Homogenization of a semilinear heat equation with a highly oscillating random potential

Etienne Pardoux

Aix–Marseille Université

joint work with Martin Hairer

The problem

- We start with the PDE (dim(x)=1)

$$\partial_t u_\varepsilon(t,x) = \partial_x^2 u_\varepsilon(t,x) + H(u_\varepsilon(t,x)) + G(u_\varepsilon(t,x))\eta_\varepsilon(t,x)$$
$$u_\varepsilon(0,x) = u_0(x), \quad u_\varepsilon(t,0) = u_\varepsilon(t,1) = 0.$$

  where

$$\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

  and $\eta(t,x)$ is a stationary zero–mean generalized random field with "good" mixing properties.
- This problem has been studied in case $H = 0$ and $G(u) = u$ in P., Piatnitski '12 and Hairer, P., Piatnitski '13, with different respective scalings of t and x. Those papers establish the LLN $u_\varepsilon \to u$, with a limiting PDE which depends upon the specific scaling.
- Bal '11 proves both the LLN and the CLT in the linear case, with a Gaussian perturbation η_ε.
- Here we prove both the LLN and the CLT in the semilinear case, with a non–Gaussian η_ε.

The problem

- We start with the PDE (dim(x)=1)

$$\partial_t u_\varepsilon(t,x) = \partial_x^2 u_\varepsilon(t,x) + H(u_\varepsilon(t,x)) + G(u_\varepsilon(t,x))\eta_\varepsilon(t,x)$$
$$u_\varepsilon(0,x) = u_0(x), \quad u_\varepsilon(t,0) = u_\varepsilon(t,1) = 0.$$

  where

$$\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

  and $\eta(t,x)$ is a stationary zero–mean generalized random field with "good" mixing properties.
- This problem has been studied in case $H = 0$ and $G(u) = u$ in P., Piatnitski '12 and Hairer, P., Piatnitski '13, with different respective scalings of t and x. Those papers establish the LLN $u_\varepsilon \to u$, with a limiting PDE which depends upon the specific scaling.
- Bal '11 proves both the LLN and the CLT in the linear case, with a Gaussian perturbation η_ε.
- Here we prove both the LLN and the CLT in the semilinear case, with a non–Gaussian η_ε.

The problem

- We start with the PDE (dim(x)=1)

$$\partial_t u_\varepsilon(t,x) = \partial_x^2 u_\varepsilon(t,x) + H(u_\varepsilon(t,x)) + G(u_\varepsilon(t,x))\eta_\varepsilon(t,x)$$
$$u_\varepsilon(0,x) = u_0(x), \quad u_\varepsilon(t,0) = u_\varepsilon(t,1) = 0.$$

  where

$$\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

  and $\eta(t,x)$ is a stationary zero–mean generalized random field with "good" mixing properties.
- This problem has been studied in case $H = 0$ and $G(u) = u$ in P., Piatnitski '12 and Hairer, P., Piatnitski '13, with different respective scalings of t and x. Those papers establish the LLN $u_\varepsilon \to u$, with a limiting PDE which depends upon the specific scaling.
- Bal '11 proves both the LLN and the CLT in the linear case, with a Gaussian perturbation η_ε.
- Here we prove both the LLN and the CLT in the semilinear case, with a non–Gaussian η_ε.

The problem

- We start with the PDE (dim(x)=1)

$$\partial_t u_\varepsilon(t,x) = \partial_x^2 u_\varepsilon(t,x) + H(u_\varepsilon(t,x)) + G(u_\varepsilon(t,x))\eta_\varepsilon(t,x)$$
$$u_\varepsilon(0,x) = u_0(x), \quad u_\varepsilon(t,0) = u_\varepsilon(t,1) = 0.$$

  where

$$\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

  and $\eta(t,x)$ is a stationary zero–mean generalized random field with "good" mixing properties.
- This problem has been studied in case $H = 0$ and $G(u) = u$ in P., Piatnitski '12 and Hairer, P., Piatnitski '13, with different respective scalings of t and x. Those papers establish the LLN $u_\varepsilon \to u$, with a limiting PDE which depends upon the specific scaling.
- Bal '11 proves both the LLN and the CLT in the linear case, with a Gaussian perturbation η_ε.
- Here we prove both the LLN and the CLT in the semilinear case, with a non–Gaussian η_ε.

Comparison with earlier results

- Wong–Zakai, see Hairer, P.'15. Consider for $x \in S^1$

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + H(u_\varepsilon) - C_\varepsilon G' G(u_\varepsilon) + G(u_\varepsilon)\xi_\varepsilon$$

where $\xi_\varepsilon(t,x) = \varepsilon^{-3/2}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x)$ and $C_\varepsilon \sim \varepsilon^{-1}$. $u_\varepsilon \to u$

$$\partial_t u = \Delta u + \overline{H}(u) + G(u)\xi.$$

- Homogenization. Consider for $x \in [0,1]$, with $\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x)$,

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + H(u_\varepsilon) + G(u_\varepsilon)\eta_\varepsilon.$$

LLN $u_\varepsilon \to \overline{u}$ in probability, where

$$\partial_t \overline{u} = \Delta \overline{u} + H(\overline{u}) + c_\eta GG'(\overline{u}).$$

CLT Let $v_\varepsilon = \varepsilon^{-1/2}(u_\varepsilon - \overline{u})$. $v_\varepsilon \Rightarrow v$, where

$$\partial_t v = \Delta v + (H + c_\eta GG')'(\overline{u})v + G(\overline{u})\xi.$$

Comparison with earlier results

- Wong–Zakai, see Hairer, P.'15. Consider for $x \in S^1$

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + H(u_\varepsilon) - C_\varepsilon G' G(u_\varepsilon) + G(u_\varepsilon)\xi_\varepsilon$$

where $\xi_\varepsilon(t,x) = \varepsilon^{-3/2}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x)$ and $C_\varepsilon \sim \varepsilon^{-1}$. $u_\varepsilon \to u$

$$\partial_t u = \Delta u + \overline{H}(u) + G(u)\xi.$$

- Homogenization. Consider for $x \in [0,1]$, with $\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x)$,

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + H(u_\varepsilon) + G(u_\varepsilon)\eta_\varepsilon.$$

LLN $u_\varepsilon \to \overline{u}$ in probability, where

$$\partial_t \overline{u} = \Delta \overline{u} + H(\overline{u}) + c_\eta GG'(\overline{u}).$$

CLT Let $v_\varepsilon = \varepsilon^{-1/2}(u_\varepsilon - \overline{u})$. $v_\varepsilon \Rightarrow v$, where

$$\partial_t v = \Delta v + (H + c_\eta GG')'(\overline{u})v + G(\overline{u})\xi.$$

