs

Consider

$$du_t(x) = \mathcal{L}_t u_t(x)\,dt + \mathcal{M}^k_t u_t(x)\,dW^k_t, \quad t \in (0,T],\ x \in \mathbb{R}^d \quad (2)$$

$$u_0(x) = \psi(x), \quad x \in \mathbb{R}^d, \quad (3)$$

where

$$\mathcal{L}_t = a^{ij}_t(x) D_i D_j + b^i_t(x) D_i + c_t(x), \quad \mathcal{M}^k_t = \sigma^{ki}_t(x) D_i + \mu^k_t(x)$$

with random initial value and coefficients

$$\mathcal{D} := (\psi, a, b, c, \sigma, \mu).$$

Assumption I.(stochastic parabolicity)

$$(\alpha^{ij}) := (2a^{ij} - \sigma^{ik}\sigma^{jk}) \geq 0$$

Assumption II. There is an integer $m \geq 0$ and a constant K such that the derivatives in x of (a, b, c) up to order m and of (σ, μ) up to order $m+1$ are continuous in (t, x), are predictable in (ω, t) and are bounded by a constant K.
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Theorem 1. Let Assumptions I-II hold with $m > 2 + d/p$. Then (2)-(3) has a unique classical solution u, i.e., u is predictable in (ω, t), almost surely $u \in C_b^{0,2}([0,T] \times \mathbb{R}^d)$, and (2)-(3) hold almost surely for all $x \in \mathbb{R}^d$ and $t \in [0, T]$.

Consider also

$$\begin{aligned} d\bar{u}_t(x) =& \bar{\mathcal{L}}_t \bar{u}_t(x)\, dt + \bar{\mathcal{M}}^k_t \bar{u}_t(x)\, dW^k_t, \quad t \in (0, T], x \in \mathbb{R}^d \quad (4) \\ \bar{u}_0(x) =& \bar{\psi}(x), \quad x \in \mathbb{R}^d, \quad (5) \end{aligned}$$

with

$$\bar{\mathcal{L}} = \bar{a}^{ij}_t(x) D_i D_j + \bar{b}^i_t(x) D_i + \bar{c}_t(x), \quad \bar{\mathcal{M}}^k = \bar{\sigma}^{ki}_t(x) D_i + \bar{\mu}^k_t(x)$$

such that almost surely

$$(\bar{\psi}, \bar{a}, \bar{b}, \bar{c}, \bar{\sigma}, \bar{\mu}) = (\psi, a, b, c, \sigma, \mu) \quad \text{for } (t, x) := [0, T] \times B_R, \quad (6)$$

where $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$.

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Assumption III. The derivatives in x of $\theta := \sqrt{\alpha}$ and $\bar{\theta} := \sqrt{\bar{\alpha}}$ up to order $m+1$ are continuous in x and are bounded by K.

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Theorem 2.(L. Gerencsér-I.G. 2017) Let problems (2)-(3) and (4)-(5) satisfy Assumptions I, II and III with $m > d + p/2$. Assume $\bar{\mathcal{D}} = \mathcal{D}$ a.s. on $[0, T] \times B_R$ for some $R > 0$. Then for $r > 1$ and $\nu \in (0, 1)$

$$E \sup_{t \in [0,T]} \sup_{x \in B_{\nu R}} |u_t(x) - \bar{u}_t(x)|^q \leq N e^{-\delta R^2} E^{1/r}(|\psi|^{qr}_{W^m_p} + |\bar{\psi}|^{qr}_{W^m_p}),$$

where N and δ are positive constants, depending on K, d, T, q, r, p and ν.

