Stochastic heat equation driven by a rough time-fractional noise

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[1] Joint work with Le Chen, Yaozhong Hu and Kamran Kalbasi

Parabolic Anderson model

Consider the stochastic heat equation on $\mathbb{R}^d$:

$$\boxed{\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial W}{\partial t}} \qquad (1)$$

with initial condition u_0.

- $W = \{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a centered Gaussian family with covariance

$$E[W(t,x)W(s,y)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) Q(x,y),$$

where $H \in (0, \frac{1}{2})$ and Q is a covariance function satisfying:

(H1) There exist constants $C_0 > 0$ and $\alpha \in (0,1]$, such that

$$Q(x,x) + Q(y,y) - 2Q(x,y) \leq C_0|x-y|^{2\alpha}$$

for all $x, y \in \mathbb{R}^d$.

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Example

If $d = 1$, the covariance of a fractional Brownian motion with Hurst parameter $H_0 \in (0, 1)$:

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satisfies (H1) with $\alpha = H_0$.

- More generally, (H1) is equivalent to saying that if $\{Y(x), x \in \mathbb{R}^d\}$ is a Gaussian centered process with covariance Q, then

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Nonlinear integration with respect to W

If $\phi : [0, T] \to \mathbb{R}$ is continuous, we define

$$\int_0^t W(ds, \phi_s) := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t (W(s+\epsilon, \phi_s) - W(s-\epsilon, \phi_s))\, ds,$$

if the limit exists in $L^2(\Omega)$.

Theorem

Assume that Q satisfies (H1). Then, if $\phi \in C^\kappa([0, T])$ with $\alpha\kappa + H > \frac{1}{2}$, the stochastic integral $I_t(\phi) := \int_0^t W(ds, \phi_s)$ exists for any $t \in [0, T]$ and

$$\begin{aligned} E\left[I_t(\phi)^2\right] &= H \int_0^t \theta^{2H-1} \left[Q(\phi_\theta, \phi_\theta) + Q(\phi_{t-\theta}, \phi_{t-\theta})\right] d\theta \\ &- \frac{\alpha_H}{2} \int_0^t \int_0^\theta r^{2H-2} \left[Q(\phi_\theta, \phi_\theta) + Q(\phi_{\theta-r}, \phi_{\theta-r}) - 2Q(\phi_\theta, \phi_{\theta-r})\right] d\theta, \end{aligned}$$

where $\alpha_H = H(2H-1) < 0$.

Remarks

- In the semimartingale context, nonlinear stochastic integrals defined as limit of forward Riemann sums:

$$\int_0^t X(ds, u_s) = \lim_{n\to\infty} \sum_{i=0}^{n-1} [X(t_{i+1}, u_{t_i}) - X(t_i, u_{t_i})],$$

  where studied by Kunita '90 and Carmona-N. '90.

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Hölder continuity of the stochastic integral

Proposition

For any $s, t \in [0, T]$, *we have:*

$$E[(I_t(\phi) - I_s(\phi))^2] \leq C'(1 + \|\phi\|_\infty)^{2\alpha}|t - s|^{2H} + C''\|\phi\|_\kappa^{2\alpha}|t - s|^{2(H+\alpha\kappa)}.$$

As a consequence, the process $\{I_t(\phi), t \in [0, T]\}$ *has a version with* $(H - \epsilon)$*-Hölder continuous trajectories for any* $\epsilon > 0$.

Proof:

- Upper bound for the variance:

$$E\left[I_t(\phi)^2\right] \leq 2H\int_0^t \theta^{2H-1} Q(\phi_\theta,\phi_\theta)d\theta + \frac{|\alpha_H|C_0}{2}\int_0^t\int_0^\theta r^{2H-2}|\phi_\theta-\phi_{\theta-r}|^{2\alpha}drd\theta.$$

- Taking into account that $Q(x,x)\leq C_1(1+|x|^{2\alpha})$, we get

$$E\left[I_t(\phi)^2\right] \leq C_1\|\phi\|_\infty t^{2H} + \frac{|\alpha_H|C_0}{2(2H+2\alpha\kappa-1)(2H+2\alpha\kappa)}\|\phi\|_\kappa^{2\alpha} t^{2H+2\alpha\kappa}.$$

Feynman-Kac formula

The solution to equation (1) should have the representation

$$u(t,x) = E^B\left[u_0(B_t^x)\exp\left(\int_0^t W(ds, B_{t-s}^x)\right)\right],$$

where B^x is a d-dimensional Brownian motion independent of W, starting from x.

