Euler-Maruyama approximation and Greeks

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$(\Omega, \mathcal{F}, P)$ Probability Space

$B(t) = (B^1(t), \ldots, B^d(t))$, $t \geq 0$,

d-dimensinal Standard Wiener Process

$B^0(t) = t$,

$N, M \geqq 1$

$V_k \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M; \mathbf{R}^N)$, $i = 0, 1, \ldots, d$,

Stratonovich type SDE on $\mathbf{R}^N$ with M-dimensinal parameters

$$dX^i(t,x,\theta) = \sum_{k=0}^{d} V_k(X(t,x,\theta),\theta) \circ dB^i(t)$$

$$X(0,x,\theta) = x \in \mathbf{R}^N, \qquad \theta \in \mathbf{R}^M$$

Problems

$f : \mathbf{R}^N \to \mathbf{R}$

(1) Compute $E[f(X(T, x_0, \theta))]$, $\theta \in \mathbf{R}^M$, numerically

to find a good θ

(2) Compute numeriaclly

$$\frac{\partial^{|\alpha|}}{\partial \theta^\alpha} E[f(X(T, x_0, \theta_0))]$$

for $\alpha \in \mathbf{Z}^M_{\geqq 0}$, $x_0 \in \mathbf{R}^N$, $\theta_0 \in \mathbf{R}^M$

Greeks

(Euler-Maruyama Approximation)

$b \in C_b^{\infty}(\mathbf{R}^N \times \mathbf{R}^M; \mathbf{R}^N)$

$$b^i(x,\theta) = \frac{1}{2}\sum_{k=1}^{d}\sum_{j=1}^{N} V_k^j(x,\theta)\frac{\partial V_k^i}{\partial x^j}(x,\theta), \qquad i = 1,\ldots,N,\ x \in \mathbf{R}^N,\ \theta \in \mathbf{R}^M$$

Ito type SDE

$$X(t,x,\theta) = x + \sum_{k=1}^{d}\int_0^t V_k(X(s,x;\theta),\theta)dB^k(s) + \int_0^t b(X(s,x;\theta))ds$$

$$X_h : [0,\infty) \times \mathbf{R}^N \times \mathbf{R}^M \times \Omega \to \mathbf{R}^N,\ h > 0,$$

$$X_h(0,x,\theta) = x, \qquad x \in \mathbf{R}^N,\ \theta \in \mathbf{R}^M$$

$$X_h(t,x,\theta)$$

$$= X_h((n-1)h,x,\theta) + \sum_{k=1}^{d} V_k(X_h((n-1)h,x,\theta),\theta)(B^k(t) - B^k((n-1)h))$$

$$+ b(X_h((n-1)h,x,\theta),\theta)(t-(n-1)h),$$

$$t \in ((n-1)h, nh],\ n = 1,2,\ldots,\ x \in \mathbf{R}^N,\ \theta \in \mathbf{R}^M.$$

X_h : Euler-Maruyama Approximation

Theorem 1 (Maruyama) $\forall p \in [2,\infty)$ $\forall T > 0$ $\exists C \in (0,\infty)$

$$E[\sup_{t\in[0,T]} |X(t,x) - X_h(t,x)|^p]^{1/p} \leqq Ch^{1/2}, \qquad h \in (0,1],\ x \in \mathbf{R}^N$$

(1) Computation of $E[f(X(T,x_0,\theta)]$

$$E[f(X_{T/n}(T,x_0,\theta))],\ n \geqq 1$$

$$X_{T/n}(T,x_0,\theta)) = \tilde{X}_n(\theta, \{B(kT/n) - B((k-1)T/n);\ k=1,\dots,n\})$$

function of $\theta \in \mathbf{R}^M$, and

$\{B(kT/n) - B((k-1)T/n);\ k=1,\dots,n\} \in \mathbf{R}^{dn}$

$$\hat{E}_n(\theta) = \frac{1}{L}\sum_{\ell=1}^{L} f(\tilde{X}_n(\theta, (T/n)^{1/2} Z_\ell))$$

$Z_1, Z_2, \ldots$ i.i.d. $N(0, I_{dn})$-distributed ranndom variables

$\hat{E}_{n,L}(\theta)$: estimater of $E[f(X_{T/n}(T, x_0, \theta))]$

$$\frac{1}{(LVar(X_{T/n}(T, x_0, \theta))^{1/2}}(\hat{E}_{n,L}(\theta) - E[f(X_{T/N}(T, x_0, \theta)])$$

$$\to N(0,1) \text{ in law}$$

$$|\hat{E}_{n,L}(\theta) - E[f(X_{T/N}(T, x_0, \theta)]| \sim L^{-1/2} Var(X_{T/n}(T, x_0, \theta)^{1/2}$$

Let $e_j = (0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0)) \in \mathbf{R}^M$, $j = 1, \ldots, M$.