Idea of the proof. Consider first the simpler case:

$$du_t(x) + \mathcal{L}u_t(x)dt = 0, \quad t \in [0, T), \, x \in \mathbb{R}^d$$

$$u_T(x) = \psi(x), \quad x \in \mathbb{R}^d$$

with nonrandom terminal value ψ and nonrandom operator

$$\mathcal{L} = a_t^{ij}(x)D_iD_j + b_t^i(x)D_i.$$

Then by Feynman-Kac

$$u_t(x) = E\psi(X_T^{t,x}),$$

where $(X^{t,x})_{s\in[t,T]}$ is given by

$$dX_s = \theta_s(X_s)\,dW_s + b_s(X_s)\,ds, \quad s \in [t, T], \quad X_t = x.$$

By standard estimates for $\hat{X}_s^{t,x} := X_s^{t,x} - x$ we have

$$P(\sup_{0\le t\le s\le T}\sup_{|x|\le R}|\hat{X}_s^{t,x}| > r) \le Ne^{-\delta r^2}(1 + R^{d+1/2}), \qquad (7)$$

for any $r, R > 0$, with $N = N(d, K, M, T)$ and $\delta = \delta(d, K, M, T) > 0$, where K is the bound and M is the Lipschitz constant for a and b.

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Thus when $|x| \le \nu R$, $\nu \in (0, 1)$, for

$$\tau_{t,x} = \inf\{s \ge t : |X_s^{t,x}| \ge R\}$$

we have

$$P(\tau_{t,x} \le T) \le Ne^{-\delta R^2}$$

with $N = N(d, K, M, T, \nu)$ and $\delta = \delta(d, K, M, T, \nu) > 0$.

Hence using the stopping times

$$\tau_{t,x} = \inf\{s \geq t : |X_s^{t,x}| \geq R\}, \quad \bar{\tau}_{t,x} = \inf\{s \geq t : |\bar{X}_s^{t,x}| \geq R\}$$

we have

$$|u_t(x) - \bar{u}_t(x)| = |E(\psi(X_T^{t,x}) - \bar{\psi}(\bar{X}_T^{t,x}))| =$$

$$E\{\mathbf{1}_{\tau_{t,x} \wedge \bar{\tau}_{t,x} \leq T}(\psi(X_T^{t,x}) - \bar{\psi}(\bar{X}_T^{t,x}))\},$$

$$\leq \{P(\tau_{t,x} \leq T) + P(\bar{\tau}_{t,x} \leq T)\}(\sup_x |\psi(x)| + \sup_x |\bar{\psi}(x)|)$$

$$\leq Ne^{-\delta R^2} \sup_x(|\psi(x)| + \bar{\psi}(x)|).$$

The case of SPDEs.

Instead of (2)-(3) consider

$$dv_t(x) = \mathcal{L}v_t(x)\,dt + \mathcal{M}^k v_t(x)\,dW_t^k + \theta^{ri}(x)D_i v_t(x)\,d\hat{W}_t^r,$$

$$v_0(x) = \psi(x),$$

where $\theta = (2a - \sigma\sigma^*)^{1/2}$, $\hat{W}$ is a d-dimensional Wiener, independent of $\mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$. By Theorem 1 there is a classical solution $v = (v_t(x))$, and one can show that

$$u_t(x) = E(v_t(x)|\mathcal{F}_t).$$

Together with the above SPDE consider

$$
\begin{aligned}
dY_t =& \beta_t(Y_t)\,dt - \sigma_t^k(Y_t)\,dW_t^k - \theta_t^r(Y_t)\,d\hat{W}_t^r, \quad 0 \le t \le T, && (8)\\
Y_0 =& y, && (9)
\end{aligned}
$$

with

$$
\beta_t(y) := -b_t(y) + \sigma_t^{ik}(y) D_i \sigma_t^k(y) + \theta_t^{ri}(y) D_i \theta_t^r(y) + \sigma_t^k(y)\mu_t^k(y),
$$

for $t \in [0, T]$ and $y \in \mathbb{R}^d$.

By the Itô-Wentzell formula for $U_t(y) := v_t(Y_t(y))$ we get

$$
dU_t(y) = \gamma_t(Y_t(y))U_t(y)\,dt + \mu^k(Y_t(y))U_t(y)\,dW_t^k, \quad U_0(y) = \psi(y)
$$

with

$$
\gamma_t(x) = c_t(x) - \sigma_t^{ki}(x) D_i \mu_t^k(x).
$$

Hence

$$v_t(x) = U_t(Y_t^{-1}(x)) \quad \text{and } \bar{v}_t(x) = \bar{U}_t(\bar{Y}_t^{-1}(x)),$$

where $\bar{v}$, $\bar{Y}$ and $\bar{U}$ obtained by replacing $\mathcal{D}$ with $\bar{\mathcal{D}}$.