- Notice that the integral $\int_0^t W(ds, B_{t-s}^x)$ is well defined provided

$$\boxed{\alpha + 2H > 1}$$

Theorem

Suppose that Q satisfies (H1) with $2H + \alpha > 1$ *and* u_0 *is bounded. Then for all* $t > 0$ *and* $x \in \mathbb{R}^d$, *the random variable* $\int_0^t W(ds, B^x_{t-s})$ *is exponentially integrable and* $u(t,x) \in L^p(\Omega)$ *for all* $p \geq 1$. *Moreover, for some constants* $C = C(d, H, \alpha, \|u_0\|_\infty) > 0$ *and* $C_x = C_x(d, H, \alpha, \|u_0\|_\infty, x) > 0$,

$$E\left[|u(t,x)|^k\right] \leq C_x \exp\left(Ck^{\frac{2-\alpha}{1-\alpha}}\, t^{\frac{2H+\alpha}{1-\alpha}}\right)$$

for all $t \geq 1$ *and* $x \in \mathbb{R}^d$.

Sketch of the proof: Suppose $u_0 = 1$ and $t \geq 1$.

$$\begin{aligned}
E[u(t,x)^k] &= E^W E^B \exp\left\{ \sum_{j=1}^{k} \int_0^t W(ds, B_{t-s}^{j,x}) \right\} \\
&= E^B \exp\left\{ \frac{1}{2} \sum_{i,j=1}^{k} E^W \left[\int_0^t W(ds, B_{t-s}^{i,x}) \int_0^t W(ds, B_{t-s}^{j,x}) \right] \right\} \\
&\leq E^B \exp\left\{ 2k \sum_{i=1}^{k} E^W \left[\left(\int_0^t W(ds, B_{t-s}^{i,x}) \right)^2 \right] \right\} \\
&= \left[E^B \exp\left\{ 2k E^W \left[\left(\int_0^t W(ds, B_{t-s}^{1,x}) \right)^2 \right] \right\} \right]^k .
\end{aligned}$$

Therefore

$$\begin{aligned}\|u(t,x)\|_k &\leq E^B \exp\left\{kC_0|\alpha_H|\int_0^t\int_0^t |B_u-B_v|^{2\alpha}|u-v|^{2H-2}dudv\right.\\ &\quad \left. +4kHC_1t^{2H}\left(1+|x|+\sup_{0\leq s\leq t}|B_s|\right)^{2\alpha}\right\}\\ &\leq \left\{E[I_1^2]E[I_2^2]\right\}^{1/2},\end{aligned}$$

where

$$I_1=\exp\left\{2C_0|\alpha_H|kt^{2H+\alpha}\int_0^1\int_0^1 |B_u^1-B_v^1|^{2\alpha}|u-v|^{2H-2}dudv\right\}$$

and

$$I_2=\exp\left\{8kHC_1t^{2H}\left(1+|x|+\sup_{0\leq s\leq t}|B_s|\right)^{2\alpha}\right\}.$$

- Set

$$U = \int_0^1 |B_u^1 - B_v^1|^{2\alpha}|u-v|^{2H-2}du.$$

We have

$$E^B \exp\left(Ckt^{2H+\alpha}U\right) \le \exp\left(\left(C'k^{\frac{1}{1-\alpha}}t^{\frac{2H+\alpha}{1-\alpha}}\right)\right).$$

This follows from

$$\lambda U \le (1-\alpha)\left(\frac{\lambda}{\delta}\right)^{\frac{1}{1-\alpha}} + \alpha(\delta U)^{\frac{1}{\alpha}}$$

and by Fernique's theorem:

$$E[\exp(\epsilon U^{\frac{1}{\alpha}})] < \infty$$

for ϵ small enough.