For any $f : \mathbf{R}^M \to \mathbf{R}$, we define $\partial_{\varepsilon,i} f : \mathbf{R}^M \to \mathbf{R}$, by

$$\partial_{\varepsilon,i} f(\theta) = \frac{1}{\varepsilon}(f(\theta + \varepsilon e_i) - f(\theta))$$

(2-1) Computation of $\dfrac{\partial}{\partial \theta^j} E[f(X(T, x_0, \theta_0)]$

$$\Longrightarrow \qquad (\partial_{\varepsilon,i} \hat{E}_{n,L})(\theta_0)$$

(2-2) Computation of $\dfrac{\partial^2}{\partial \theta^i \partial \theta^j} E[f(X(T, x_0, \theta_0)]$

$$\Longrightarrow \qquad (\partial_{\varepsilon,i} \partial_{\varepsilon,j} \hat{E}_{n,L})(\theta_0)$$

How accurate is it?

$V_{[\alpha]} \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M; \mathbf{R}^N)$, $\alpha = (k_1, k_2, \ldots, k_n) \in \{0, 1, \ldots, d\}^n$, $n \geqq 1$,

$$V_{[k]}(x, \theta) = V_k(x, \theta), \qquad k = 0, 1, \ldots, d,$$

$$V_{[(k_1, k_2, \ldots, k_n, k_{n+1})]}(x, \theta) = [V_{k_1}, V_{(k_2, \ldots, k_n, k_{n+1})}](x, \theta)$$

$$= \sum_{i=1}^N V_{k_1}^i(x, \theta) \frac{\partial V_{[(k_2, \ldots, k_n, k_{n+1})]}}{\partial x^i}(x, \theta) - \sum_{i=1}^N V_{[(k_2, \ldots, k_n, k_{n+1})]}^i(x, \theta) \frac{\partial V_{k_1}}{\partial x^i}(x, \theta)$$

$k_1, \ldots, k_{n+1} = 0, 1, \ldots, d$, $n \geqq 1$.

$A'_\ell = \bigcup_{j=1}^\ell \{0, 1, \ldots, d\}^j \setminus \{0\}$, $\ell \geqq 1$

(UH) (Uniform Hörmander Condition)

There is an $\ell_0 \geqq 1$ and $c_0 > 0$ such that

$$\sum_{\alpha \in A'_{\ell_0}} (V_{[\alpha]}(x, \theta), \xi)_{\mathbf{R}^N}^2 \geqq c_0 |\xi|^2 \qquad x, \xi \in \mathbf{R}^N,\ \theta \in \mathbf{R}^M$$

Known result

Theorem 2 Assume that $\{V_0, V_1, \ldots, V_d\}$ satisfies (UH)

Then for any $T > 0$ there exists a $C \in (0, \infty)$ such that

$$\sup_{x \in \mathbf{R}^N, \theta \in \mathbf{R}^N} |E[f(X(T,x,\theta))] - E[f(X_{T/n}(T,x,\theta))]| \leqq \frac{C}{n} ||f||_{L^\infty}$$

for any $n \geqq 1$, $f \in C_b^\infty(\mathbf{R}^N)$

$$\sup_{n \geqq 1} E[f(X_{T/n}(T, x_0, \theta_0))^2] < \infty.$$

$$|\hat{E}_{n,L}(\theta_0) - E[f(X_{T/n}(T, x_0, \theta_0)]| = O(L^{-1/2})$$

$$|\hat{E}_{n,L}(\theta) - E[f(X(T, x_0, \theta)]| = O(L^{-1/2} + n^{-1})$$

Theorem 3 Assume that $\{V_0, V_1, \ldots, V_d\}$ satisfies (UH).

Then for any $T > 0$, $\gamma > 0$, and $i_1, i_2, \ldots, i_m = 1, \ldots, M$, $m \geqq 1$, there is a $C \in (0, \infty)$ such that

$$\sup_{x \in \mathbf{R}^N, \theta \in \mathbf{R}^M} |\frac{\partial^m}{\partial \theta^{i_1} \cdots \partial \theta^{i_m}} E[f(X(T, x, \theta))]$$

$$-\partial_{\varepsilon, i_1} \cdots \partial_{\varepsilon, i_m} E[f(X_{T/n}(T, x, \cdot)(\theta)]|$$

$$\leqq C(\varepsilon + \frac{1}{n} + \frac{1}{\varepsilon^m n^\gamma})||f||_{L^\infty}$$

for any $n \geqq 1$, $\varepsilon \in (0, 1]$ and $f \in C_b^\infty(\mathbf{R}^N)$.