Lemma. Let $\nu' = (1+\nu)/2$ and set

$$H := \left[\sup_{t\in[0,T]} \sup_{|x|\le\nu R} |Y_t^{-1}(x)| \le \nu' R\right] \cap \left[\sup_{t\in[0,T]} \sup_{|x|\le\nu' R} |Y_t(x)| \le \nu R\right]$$

then $P(H^c) \le Ne^{-\delta R^2}$ and on H we have

$$v_t(x) = \bar{v}_t(x) \quad \text{for } t \in [0, T] \text{ and } |x| \le \nu R.$$

Hence by Doob's, Hölder's and Jensen's inequalities

$$E \sup_{t\in[0,T]} \sup_{|x|\le\nu R} |u_t(x) - \bar{u}_t(x)|^q$$

$$\le E \sup_{t\in[0,T]\cap\mathbb{Q}} |E(\mathbf{1}_{H^c} \sup_{s\in[0,T]} \sup_{|x|\le\nu R} |v_s(x) - \bar{v}_s(x)||\mathcal{F}_t)|^q$$

$$\le \frac{q}{q-1}(P(H^c))^{1/r} E^{1/r'}(\sup_{t\in[0,T]} \sup_x |v_t(x) - \bar{v}_t(x)|^{qr'}) \qquad (10)$$

$$\le \frac{2^{q-1}q}{q-1}(P(H))^{1/r} V_T \qquad (11)$$

with

$$V_T := E^{1/r'} \sup_{t\in[0,T]} \sup_x |v_t(x)|^{qr'} + E^{1/r'} \sup_{t\in[0,T]} \sup_x |\bar{v}_t(x)|^{qr'}$$

for $r > 1$, $r' = r/(r-1)$. □

3. Spatial Finite Difference Schemes for the Zakai equation

Lattice: $\mathbb{G}_h = h\mathbb{Z}^d$, $h > 0$

$$du^h_t(x) = L^h_t u^h(x)\, dt + M^{r,h}_t u^h_t(x)\, dY^r_t, \quad u^h_0(x) = \pi_0(x), \quad (12)$$

for $t \in [0, T]$, $x \in \mathbb{G}_h$, where

$$L^h_t \sim L^*_t, \quad M^{h,r}_t \sim M^{r*}_t, \quad D_i \sim \delta^h_i,$$

$$\delta_i u(x) = (u(x + he_i) - u(x - he_i))/(2h)$$

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Theorem 3. If b, B, θ, ρ have bounded derivatives in x up to sufficiently high order, and $E|\pi_0|^p_{W_2^m} < \infty$ for some $p > 0$ and sufficiently large m, then

$$E \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |u_t^h(x) - u_t(x)|^p \leq N h^{2p} E|\pi_0|^p_{W_2^m}.$$

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Remark 1. (2) is an infinite system of SDEs

Discretise further in time: $\tau := T/n$,

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$x \in \mathbb{G}_h$, where $\xi_i := Y_{t_i} - Y_{t_{i-1}}$.

This is an infinite system of equations.
For sufficiently $h > 0$ it has a unique solution $(v_i^{h,\tau})_{i=0}^n$.

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For sufficiently $h > 0$ it has a unique solution $(v_i^{h,\tau})_{i=0}^n$.

Theorem 4. (Gerencsér-I.G. 2017) If b, θ, ρ, B have bounded derivatives in x up to sufficiently high order, and $E|\pi_0|^2_{W_2^m} < \infty$ for sufficiently large m, then

$$E \max_{0 \le i \le n} \sup_{x \in \mathbb{G}_h} |u_{t_i}(x) - v_i^{h,\tau}(x)|^2 \le N(\tau + h^4) E(|\pi_0|^2_{W_2^m} + 1).$$

Clearly if $B(x, y) = 0$, $b(x, y) = 0$, $\theta(x, y) = 0$, $\rho(x, y) = 0$ and $\pi_0(x) = 0$ for $|x| \geq r$ for some $r > 0$, then (3) is a finite system.

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This suggests truncating: For $R > 0$ set

$$(B^R, b^R, \theta^R, \rho^R, \pi_0^R) := \zeta_R(B, b, \theta, \rho, \pi_0),$$

where $\zeta_R = \zeta_R(x)$, $x \in \mathbb{R}^d$, is a sufficiently smooth function with compact support such that $\zeta_R = 1$ for $|x| \leq R$.