Our assumptions

- We assume that the noise $\eta(t,x)$ is zero–mean, stationary, has finite moments of all order, and moreover that for any $\ell \geq 1$, the ℓ–th joint cumulant $\kappa_\ell(z_1,\ldots,z_\ell)$ of the random variables $\eta(z_1),\ldots,\eta(z_\ell)$ satisfies certain bounds ($z=(t,x)$).
- Let us recall what are the cumulants. Formally, the joint cumulant of the random variables $X_1,\ldots,X_\ell$ is

$$\kappa_\ell(X_1,\ldots,X_\ell) = (-i)^\ell \frac{\partial^\ell}{\partial z_1\cdots\partial z_\ell} \log \mathbf{E}\left[\exp\left(i\sum_{j=1}^{\ell} z_j X_j\right)\right]\Big|_{z_1=\cdots=z_\ell=0}.$$

- Cumulants can be expressed in terms of moments

$$\kappa_\ell(X_1,\ldots,X_\ell) = \sum_{\{a_1,\ldots,a_r\}\in\mathcal{P}([n])} (-1)^{r-1}(r-1)!\mathbf{E}(X^{a_1})\times\cdots\times\mathbf{E}(X^{a_r}),$$

where $[n]=\{1,2,\ldots,n\}$ and if $b\subset[n]$, $X^b=\prod_{j\in b}X_j$.

Our assumptions

- We assume that the noise $\eta(t,x)$ is zero–mean, stationary, has finite moments of all order, and moreover that for any $\ell \geq 1$, the ℓ–th joint cumulant $\kappa_\ell(z_1,\ldots,z_\ell)$ of the random variables $\eta(z_1),\ldots,\eta(z_\ell)$ satisfies certain bounds ($z=(t,x)$).
- Let us recall what are the cumulants. Formally, the joint cumulant of the random variables $X_1,\ldots,X_\ell$ is

$$\kappa_\ell(X_1,\ldots,X_\ell) = (-i)^\ell \frac{\partial^\ell}{\partial z_1\cdots\partial z_\ell} \log \mathbf{E}\left[\exp\left(i\sum_{j=1}^{\ell} z_j X_j\right)\right]\Big|_{z_1=\cdots=z_\ell=0}.$$

- Cumulants can be expressed in terms of moments

$$\kappa_\ell(X_1,\ldots,X_\ell) = \sum_{\{a_1,\ldots,a_r\}\in\mathcal{P}([n])} (-1)^{r-1}(r-1)!\mathbf{E}(X^{a_1})\times\cdots\times\mathbf{E}(X^{a_r}),$$

where $[n]=\{1,2,\ldots,n\}$ and if $b\subset[n]$, $X^b=\prod_{j\in b}X_j$.

Our assumptions

- We assume that the noise $\eta(t,x)$ is zero–mean, stationary, has finite moments of all order, and moreover that for any $\ell \geq 1$, the ℓ–th joint cumulant $\kappa_\ell(z_1,\ldots,z_\ell)$ of the random variables $\eta(z_1),\ldots,\eta(z_\ell)$ satisfies certain bounds ($z=(t,x)$).
- Let us recall what are the cumulants. Formally, the joint cumulant of the random variables $X_1,\ldots,X_\ell$ is

$$\kappa_\ell(X_1,\ldots,X_\ell) = (-i)^\ell \frac{\partial^\ell}{\partial z_1\cdots\partial z_\ell} \log \mathbf{E}\left[\exp\left(i\sum_{j=1}^{\ell} z_j X_j\right)\right]\Bigg|_{z_1=\cdots=z_\ell=0}.$$

- Cumulants can be expressed in terms of moments

$$\kappa_\ell(X_1,\ldots,X_\ell) = \sum_{\{a_1,\ldots,a_r\}\in\mathcal{P}([n])} (-1)^{r-1}(r-1)!\mathbf{E}(X^{a_1})\times\cdots\times\mathbf{E}(X^{a_r}),$$

where $[n]=\{1,2,\ldots,n\}$ and if $b\subset[n]$, $X^b=\prod_{j\in b}X_j$.

More on cumulants

- $\kappa_1(X) = \mathbf{E}(X)$, $\kappa_2(X, Y) = \mathrm{Cov}(X, Y)$, etc...
- $\{X_j, j \in J\}$ is Gaussian $\Leftrightarrow \kappa_\ell(X_{j_1}, \ldots, X_{j_\ell}) = 0$ whenever $\ell \geq 3$.
- If $X \simeq \mathrm{Poisson}(\lambda)$, $\kappa_\ell(X, \ldots, X) = \lambda$, for all $\ell \geq 1$.
- If $(X_1, \ldots, X_j)$ and $(X_{j+1}, \ldots, X_\ell)$ are independent, then $\kappa_\ell(X_1, \ldots, X_\ell) = 0$.
- If $c_1, \ldots, c_\ell$ are constants, $\ell \geq 2$, $\kappa_\ell(X_1 + c_1, \ldots, X_\ell + c_\ell) = \kappa_\ell(X_1, \ldots, X_\ell)$.
- If the two vectors $(X_1, \ldots, X_\ell)$ and $(X'_1, \ldots, X'_\ell)$ are independent, then $\kappa_\ell(X_1 + X'_1, \ldots, X_\ell + X'_\ell) = \kappa_\ell(X_1, \ldots, X_\ell) + \kappa_\ell(X'_1, \ldots, X'_\ell)$.
- If N is a Poisson point measure on $\mathbf{R}^d$ with mean measure μ, $f_1, \ldots, f_\ell$ are continuous and have compact support, then

$$\kappa_\ell(N(f_1), \ldots, N(f_\ell)) = \int_{\mathbf{R}^d} f_1(x) \times \cdots \times f_\ell(x)\, \mu(dx).$$

More on cumulants

- $\kappa_1(X) = \mathbf{E}(X)$, $\kappa_2(X, Y) = \mathrm{Cov}(X, Y)$, etc...
- $\{X_j,\, j \in J\}$ is Gaussian $\Leftrightarrow \kappa_\ell(X_{j_1}, \ldots, X_{j_\ell}) = 0$ whenever $\ell \geq 3$.
- If $X \simeq \mathrm{Poisson}(\lambda)$, $\kappa_\ell(X, \ldots, X) = \lambda$, for all $\ell \geq 1$.
- If $(X_1, \ldots, X_j)$ and $(X_{j+1}, \ldots, X_\ell)$ are independent, then $\kappa_\ell(X_1, \ldots, X_\ell) = 0$.
- If $c_1, \ldots, c_\ell$ are constants, $\ell \geq 2$, $\kappa_\ell(X_1 + c_1, \ldots, X_\ell + c_\ell) = \kappa_\ell(X_1, \ldots, X_\ell)$.
- If the two vectors $(X_1, \ldots, X_\ell)$ and $(X'_1, \ldots, X'_\ell)$ are independent, then $\kappa_\ell(X_1 + X'_1, \ldots, X_\ell + X'_\ell) = \kappa_\ell(X_1, \ldots, X_\ell) + \kappa_\ell(X'_1, \ldots, X'_\ell)$.
- If N is a Poisson point measure on $\mathbf{R}^d$ with mean measure μ, $f_1, \ldots, f_\ell$ are continuous and have compact support, then

$$\kappa_\ell(N(f_1), \ldots, N(f_\ell)) = \int_{\mathbf{R}^d} f_1(x) \times \cdots \times f_\ell(x)\, \mu(dx).$$