If $f : \mathbf{R}^N \to \mathbf{R}$ is Lipschtz continuous, then

$$\sup_{\varepsilon\in(0,1],n\geqq 1} E[(\partial_{\varepsilon,i}(f(X_{T/n}(T,x,\cdot)(\theta))^2] < \infty.$$

$$|\frac{\partial}{\partial\theta^i}E[f(X(T,x,\theta))] - \frac{1}{n^{-m}}(\hat{E}_{n,L}(\theta+n^{-m}e_i) - \hat{E}_{n,L}(\theta))| = O(L^{-1/2}+n^{-1}).$$

if $m \geqq 1$. However,

$$\sup_{n\geqq 1} E[(\partial_{\varepsilon,i}\partial_{\varepsilon,j}(f(X_{T/n}(T,x,\cdot)(\theta))^2] = O(\varepsilon^{-2})$$

We can only show that

$$|\frac{\partial}{\partial\theta^i}\frac{\partial}{\partial\theta^j}E[f(X(T,x,\theta))] - \partial_{\varepsilon,i}\partial_{\varepsilon,j}\hat{E}_{n,L}(\theta)|$$

$$= O(\frac{1}{\varepsilon L^{1/2}} + n^{-1} + \frac{1}{n^\gamma \varepsilon})$$

Idea of Proof

Let

$$M(T,x,\theta) = (M^{ij}(T,x,\theta))_{i,j=1,\dots,N}$$

$$= ((DX^i(T,x,\theta), DX^j(T,x,\theta)))_{i,j=1,\dots,N},$$

and

$$M_h(T,x,\theta) = (M_h^{ij}(T,x,\theta))_{i,j=1,\dots,N}$$

$$= ((DX_h^i(T,x,\theta), DX_h^j(T,x,\theta)))_{i,j=1,\dots,N}.$$

Malliavin Covariance

Under (UH)

$$\sup_{t\in[T_0,T_1],x\in\mathbf{R}^N,\theta\in\mathbf{R}^M} E[(\det M(t,x,\theta)^{-p}] < \infty$$

for any $p \in (1,\infty)$, and $T_1 > T_0 > 0$.

$$(P_t^\theta f)(x) = E[f(X(t,x;\theta))]$$

$$(Q_{(h)}^\theta(t)f)(x) = E[f(X_h(t,x;\theta))]$$

$t \geqq 0$, $x \in \mathbf{R}^N$, $\theta \in \mathbf{R}^M$, $f \in C_b^\infty(\mathbf{R}^N)$.

$$\rho_h(t,x;\theta) = detM(t,x;\theta)^{-1} detM_h(t,x;\theta)$$

$\varphi_0 \in C_0^\infty(\mathbf{R})$ such that $0 \leqq \varphi_0 \leqq 1$, $\varphi_0(s) = 1$, $s \in (2/3, 4/3)$, and $\varphi_0(s) = 0$, $s \in (-\infty, 1/3)$.

$$(Q_{(h,0)}^\theta(t)f)(x) = E[\varphi_0(\rho_h(t,x,\theta))f(X_h(t,x,\theta))]$$

$$(Q_{(h,1)}^\theta(t)f)(x) = E[(1-\varphi_0(\rho_h(t,x,\theta)))f(X_h(t,x,\theta))]$$

$t, h > 0$, $x \in \mathbf{R}^N$, $\theta \in \mathbf{R}^M$, $f \in C_b^\infty(\mathbf{R}^N)$.

Let

$$J_h(t,x;\theta) = (\frac{\partial}{\partial x^i} X_h^j(t,x;\theta))_{i,j=1,\ldots,N}$$

$$J_h(t,x;\theta) = I_N + \sum_{k=0}^{d} \nabla_x \sigma_k(x;\theta) B^k(t), \quad t \in [0,h]$$

is not invertible in general

$$(\hat{Q}^{\theta}_{(h)} f)(x) = E[f(X(h \wedge \tau, x, \theta)]$$

$x \in \mathbf{R}^N$, $\theta \in \mathbf{R}^M$, $f \in C_b^\infty(\mathbf{R}^N)$.

Here

$$\tau = \inf\{t > 0;\ |B(t)| + t > \varepsilon_0\}$$

$$\varepsilon_0 = \frac{1}{2}(1 + \sum_{k=1}^{d} \sum_{i=1}^{N} ||\frac{\partial}{\partial x^i} \sigma_k||_\infty)^{-1}$$

$$det M_h(t,x;\theta)^{-1}\varphi_0(\rho_h(t,x))$$

$$= det M(t,x;\theta)^{-1}\varphi_1(\rho_h(t,x))$$

$\varphi_1(z) = z^{-1}\varphi_0(z) \in C_0^\infty(\mathbf{R})$

For any $T_1 > T_0 > 0$, $\alpha, \beta \in \mathbf{Z}^N_{\geqq 0}$, $\gamma \in \mathbf{Z}^M_{\geqq 0}$, there is a $C \in (0,\infty)$ such that