Truncated schemes

$v_0 := \pi^R,$

$$v_i(x) = v_{i-1}(x) + L^{h,R}_{t_i} v_i(x)\tau + M^{h,k,R}_{t_{i-1}} v_{i-1}(x)\xi^r_i, \quad i = 1, 2, ..., n, \tag{14}$$

For sufficiently small $\tau > 0$ for all h and R it has a unique solution $v_i = v_i^{h,R}$, $i = 0, 1, ..., n$.
Set $\mathbb{G}^R_h := \mathbb{G}_h \cap \operatorname{supp} \zeta$.

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For sufficiently small $\tau > 0$ for all h and R it has a unique solution $v_i = v_i^{h,R}$, $i = 0, 1, ..., n$.
Set $\mathbb{G}_h^R := \mathbb{G}_h \cap \operatorname{supp} \zeta$.

Theorem 5. (Gerencsér-I.G. 2017) Under the conditions of Thm 4, for every $R > 0$ and $\nu \in (0, 1)$ there are constants $\delta > 0$ and N such that

$$E \max_{0 \le i \le n} \max_{x \in \mathbb{G}_h^{\nu R}} |u_{t_i}(x) - v_i^{h,\tau,R}(x)|^2 \le N(e^{-\delta R^2} + \tau + h^4)(E|\pi_0|^2_{W_2^m} + 1).$$

Accelerated schemes

For an integer $k \geq 1$ set $\tilde{k} := \lfloor k/2 \rfloor$

$$\tilde{v}^{h,\tau,R} := \sum_{j=0}^{\lfloor k/2 \rfloor} c_j v^{h/2^j,\tau,R},$$

$$(c_0, ..., c_{\tilde{k}}) = (1, 0, ..., 0)V^{-1}, \quad \tilde{V}^{ij} = 4^{-(i-1)(j-1)}, \quad i, j = 1, ..., \tilde{k}+1.$$

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Theorem 6. (L. Gerencsér-I.G.) Let $k \geq 0$ be an integer. If b, θ, ρ, B have bounded derivatives in x up to sufficiently high order, and $E|\pi_0|^2_{W_2^m} < \infty$ for sufficiently large m, then for $\nu \in (0, 1)$, $R > 0$ we have constants $\delta > 0$, N such that

$$E \max_{0 \leq i \leq n} \max_{x \in \mathbb{G}_h^{\nu R}} |u_{t_i}(x) - v_i^{h,\tau,R}(x)|^2 \leq N(e^{-\delta R^2} + \tau + h^{2k+2})(E|\pi_0|^2_{W_2^m} + 1).$$

Sketch of proof: We write

$$|u_{\tau i}(x) - \bar{v}_i^{h,R,\tau}(x)| \leq |u_{\tau i}(x) - u_{\tau i}^{0,R}(x)| + |u_{\tau i}^{0,R}(x) - \bar{u}_{\tau i}^{h,R}(x)| + \sum_{j=0}^{r} c_j |u_{\tau i}^{h/2^j,R}(x) - v_i^{h/2^j,R,\tau}(x)|,$$

where $u^{0,R}$ denotes the solution of the truncated Zakai equation and $\bar{u}^{h,R} = \sum_{j=0}^{r} c_j u^{h/2^j,R}$.

The first term is estimated by Theorem 2 on localisation error, the second by a theorem from (I.G. 2015) on accelerated finite difference schemes, and for each term in the sum we prove

$$E \max_{0\leq i\leq n} \max_{x\in\mathbb{G}_h} |u_{\tau i}^{h',R} - v_i^{h',R,\tau}|^2 \leq N\tau(1 + E|\psi_m|^2_{W_2^m})$$

with $h' = h/2^j$. $\square$.

Conclusion

- The truncation error for parabolic (possibly degenerate) PDEs and SPDEs is exponentially small.
- The finite different schemes for the truncated systems are fully implementable. We estimated their error independently of the truncation.
- We have shown that the error coming form the space discretisation can be made as small as we wish by Richardson's extrapolation, provided the data are sufficiently smooth.

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