More on cumulants

- $\kappa_1(X) = \mathbf{E}(X)$, $\kappa_2(X, Y) = \text{Cov}(X, Y)$, etc...
- $\{X_j,\, j \in J\}$ is Gaussian $\Leftrightarrow \kappa_\ell(X_{j_1}, \ldots, X_{j_\ell}) = 0$ whenever $\ell \geq 3$.
- If $X \simeq \text{Poisson}(\lambda)$, $\kappa_\ell(X, \ldots, X) = \lambda$, for all $\ell \geq 1$.
- If $(X_1, \ldots, X_j)$ and $(X_{j+1}, \ldots, X_\ell)$ are independent, then $\kappa_\ell(X_1, \ldots, X_\ell) = 0$.
- If $c_1, \ldots, c_\ell$ are constants, $\ell \geq 2$, $\kappa_\ell(X_1 + c_1, \ldots, X_\ell + c_\ell) = \kappa_\ell(X_1, \ldots, X_\ell)$.
- If the two vectors $(X_1, \ldots, X_\ell)$ and $(X'_1, \ldots, X'_\ell)$ are independent, then $\kappa_\ell(X_1 + X'_1, \ldots, X_\ell + X'_\ell) = \kappa_\ell(X_1, \ldots, X_\ell) + \kappa_\ell(X'_1, \ldots, X'_\ell)$.
- If N is a Poisson point measure on $\mathbf{R}^d$ with mean measure μ, $f_1, \ldots, f_\ell$ are continuous and have compact support, then

$$\kappa_\ell(N(f_1), \ldots, N(f_\ell)) = \int_{\mathbf{R}^d} f_1(x) \times \cdots \times f_\ell(x)\, \mu(dx).$$

More on cumulants

- $\kappa_1(X) = \mathbf{E}(X)$, $\kappa_2(X, Y) = \mathrm{Cov}(X, Y)$, etc...
- $\{X_j,\ j \in J\}$ is Gaussian $\Leftrightarrow \kappa_\ell(X_{j_1}, \ldots, X_{j_\ell}) = 0$ whenever $\ell \geq 3$.
- If $X \simeq \mathrm{Poisson}(\lambda)$, $\kappa_\ell(X, \ldots, X) = \lambda$, for all $\ell \geq 1$.
- If $(X_1, \ldots, X_j)$ and $(X_{j+1}, \ldots, X_\ell)$ are independent, then $\kappa_\ell(X_1, \ldots, X_\ell) = 0$.
- If $c_1, \ldots, c_\ell$ are constants, $\ell \geq 2$, $\kappa_\ell(X_1 + c_1, \ldots, X_\ell + c_\ell) = \kappa_\ell(X_1, \ldots, X_\ell)$.
- If the two vectors $(X_1, \ldots, X_\ell)$ and $(X'_1, \ldots, X'_\ell)$ are independent, then $\kappa_\ell(X_1 + X'_1, \ldots, X_\ell + X'_\ell) = \kappa_\ell(X_1, \ldots, X_\ell) + \kappa_\ell(X'_1, \ldots, X'_\ell)$.
- If N is a Poisson point measure on $\mathbf{R}^d$ with mean measure μ, $f_1, \ldots, f_\ell$ are continuous and have compact support, then

$$\kappa_\ell(N(f_1), \ldots, N(f_\ell)) = \int_{\mathbf{R}^d} f_1(x) \times \cdots \times f_\ell(x)\, \mu(dx).$$

More on cumulants

- $\kappa_1(X) = \mathbf{E}(X)$, $\kappa_2(X, Y) = \text{Cov}(X, Y)$, etc...
- $\{X_j, j \in J\}$ is Gaussian $\Leftrightarrow \kappa_\ell(X_{j_1}, \ldots, X_{j_\ell}) = 0$ whenever $\ell \geq 3$.
- If $X \simeq \text{Poisson}(\lambda)$, $\kappa_\ell(X, \ldots, X) = \lambda$, for all $\ell \geq 1$.
- If $(X_1, \ldots, X_j)$ and $(X_{j+1}, \ldots, X_\ell)$ are independent, then $\kappa_\ell(X_1, \ldots, X_\ell) = 0$.
- If $c_1, \ldots, c_\ell$ are constants, $\ell \geq 2$, $\kappa_\ell(X_1 + c_1, \ldots, X_\ell + c_\ell) = \kappa_\ell(X_1, \ldots, X_\ell)$.
- If the two vectors $(X_1, \ldots, X_\ell)$ and $(X'_1, \ldots, X'_\ell)$ are independent, then $\kappa_\ell(X_1 + X'_1, \ldots, X_\ell + X'_\ell) = \kappa_\ell(X_1, \ldots, X_\ell) + \kappa_\ell(X'_1, \ldots, X'_\ell)$.
- If N is a Poisson point measure on $\mathbf{R}^d$ with mean measure μ, $f_1, \ldots, f_\ell$ are continuous and have compact support, then

$$\kappa_\ell(N(f_1), \ldots, N(f_\ell)) = \int_{\mathbf{R}^d} f_1(x) \times \cdots \times f_\ell(x)\, \mu(dx).$$

More on cumulants

- $\kappa_1(X) = \mathbf{E}(X)$, $\kappa_2(X, Y) = \mathrm{Cov}(X, Y)$, etc...
- $\{X_j,\, j \in J\}$ is Gaussian $\Leftrightarrow \kappa_\ell(X_{j_1}, \dots, X_{j_\ell}) = 0$ whenever $\ell \geq 3$.
- If $X \simeq \mathrm{Poisson}(\lambda)$, $\kappa_\ell(X, \dots, X) = \lambda$, for all $\ell \geq 1$.
- If $(X_1, \dots, X_j)$ and $(X_{j+1}, \dots, X_\ell)$ are independent, then $\kappa_\ell(X_1, \dots, X_\ell) = 0$.
- If $c_1, \dots, c_\ell$ are constants, $\ell \geq 2$, $\kappa_\ell(X_1 + c_1, \dots, X_\ell + c_\ell) = \kappa_\ell(X_1, \dots, X_\ell)$.
- If the two vectors $(X_1, \dots, X_\ell)$ and $(X'_1, \dots, X'_\ell)$ are independent, then $\kappa_\ell(X_1 + X'_1, \dots, X_\ell + X'_\ell) = \kappa_\ell(X_1, \dots, X_\ell) + \kappa_\ell(X'_1, \dots, X'_\ell)$.
- If N is a Poisson point measure on $\mathbf{R}^d$ with mean measure μ, $f_1, \dots, f_\ell$ are continuous and have compact support, then

$$\kappa_\ell(N(f_1), \dots, N(f_\ell)) = \int_{\mathbf{R}^d} f_1(x) \times \cdots \times f_\ell(x)\, \mu(dx).$$