$$\sup_{t\in[T_0,T_1]} ||\frac{\partial^\gamma}{\partial\theta^\gamma}\frac{\partial^\alpha}{\partial x^\alpha}\tilde{Q}^\theta_{(h,0)}(t)\frac{\partial^\beta f}{\partial x^\beta}||_\infty \leqq C||f||_\infty,$$

$$\theta \in \mathbf{R}^M,\ f \in C_b^\infty(\mathbf{R}^N),\ h \in (0,1]$$

$$E[f(X(T,x;\theta)] - E[f(X_h(t,x;\theta)] = (P_T^\theta f)(x) - (Q^\theta_{(T/n)}(T)f)(x)$$

$$= \sum_{k=1}^{n} (Q^\theta_{(T/n)}((k-1)T/n)(P^\theta_{T/n} - Q^\theta_{(T/n)}(T/n))P^\theta_{(n-k)T/n} f)(x)$$

$$= (R^\theta_{n,T,0} f)(x)) + (R^\theta_{n,T,1} f)(x)$$

$$R^\theta_{n,T,0}$$

$$= \sum_{k=1}^{[n/2]} Q^\theta_{(T/n)}((k-1)T/n)(P^\theta_{T/n} - Q^\theta_{T/n}(T/n))P^\theta_{(n-k)T/n}$$

$$+ \sum_{k=[n/2]+1}^{n} Q^\theta_{(T/n,0)}((k-1)T/n)(P^\theta_{T/n} - \hat{Q}^\theta_{T/n})P^\theta_{(n-k)T/n}$$

For any $\alpha \in \mathbf{Z}^M_{\geqq 0}$, there is a $C \in (0,\infty)$ such that

$$\sup_{x\in\mathbf{R}^N,\ \theta\in\mathbf{R}^M} |\frac{\partial^\alpha}{\partial\theta^\alpha}(R^\theta_{n,T,0}f)(x)| \leqq \frac{C}{n}||f||_\infty$$

for any $n \geqq 1$ and $f \in C^\infty_b(\mathbf{R}^N)$.

For any $\gamma > 0$, there is a $C \in (0,\infty)$ such that

$$\sup_{x\in\mathbf{R}^N,\ \theta\in\mathbf{R}^M} |(R^\theta_{n,T,1}f)(x)| \leqq \frac{C}{n^\gamma}||f||_\infty$$

for any $n \geqq 1$ and $f \in C^\infty_b(\mathbf{R}^N)$.

$$
\begin{aligned}
&|\frac{\partial^m}{\partial\theta^{i_1}\cdots\partial\theta^{i_m}}f(\theta)-\partial_{\varepsilon,i_1}\cdots\partial_{\varepsilon,i_m}f(\theta)| \\
&=|\int_0^1\cdots\int_0^1 \frac{\partial^m}{\partial\theta^{i_1}\cdots\partial\theta^{i_m}}f(\theta) \\
&\quad -\frac{\partial^m f}{\partial\theta^{i_1}\cdots\partial\theta^{i_m}}theta+t_1\varepsilon e_{i_1}+\cdots t_m\varepsilon e_{i_m})dt_1\cdots dt_m| \\
&\leqq m\varepsilon\sum_{\alpha\in\mathbf{Z}_{\geqq 0},|\alpha|=m+1}||\frac{\partial^\alpha f}{\partial\theta^\alpha}||_\infty
\end{aligned}
$$

Ninomiya-Victoir,, Ninomiya-Minomiya approximation

(special KLNV method, Gaussian K-scheme)

Modified UFG condition

(1) There are an ℓ_0,

and $\varphi_{\alpha,\beta} \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M; \mathbf{R}^N)$, $\alpha \in A'_{\ell_0+1}$, $\beta \in A'_{\ell_0}$, such that

$$V_{[\alpha]}(x,\theta) = \sum_{\beta \in A'_{\ell_0}} \varphi_{\alpha,\beta}(x,\theta) V_{[\beta]}(x,\theta), \qquad \alpha \in A'_{\ell_0+1}.$$

(2) There are an ℓ_1, and $\psi_{k,j,\beta} \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M; \mathbf{R}^N)$, $k = 0, 1, \ldots, d$, $j = 1, \ldots, M$, $\beta \in A'_{\ell_0}$, such that

$$\frac{\partial}{\partial \theta^j} V_k(x,\theta) = \sum_{\beta \in A'_{\ell_0}} \varphi_{k,j,\beta}(x,\theta) V_{[\beta]}(x,\theta), \qquad k = 0, \ldots, d, j = 1, \ldots, M$$