- Set
$$V_t = 1 + |x| + \sup_{0\le s\le t} |B_s|.$$
Then, using $t \ge 1$ and scaling,
$$\begin{aligned} E^B \exp\left(Ckt^{2H} V_t^{2\alpha}\right) &\le \exp\left(C' kt^{2H+\alpha}|x|^{2\alpha}\right) \\ &\quad \times E^B \exp\left(C'' kt^{2H+\alpha} \left(1 + \sup_{x\in[0,1]} |B_t|\right)^{2\alpha}\right) \\ &\le C_4 \exp\left(C' kt^{2H+\alpha}|x|^{2\alpha}\right) \exp\left(C''' k^{\frac{1}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right). \end{aligned}$$
Finally,
$$\exp\left(C' kt^{2H+\alpha}|x|^{2\alpha}\right) \le C'_{\alpha,d,x} \exp\left(C''_{\alpha,d} k^{\frac{1}{1-\alpha}} t^{\frac{2H+\alpha}{1-\alpha}}\right).$$

Definition

Given a random field $v = \{v(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that

$$\int_0^t \int_{\mathbb{R}^d} |v(s,x)| dxds < \infty \quad \text{a.s. for all } t > 0,$$

the *Stratonovich integral* is defined as the following limit in probability if it exists

$$\int_0^t \int_{\mathbb{R}^d} v(s,x) W(ds,dx)$$
$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \int_{\mathbb{R}^d} v(s,x)[W(s+\epsilon,x) - W(s,x)] dsdx.$$

Definition

$u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a *weak solution* to (1) if for any $\phi \in C_0^\infty(\mathbb{R}^d)$ and for all $t \geq 0$,

$$\int_{\mathbb{R}^d} [u(t,x) - u_0(x)]\,\phi(x)dx = \int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\phi(x)dxds + \int_0^t \int_{\mathbb{R}^d} u(s,x)\phi(x)W(ds,x)dx.$$

Theorem

Suppose that Q satisfies condition (H1) with $2H + \alpha > 1$ and u_0 is bounded. Let $u(t,x)$ be the random field defined by the Feynman-Kac formula. Then for any $\phi \in C_0^\infty(\mathbb{R}^d)$, $u(t,x)\phi(x)$ is Stratonovich integrable and $u(t,x)$ is a weak solution to (1).

- The case $H \in (\frac{1}{4}, \frac{1}{2})$ was proved by Hu, Lu and N. in 2012 under different conditions on Q.

Lower bounds of moments

Theorem

Suppose that:

(i) *Q satisfies (H1) with $2H+\alpha>1$.*

(ii) $\inf_x u_0(x)>0$.

(iii) $Q(x,y)\geq C_2M^{2\beta}$, *if* $\min_{1\leq i\leq d}(|x_i|\wedge|y_i|)\geq M$, $1\leq i\leq d$ *for some* $\beta\in[0,1)$.

Then there exist some constants $C=C(d,H,\alpha,\beta,\inf_{x\in\mathbb{R}^d}u_0(x))>0$ *and* $C_x=C_x(d,H,\alpha,\beta,\inf_{x\in\mathbb{R}^d}u_0(x),x)>0$, *such that for all* $t\geq 1$, $x\in\mathbb{R}^d$ *and* $k\geq 1$,

$$E\left[u(t,x)^k\right]\geq C_x\exp\left(Ck^{\frac{2-\beta}{1-\beta}}t^{\frac{2H+\beta}{1-\beta}}\right).$$

Remark: The fractional Brownian motion satisfies (iii) with $\beta=\alpha=H_0$.

Sketch of the proof:

- Suppose $u_0 = 1$. Then

$$E\left[u(t,x)^k\right] = E^B \exp\left(E^{Y,\widehat{B}}\left|\int_0^t \sum_{i=1}^k Y(B_{t-s}^{i,x})d\widehat{B}_s\right|^2\right),$$

where $Y = \{Y(x), x \in \mathbb{R}^d\}$ is a centered Gaussian process with covariance Q and $\widehat{B}$ is a fractional Brownian motion with Hurst parameter H, and B, Y and $\widehat{B}$ are independent.