More on cumulants

- $\kappa_1(X) = \mathbf{E}(X)$, $\kappa_2(X, Y) = \mathrm{Cov}(X, Y)$, etc...
- $\{X_j,\ j \in J\}$ is Gaussian $\Leftrightarrow \kappa_\ell(X_{j_1}, \ldots, X_{j_\ell}) = 0$ whenever $\ell \geq 3$.
- If $X \simeq \mathrm{Poisson}(\lambda)$, $\kappa_\ell(X, \ldots, X) = \lambda$, for all $\ell \geq 1$.
- If $(X_1, \ldots, X_j)$ and $(X_{j+1}, \ldots, X_\ell)$ are independent, then $\kappa_\ell(X_1, \ldots, X_\ell) = 0$.
- If $c_1, \ldots, c_\ell$ are constants, $\ell \geq 2$, $\kappa_\ell(X_1 + c_1, \ldots, X_\ell + c_\ell) = \kappa_\ell(X_1, \ldots, X_\ell)$.
- If the two vectors $(X_1, \ldots, X_\ell)$ and $(X'_1, \ldots, X'_\ell)$ are independent, then $\kappa_\ell(X_1 + X'_1, \ldots, X_\ell + X'_\ell) = \kappa_\ell(X_1, \ldots, X_\ell) + \kappa_\ell(X'_1, \ldots, X'_\ell)$.
- If N is a Poisson point measure on $\mathbf{R}^d$ with mean measure μ, $f_1, \ldots, f_\ell$ are continuous and have compact support, then

$$\kappa_\ell(N(f_1), \ldots, N(f_\ell)) = \int_{\mathbf{R}^d} f_1(x) \times \cdots \times f_\ell(x)\, \mu(dx).$$

Precise assumptions

- H and G are of class C^4 and C^5 resp., H, G and GG' having at most linear growth at infinity.
- Denote by $\kappa_\ell(z_1, \dots, z_\ell)$ the joint cumulant of $\eta(z_1), \dots, \eta(z_\ell)$. We assume that uniformly over all $z_1, \dots, z_\ell \in \mathbf{R}^2$,

$$|\kappa_\ell(z_1, \dots, z_\ell)| \lesssim 2^{c(\Omega_\ell)\mathbf{n}(\Omega_\ell)} \prod_{A \in \mathring{V}} 2^{c(A)\mathbf{n}(A)},$$

- where $\mathring{V}$ denotes the set of interior nodes of the minimal spanning tree of the complete graph with vertices $\{z_1, \dots, z_\ell\}$, Ω_ℓ is the root of that tree, $\mathbf{n}(A) = -\lceil \log_2 d_z(A_1, A_2) \rceil$ and $c(A) = 1/2$ if $\mathbf{n}(A) \geq 0$, $c(A) = 3/2 + \delta$ otherwise.

Precise assumptions

- H and G are of class C^4 and C^5 resp., H, G and GG' having at most linear growth at infinity.
- Denote by $\kappa_\ell(z_1, \dots, z_\ell)$ the joint cumulant of $\eta(z_1), \dots, \eta(z_\ell)$. We assume that uniformly over all $z_1, \dots, z_\ell \in \mathbf{R}^2$,

$$|\kappa_\ell(z_1, \dots, z_\ell)| \lesssim 2^{c(\Omega_\ell)\mathbf{n}(\Omega_\ell)} \prod_{A \in \mathring{V}} 2^{c(A)\mathbf{n}(A)},$$

- where $\mathring{V}$ denotes the set of interior nodes of the minimal spanning tree of the complete graph with vertices $\{z_1, \dots, z_\ell\}$, Ω_ℓ is the root of that tree, $\mathbf{n}(A) = -\lceil \log_2 d_z(A_1, A_2) \rceil$ and $c(A) = 1/2$ if $\mathbf{n}(A) \geq 0$, $c(A) = 3/2 + \delta$ otherwise.

Precise assumptions

- H and G are of class C^4 and C^5 resp., H, G and GG' having at most linear growth at infinity.
- Denote by $\kappa_\ell(z_1, \dots, z_\ell)$ the joint cumulant of $\eta(z_1), \dots, \eta(z_\ell)$. We assume that uniformly over all $z_1, \dots, z_\ell \in \mathbf{R}^2$,

$$|\kappa_\ell(z_1, \dots, z_\ell)| \lesssim 2^{c(\Omega_\ell)\mathbf{n}(\Omega_\ell)} \prod_{A \in \mathring{V}} 2^{c(A)\mathbf{n}(A)},$$

- where $\mathring{V}$ denotes the set of interior nodes of the minimal spanning tree of the complete graph with vertices $\{z_1, \dots, z_\ell\}$, Ω_ℓ is the root of that tree, $\mathbf{n}(A) = -\lceil \log_2 d_z(A_1, A_2) \rceil$ and $c(A) = 1/2$ if $\mathbf{n}(A) \geq 0$, $c(A) = 3/2 + \delta$ otherwise.

An example satisfying our assumptions

- Suppose $\eta(z) = \bar{N}(\varrho(z - \cdot)$, with N a Poisson point process on $\mathbf{R}^2$ with mean measure Lebesgue, $\bar{N}(dz) = N(dz) - dz$, and $|\varrho(z)| \lesssim |z|^{-3-\delta}$ for $|z| > 1$, and $|\varrho(z)| \lesssim |z|^{-1/2}$ for $|z| \leq 1$.
- In that case,

$$\kappa_\ell(z_1, \dots, z_\ell) = \int_{\mathbf{R}^2} \varrho(z_1 - z) \times \cdots \times \varrho(z_\ell - z) dz.$$

- It is not too hard to verify the assumption thanks to that formula.

An example satisfying our assumptions

- Suppose $\eta(z) = \bar{N}(\varrho(z - \cdot)$, with N a Poisson point process on $\mathbf{R}^2$ with mean measure Lebesgue, $\bar{N}(dz) = N(dz) - dz$, and $|\varrho(z)| \lesssim |z|^{-3-\delta}$ for $|z| > 1$, and $|\varrho(z)| \lesssim |z|^{-1/2}$ for $|z| \le 1$.
- In that case,

$$\kappa_\ell(z_1, \ldots, z_\ell) = \int_{\mathbf{R}^2} \varrho(z_1 - z) \times \cdots \times \varrho(z_\ell - z) dz.$$

- It is not too hard to verify the assumption thanks to that formula.

An example satisfying our assumptions

- Suppose $\eta(z) = \bar{N}(\varrho(z - \cdot)$, with N a Poisson point process on $\mathbf{R}^2$ with mean measure Lebesgue, $\bar{N}(dz) = N(dz) - dz$, and $|\varrho(z)| \lesssim |z|^{-3-\delta}$ for $|z| > 1$, and $|\varrho(z)| \lesssim |z|^{-1/2}$ for $|z| \leq 1$.
- In that case,

$$\kappa_\ell(z_1, \ldots, z_\ell) = \int_{\mathbf{R}^2} \varrho(z_1 - z) \times \cdots \times \varrho(z_\ell - z) dz.$$

- It is not too hard to verify the assumption thanks to that formula.

Regularity structures

- We shall write **1** for the unit constant, $|\mathbf{1}| = 0$, X for an "abstract version" of the first order monomial $x \to x$. $|X|$=1.
- We have the three following elements of $\mathcal{T}$ with negative regularity :

  • Ξ stands for ξ_ε, or space–time white noise itself in the limit, $|\Xi| = -3/2 - \kappa$;

  • $\dot{\Xi}$ stands for the noise driving our approximate PDE ($= \sqrt{\varepsilon}\xi_\varepsilon$), is zero in the limit, $|\dot{\Xi}| = -1 - \kappa$;

  • $\ddot{\Xi}$ stands for $\varepsilon\xi_\varepsilon$, is zero in the limit, $|\ddot{\Xi}| = -1/2 - \kappa$.

Regularity structures

- We shall write **1** for the unit constant, $|\mathbf{1}| = 0$, X for an "abstract version" of the first order monomial $x \to x$. $|X|$=1.
- We have the three following elements of $\mathcal{T}$ with negative regularity :

  • Ξ stands for ξ_ε, or space–time white noise itself in the limit, $|\Xi| = -3/2 - \kappa$;

  • $\dot{\Xi}$ stands for the noise driving our approximate PDE $(= \sqrt{\varepsilon}\xi_\varepsilon)$, is zero in the limit, $|\dot{\Xi}| = -1 - \kappa$;

  • $\ddot{\Xi}$ stands for $\varepsilon\xi_\varepsilon$, is zero in the limit, $|\ddot{\Xi}| = -1/2 - \kappa$.

Law of Large Numbers 1

- We want to show that $u_\varepsilon \to \bar{u}$, where $\bar{u}$ solves the parabolic PDE

$$\partial_t \bar{u}(t,x) = \partial_x^2 \bar{u}(t,x) + H(\bar{u}(t,x)) + c_\eta GG'(\bar{u}(t,x)),$$
$$\bar{u}(0,x) = u_0(x), \quad \bar{u}(t,0) = \bar{u}(t,1) = 0,$$

where $c_\eta = \int_{\mathbf{R}^2} P(z)\kappa_2(0,z)dz$.

- We rewrite the equation for u_ε as

$$U = \mathcal{P}\mathbf{1}_{t>0}(\hat{H}(U) + \hat{G}(U)\dot{\Xi}) + Pu_0,$$

where if $U = u\mathbf{1} + \tilde{U}$, $\hat{H}(U) = H(u)\mathbf{1} + H'(u)\tilde{U}$.

Law of Large Numbers 1

- We want to show that $u_\varepsilon \to \bar{u}$, where $\bar{u}$ solves the parabolic PDE

$$\partial_t \bar{u}(t,x) = \partial_x^2 \bar{u}(t,x) + H(\bar{u}(t,x)) + c_\eta GG'(\bar{u}(t,x)),$$
$$\bar{u}(0,x) = u_0(x), \quad \bar{u}(t,0) = \bar{u}(t,1) = 0,$$

  where $c_\eta = \int_{\mathbf{R}^2} P(z)\kappa_2(0,z)dz$.

- We rewrite the equation for u_ε as

$$U = \mathcal{P}\mathbf{1}_{t>0}(\hat{H}(U) + \hat{G}(U)\dot{\Xi}) + Pu_0,$$

  where if $U = u\mathbf{1} + \tilde{U}$, $\hat{H}(U) = H(u)\mathbf{1} + H'(u)\tilde{U}$.

Law of Large Numbers 2

- $$U = \mathcal{P}\mathbf{1}_{t>0}(\hat{H}(U) + \hat{G}(U)\dot{\Xi}) + Pu_0.$$
- U is of the form (for some functions u and u')
$$U = u\mathbf{1} + G(u)\mathcal{I}(\dot{\Xi}) + u'X,$$
- The right hand side of the above is given as
$$H(u)\mathbf{1} + G(u)\dot{\Xi} + G'(u)G(u)\mathcal{I}(\dot{\Xi})\dot{\Xi} + G'(u)u'X\dot{\Xi}.$$
- The LLN follows from the two facts $\Pi_z^\varepsilon\dot{\Xi} \to 0$, $\Pi_z^\varepsilon\mathcal{I}(\dot{\Xi})\dot{\Xi} \to c_\eta$.

Law of Large Numbers 2

- $$U = \mathcal{P}\mathbf{1}_{t>0}(\hat{H}(U) + \hat{G}(U)\dot{\Xi}) + Pu_0.$$
- U is of the form (for some functions u and u')
$$U = u\mathbf{1} + G(u)\mathcal{I}(\dot{\Xi}) + u'X,$$
- The right hand side of the above is given as
$$H(u)\mathbf{1} + G(u)\dot{\Xi} + G'(u)G(u)\mathcal{I}(\dot{\Xi})\dot{\Xi} + G'(u)u'X\dot{\Xi}.$$
- The LLN follows from the two facts $\Pi_z^\varepsilon\dot{\Xi} \to 0$, $\Pi_z^\varepsilon\mathcal{I}(\dot{\Xi})\dot{\Xi} \to c_\eta$.

Law of Large Numbers 2

- $$U = \mathcal{P}\mathbf{1}_{t>0}(\hat{H}(U) + \hat{G}(U)\dot{\Xi}) + Pu_0.$$
- U is of the form (for some functions u and u')
$$U = u\mathbf{1} + G(u)\mathcal{I}(\dot{\Xi}) + u'X,$$
- The right hand side of the above is given as
$$H(u)\mathbf{1} + G(u)\dot{\Xi} + G'(u)G(u)\mathcal{I}(\dot{\Xi})\dot{\Xi} + G'(u)u'X\dot{\Xi}.$$
- The LLN follows from the two facts $\Pi_z^\varepsilon\dot{\Xi} \to 0$, $\Pi_z^\varepsilon\mathcal{I}(\dot{\Xi})\dot{\Xi} \to c_\eta$.