- Then, we use the inequality (Memin-Mishura-Valkeila '01)

$$\begin{aligned} E^{Y,\widehat{B}}\left|\int_0^t \sum_{i=1}^k Y\left(B_{t-s}^{i,x}\right) d\widehat{B}_s\right|^2 &\geq C_H\left(\int_0^t \left[E^Y\left|\sum_{i=1}^k Y\left(B_s^{i,x}\right)\right|^2\right]^{\frac{1}{2H}} ds\right)^{2H} \\ &= C_H\left(\int_0^t \left[\sum_{i,j=1}^k Q\left(B_s^{i,x}, B_s^{j,x}\right)\right]^{\frac{1}{2H}} ds\right)^{2H}. \end{aligned}$$

- This leads to

$$
\begin{aligned}
E\left[u(t,x)^k\right] &\geq P(\min_{s\in[\frac{t}{2},t]} |B_s^1 + x| > M)^{kd} e^{C_H k^2 M^{2\beta} t^{2H}} \\
&\geq 2^{-(kd+1)} \exp\left(C_H k^2 M^{2\beta} t^{2H} - \frac{16kM^2}{t}\right).
\end{aligned}
$$

The proof is concluded optimizing over M.

Lyapunov exponents

When $\alpha = \beta$ and u_0 is bounded away from 0 and ∞, one can define the *moment Lyapunov exponents*

$$\overline{m}_k(x) := \limsup_{t \to +\infty} t^{-\frac{2H+\alpha}{1-\alpha}} \log E\left[u(t,x)^k\right]$$

and

$$\underline{m}_k(x) := \liminf_{t \to +\infty} t^{-\frac{2H+\alpha}{1-\alpha}} \log E\left[u(t,x)^k\right],$$

and establish that for all $k \geq 2$,

$$\underline{C} k^{\frac{2-\alpha}{1-\alpha}} \leq \inf_{x \in \mathbb{R}^d} \underline{m}_k(x) \leq \sup_{x \in \mathbb{R}^d} \overline{m}_k(x) \leq \overline{C} k^{\frac{2-\alpha}{1-\alpha}}.$$

Therefore, this solution is *fully intermittent*.

Related results

1. Consider the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d},$$

where the covariance of the noise $\xi(t,x) = \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}$ is given by

$$E(\xi(t,x)\xi(s,y)) = \gamma(t-s)\Lambda(x-y),$$

where $\gamma(t) \sim |t|^{2H-2}$ and $\Lambda(x) \sim |x|^{-2\alpha}$ (Riesz kernel). Then we have (Hu-Huang-N.-Tindel '15):

- The solution exists (in the Stratonovich sense) if $H \in (\frac{1}{2}, 1)$ and $\alpha < 2H - 1$.
- We have the Feynman-Kac formula

$$u(t,x) = E^B\left[u_0(B_t^x)\exp\left(\int_0^t \int_{\mathbb{R}^d} \delta_0(B_{t-s}^x - y W(ds,dy)\right)\right],$$

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  where B^x is a d-dimensional Brownian motion independent of W, starting from x.

- We have

$$E[u(t,x)^k] = E^B \exp\left\{ \sum_{i,j=1}^{k} \int_0^t \int_0^t \gamma(t-s)\Lambda(B_t^i - B_s^j) ds dt \right\}$$

and this leads to

$$C \exp\left(C k^{\frac{2-\alpha}{1-\alpha}} t^{\frac{2H-\alpha}{1-\alpha}} \right) \le E\left[u(t,x)^k \right] \le C' \exp\left(C' k^{\frac{2-\alpha}{1-\alpha}} t^{\frac{2H-\alpha}{1-\alpha}} \right).$$

Notice the change of sign in α in the numerator of the exponent of t.

- The solution exists in the Skorohod sense if $H \in (\frac{1}{2}, 1)$ and $\alpha < \min(2, d)$. In this case the above results are true.

2. The one-dimensional equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^2 W}{\partial t \partial x},$$

where the covariance of W is given by

$$E(W(t,x)W(s,y)) = (s \wedge t)\frac{1}{2}\left(|x|^{2H_0} + |y|^{2H_0} - |x-y|^{2H_0}\right),$$

can be solved in the Itô sense, provided $\frac{1}{4} < H_0 < \frac{1}{2}$ (Hu-Huang-Lê-N.-Tindel '17).

- We have the following Feynman-Kac formula for the moments:

$$E[u(t,x)^k] = E_B \exp\left\{ c_H \sum_{1 \le i < j \le k} \int_0^t \int_{\mathbb{R}} e^{i\xi(B_s^i - B_s^j)} |\xi|^{1-2H_0} d\xi ds \right\},$$

which leads to

$$\exp\left(c_1 t k^{1+\frac{1}{H_0}}\right) \le E(u(t,x)^k) \le \exp\left(c_2 t k^{1+\frac{1}{H_0}}\right).$$