Law of Large Numbers 2

- $$U = \mathcal{P}\mathbf{1}_{t>0}(\hat{H}(U) + \hat{G}(U)\dot{\Xi}) + Pu_0.$$
- U is of the form (for some functions u and u')
$$U = u\mathbf{1} + G(u)\mathcal{I}(\dot{\Xi}) + u'X,$$
- The right hand side of the above is given as
$$H(u)\mathbf{1} + G(u)\dot{\Xi} + G'(u)G(u)\mathcal{I}(\dot{\Xi})\dot{\Xi} + G'(u)u'X\dot{\Xi}.$$
- The LLN follows from the two facts $\Pi_z^\varepsilon\dot{\Xi} \to 0$, $\Pi_z^\varepsilon\mathcal{I}(\dot{\Xi})\dot{\Xi} \to c_\eta$.

Central Limit Theorem

- Now let $v_\varepsilon(t,x) = \frac{u_\varepsilon(t,x)-\bar u(t,x)}{\sqrt{\varepsilon}}$.
- With $\zeta_\varepsilon = \sqrt{\varepsilon}\eta_\varepsilon$,

$$\begin{aligned}\partial_t v_\varepsilon &= \partial_x^2 v_\varepsilon + \frac{H(u_\varepsilon)-H(\bar u)}{\sqrt{\varepsilon}} + \frac{G(u_\varepsilon)\eta_\varepsilon - c_\eta G'G(\bar u)}{\sqrt{\varepsilon}}\\ &= \partial_x^2 v_\varepsilon + \frac{H(u_\varepsilon)-H(\bar u)}{\sqrt{\varepsilon}} + G(\bar u)\xi_\varepsilon + \frac{G(u_\varepsilon)-G(\bar u)}{\sqrt{\varepsilon}}\eta_\varepsilon - \frac{c_\eta G'G(\bar u)}{\sqrt{\varepsilon}}\\ &\simeq \partial_x^2 v_\varepsilon + H'(\bar u)v_\varepsilon + G(\bar u)\xi_\varepsilon + G'(\bar u)v_\varepsilon\eta_\varepsilon + \frac{1}{2}G''(\bar u)v_\varepsilon^2\zeta_\varepsilon - \frac{c_\eta}{\sqrt{\varepsilon}}GG'(\bar u)\end{aligned}$$

- Consider the fixed point problem

$$V = \mathcal{P}\mathbf{1}_{t>0}\Big(\mathcal{L}(H'(\bar u))V + \mathcal{L}(G(\bar u))\Xi + \mathcal{L}(G'(\bar u))V\dot{\Xi} + \frac{1}{2}\mathcal{L}(G''(\bar u))V^2\ddot{\Xi}\Big),$$

where $(\mathcal{L}f)(z) = f(z)\mathbf{1} + \partial_x f(z)\,X$, $z=(t,x)$.

Central Limit Theorem

- Now let $v_\varepsilon(t,x) = \frac{u_\varepsilon(t,x) - \bar{u}(t,x)}{\sqrt{\varepsilon}}$.
- With $\zeta_\varepsilon = \sqrt{\varepsilon}\eta_\varepsilon$,

$$\begin{aligned}\partial_t v_\varepsilon &= \partial_x^2 v_\varepsilon + \frac{H(u_\varepsilon) - H(\bar{u})}{\sqrt{\varepsilon}} + \frac{G(u_\varepsilon)\eta_\varepsilon - c_\eta G' G(\bar{u})}{\sqrt{\varepsilon}} \\ &= \partial_x^2 v_\varepsilon + \frac{H(u_\varepsilon) - H(\bar{u})}{\sqrt{\varepsilon}} + G(\bar{u})\xi_\varepsilon + \frac{G(u_\varepsilon) - G(\bar{u})}{\sqrt{\varepsilon}}\eta_\varepsilon - \frac{c\eta G' G(\bar{u})}{\sqrt{\varepsilon}} \\ &\simeq \partial_x^2 v_\varepsilon + H'(\bar{u})v_\varepsilon + G(\bar{u})\xi_\varepsilon + G'(\bar{u})v_\varepsilon\eta_\varepsilon + \frac{1}{2}G''(\bar{u})v_\varepsilon^2\zeta_\varepsilon - \frac{c_\eta}{\sqrt{\varepsilon}}GG'(\bar{u})\end{aligned}$$

- Consider the fixed point problem

$$V = \mathcal{P}\mathbf{1}_{t>0}\Big(\mathcal{L}(H'(\bar{u}))V + \mathcal{L}(G(\bar{u}))\Xi + \mathcal{L}(G'(\bar{u}))V\dot{\Xi} + \frac{1}{2}\mathcal{L}(G''(\bar{u}))V^2\ddot{\Xi}\Big),$$

where $(\mathcal{L}f)(z) = f(z)\mathbf{1} + \partial_x f(z)X$, $z = (t,x)$.

Central Limit Theorem

- Now let $v_\varepsilon(t,x) = \frac{u_\varepsilon(t,x) - \bar{u}(t,x)}{\sqrt{\varepsilon}}$.
- With $\zeta_\varepsilon = \sqrt{\varepsilon}\eta_\varepsilon$,

$$\begin{aligned}
\partial_t v_\varepsilon &= \partial_x^2 v_\varepsilon + \frac{H(u_\varepsilon) - H(\bar{u})}{\sqrt{\varepsilon}} + \frac{G(u_\varepsilon)\eta_\varepsilon - c_\eta G' G(\bar{u})}{\sqrt{\varepsilon}} \\
&= \partial_x^2 v_\varepsilon + \frac{H(u_\varepsilon) - H(\bar{u})}{\sqrt{\varepsilon}} + G(\bar{u})\xi_\varepsilon + \frac{G(u_\varepsilon) - G(\bar{u})}{\sqrt{\varepsilon}}\eta_\varepsilon - \frac{c\eta G' G(\bar{u})}{\sqrt{\varepsilon}} \\
&\simeq \partial_x^2 v_\varepsilon + H'(\bar{u})v_\varepsilon + G(\bar{u})\xi_\varepsilon + G'(\bar{u})v_\varepsilon\eta_\varepsilon + \frac{1}{2}G''(\bar{u})v_\varepsilon^2\zeta_\varepsilon - \frac{c_\eta}{\sqrt{\varepsilon}}GG'(\bar{u})
\end{aligned}$$

- Consider the fixed point problem

$$V = \mathcal{P}\mathbf{1}_{t>0}\Big(\mathcal{L}(H'(\bar{u}))V + \mathcal{L}(G(\bar{u}))\Xi + \mathcal{L}(G'(\bar{u}))V\dot{\Xi} + \frac{1}{2}\mathcal{L}(G''(\bar{u}))V^2\ddot{\Xi}\Big),$$

where $(\mathcal{L}f)(z) = f(z)\,\mathbf{1} + \partial_x f(z)\,X$, $z = (t,x)$.

- Note that we have left out the last term of the previous equation. At the end we renormalize the equation, which means bring back in that term.
- V must be of the form (up to terms of homogeneity > 1)

$$V = v\mathbf{1} + G(\bar{u})\,\mathcal{I}(\Xi) + G'(\bar{u})v\,\mathcal{I}(\dot{\Xi}) + v'X\,.$$

- The factor of $\mathcal{P}$ in the righthand side reads (up to terms of homogeneity > 0)

$$\begin{aligned} H'(\bar{u})v\,\mathbf{1} + G(\bar{u})\,\Xi + G'(\bar{u})\bar{u}'\,X\Xi + G'(\bar{u})v\,\dot{\Xi} \\ + G'(\bar{u})G(\bar{u})\mathcal{I}(\Xi)\dot{\Xi} + G'(\bar{u})G'(\bar{u})v\,\mathcal{I}(\dot{\Xi})\dot{\Xi} + G''(\bar{u})\bar{u}'v\,X\dot{\Xi} \\ + G'(\bar{u})v'\,X\dot{\Xi} + \frac{1}{2}G''(\bar{u})v^2\ddot{\Xi} + G''(\bar{u})G(\bar{u})v\mathcal{I}(\Xi)\ddot{\Xi}\,. \end{aligned}$$

- Note that we have left out the last term of the previous equation. At the end we renormalize the equation, which means bring back in that term.
- V must be of the form (up to terms of homogeneity > 1)

$$V = v\mathbf{1} + G(\bar{u})\,\mathcal{I}(\Xi) + G'(\bar{u})v\,\mathcal{I}(\dot{\Xi}) + v'X\,.$$

- The factor of $\mathcal{P}$ in the righthand side reads (up to terms of homogeneity > 0)

$$\begin{aligned} &H'(\bar{u})v\,\mathbf{1} + G(\bar{u})\,\Xi + G'(\bar{u})\bar{u}'\,X\Xi + G'(\bar{u})v\,\dot{\Xi} \\ &+G'(\bar{u})G(\bar{u})\mathcal{I}(\Xi)\dot{\Xi} + G'(\bar{u})G'(\bar{u})v\,\mathcal{I}(\dot{\Xi})\dot{\Xi} + G''(\bar{u})\bar{u}'v\,X\dot{\Xi} \\ &+G'(\bar{u})v'\,X\dot{\Xi} + \frac{1}{2}G''(\bar{u})v^2\ddot{\Xi} + G''(\bar{u})G(\bar{u})v\mathcal{I}(\Xi)\ddot{\Xi}\,. \end{aligned}$$

- Note that we have left out the last term of the previous equation. At the end we renormalize the equation, which means bring back in that term.
- V must be of the form (up to terms of homogeneity > 1)

$$V = v\mathbf{1} + G(\bar{u})\,\mathcal{I}(\Xi) + G'(\bar{u})v\,\mathcal{I}(\dot{\Xi}) + v'X\ .$$

- The factor of $\mathcal{P}$ in the righthand side reads (up to terms of homogeneity > 0)

$$\begin{aligned} H'(\bar{u})v\,\mathbf{1} + G(\bar{u})\,\Xi + G'(\bar{u})\bar{u}'\,X\Xi + G'(\bar{u})v\,\dot{\Xi} \\ + G'(\bar{u})G(\bar{u})\,\mathcal{I}(\Xi)\dot{\Xi} + G'(\bar{u})G'(\bar{u})v\,\mathcal{I}(\dot{\Xi})\dot{\Xi} + G''(\bar{u})\bar{u}'v\,X\dot{\Xi} \\ + G'(\bar{u})v'\,X\dot{\Xi} + \frac{1}{2}G''(\bar{u})v^2\ddot{\Xi} + G''(\bar{u})G(\bar{u})v\mathcal{I}(\Xi)\ddot{\Xi}\ . \end{aligned}$$

- The canonical model satisfies $\Pi^\varepsilon \Xi = \xi_\varepsilon$ and also $\mathbf{E}\Big(\Pi^\varepsilon \mathcal{I}(\Xi)\dot{\Xi}\Big) = \varepsilon^{-1/2} c_\eta$.
- We define a renormalized model $\hat{\Pi}^\varepsilon$ by setting

$$\hat{\Pi}^\varepsilon_z \mathcal{I}(\Xi)\dot{\Xi} = \Pi^\varepsilon_z \mathcal{I}(\Xi)\dot{\Xi} - \frac{c_\eta}{\sqrt{\varepsilon}} , \qquad \hat{\Pi}^\varepsilon_z \tau = \Pi^\varepsilon_z \tau$$

  for all other basis vectors τ.
- The core result says

Theorem

The random models $\hat{\Pi}^\varepsilon$ converge weakly to a limiting admissible model $\hat{\Pi}$ such that with ξ = space-time white noise,

$$\hat{\Pi}\,\Xi = \xi , \qquad \hat{\Pi}\,\dot{\Xi} = 0 , \qquad \hat{\Pi}\,\ddot{\Xi} = 0 ,$$
$$\hat{\Pi}\,\mathcal{I}(\Xi)\dot{\Xi} = 0 , \qquad \hat{\Pi}\,\mathcal{I}(\dot{\Xi})\dot{\Xi} = c_\eta , \qquad \hat{\Pi}\,\mathcal{I}(\Xi)\ddot{\Xi} = c_\eta$$

- The canonical model satisfies $\Pi^\varepsilon \Xi = \xi_\varepsilon$ and also $\mathbf{E}\Big(\Pi^\varepsilon \mathcal{I}(\Xi)\dot{\Xi}\Big) = \varepsilon^{-1/2} c_\eta$.
- We define a renormalized model $\hat{\Pi}^\varepsilon$ by setting

$$\hat{\Pi}^\varepsilon_z \mathcal{I}(\Xi)\dot{\Xi} = \Pi^\varepsilon_z \mathcal{I}(\Xi)\dot{\Xi} - \frac{c_\eta}{\sqrt{\varepsilon}}\,, \qquad \hat{\Pi}^\varepsilon_z \tau = \Pi^\varepsilon_z \tau$$

for all other basis vectors τ.

- The core result says

Theorem

The random models $\hat{\Pi}^\varepsilon$ converge weakly to a limiting admissible model $\hat{\Pi}$ such that with ξ = space-time white noise,

$$\hat{\Pi}\,\Xi = \xi\,, \qquad \hat{\Pi}\,\dot{\Xi} = 0\,, \qquad \hat{\Pi}\,\ddot{\Xi} = 0\,,$$

$$\hat{\Pi}\,\mathcal{I}(\Xi)\dot{\Xi} = 0\,, \qquad \hat{\Pi}\,\mathcal{I}(\dot{\Xi})\dot{\Xi} = c_\eta\,, \qquad \hat{\Pi}\,\mathcal{I}(\Xi)\ddot{\Xi} = c_\eta$$

- The canonical model satisfies $\Pi^\varepsilon \Xi = \xi_\varepsilon$ and also $\mathbf{E}\Big(\Pi^\varepsilon \mathcal{I}(\Xi)\dot{\Xi}\Big) = \varepsilon^{-1/2} c_\eta$.
- We define a renormalized model $\hat{\Pi}^\varepsilon$ by setting

$$\hat{\Pi}^\varepsilon_z \mathcal{I}(\Xi)\dot{\Xi} = \Pi^\varepsilon_z \mathcal{I}(\Xi)\dot{\Xi} - \frac{c_\eta}{\sqrt{\varepsilon}}\,, \qquad \hat{\Pi}^\varepsilon_z \tau = \Pi^\varepsilon_z \tau$$

  for all other basis vectors τ.
- The core result says

Theorem

The random models $\hat{\Pi}^\varepsilon$ converge weakly to a limiting admissible model $\hat{\Pi}$ such that with ξ = space-time white noise,

$$\hat{\Pi}\,\Xi = \xi\,, \qquad \hat{\Pi}\,\dot{\Xi} = 0\,, \qquad \hat{\Pi}\,\ddot{\Xi} = 0\,,$$
$$\hat{\Pi}\,\mathcal{I}(\Xi)\dot{\Xi} = 0\,, \qquad \hat{\Pi}\,\mathcal{I}(\dot{\Xi})\dot{\Xi} = c_\eta\,, \qquad \hat{\Pi}\,\mathcal{I}(\Xi)\ddot{\Xi} = c_\eta$$

- As a consequence

Corollary

The sequence v_ε converges weakly to the limit v given by the solution to

$$\partial_t v = \partial_x^2 v + \big(H + c_\eta G G'\big)'(\bar{u})\, v + G(\bar{u})\,\xi\;, \quad v(0,\cdot) = 0\;.$$

- Among the various technical aspects which I have left under the rug is the treatment of the boundary condition, for which we need the very recent work of Màtè Derencsér and Martin Hairer.

- As a consequence

Corollary

The sequence v_ε converges weakly to the limit v given by the solution to

$$\partial_t v = \partial_x^2 v + \big(H + c_\eta GG'\big)'(\bar{u})\, v + G(\bar{u})\,\xi\ , \quad v(0,\cdot) = 0\ .$$

- Among the various technical aspects which I have left under the rug is the treatment of the boundary condition, for which we need the very recent work of Màtè Derencsér and Martin Hairer.

$d > 1$

- After all, the above result makes sense in dimension $d > 1$. The following is work in progress.
- Consider in dimension $d = 1, 2, 3$ the SPDE

$$\partial_t u_\varepsilon(t,x) = \Delta u_\varepsilon(t,x) + H(u_\varepsilon(t,x)) + G(u_\varepsilon(t,x))\eta_\varepsilon(t,x)$$
$$u_\varepsilon(0,x) = u_0(x),\ x \in D \quad u_\varepsilon(t,x) = 0,\ x \in \partial D.$$

where

$$\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

and $\eta(t,x)$ is a stationary zero–mean generalized random field with "good" mixing properties.

$d > 1$

- After all, the above result makes sense in dimension $d > 1$. The following is work in progress.
- Consider in dimension $d = 1, 2, 3$ the SPDE

$$\partial_t u_\varepsilon(t,x) = \Delta u_\varepsilon(t,x) + H(u_\varepsilon(t,x)) + G(u_\varepsilon(t,x))\eta_\varepsilon(t,x)$$
$$u_\varepsilon(0,x) = u_0(x),\ x \in D \quad u_\varepsilon(t,x) = 0,\ x \in \partial D.$$

where

$$\eta_\varepsilon(t,x) = \varepsilon^{-1}\eta(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

and $\eta(t,x)$ is a stationary zero–mean generalized random field with "good" mixing properties.

Results in case $1 \leq d \leq 3$

- Consider the following deterministic PDEs ($H_\eta = H + c_\eta GG'$)

$$\partial_t \overline{u}^0 = \Delta \overline{u}^0 + H_\eta(\overline{u}^0).$$

$$\partial_t \overline{u}^1 = \Delta \overline{u}^1 + \Psi(\overline{u}^0, \nabla \overline{u}^0),$$
$$\Psi(\overline{u}^0, \nabla \overline{u}^0) = c_3 (G^2 G'')(\overline{u}^0) + c_3' (G(G')^2)(\overline{u}^0) + G'(\overline{u}^0)^2 c_2^i \partial_i \overline{u}^0.$$

- LLN $u_\varepsilon \to \overline{u}^0$ in probability.
- CLT $\varepsilon^{-d/2}(u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1) \Rightarrow v$, where

$$\partial_t v = \Delta v + H_\eta'(\overline{u}^0) v + G(\overline{u}^0)\xi,$$

where ξ is space–time white noise.

Results in case $1 \le d \le 3$

- Consider the following deterministic PDEs ($H_\eta = H + c_\eta G G'$)

$$\partial_t \overline{u}^0 = \Delta \overline{u}^0 + H_\eta(\overline{u}^0).$$

$$\partial_t \overline{u}^1 = \Delta \overline{u}^1 + \Psi(\overline{u}^0, \nabla \overline{u}^0),$$
$$\Psi(\overline{u}^0, \nabla \overline{u}^0) = c_3 (G^2 G'')(\overline{u}^0) + c_3' (G (G')^2)(\overline{u}^0) + G'(\overline{u}^0)^2 c_2^i \partial_i \overline{u}^0.$$

- LLN $u_\varepsilon \to \overline{u}^0$ in probability.
- CLT $\varepsilon^{-d/2}(u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1) \Rightarrow v$, where

$$\partial_t v = \Delta v + H_\eta'(\overline{u}^0) v + G(\overline{u}^0) \xi,$$

where ξ is space–time white noise.

Results in case $1 \leq d \leq 3$

- Consider the following deterministic PDEs ($H_\eta = H + c_\eta GG'$)

$$\partial_t \overline{u}^0 = \Delta \overline{u}^0 + H_\eta(\overline{u}^0).$$

$$\begin{aligned} \partial_t \overline{u}^1 &= \Delta \overline{u}^1 + \Psi(\overline{u}^0, \nabla \overline{u}^0), \\ \Psi(\overline{u}^0, \nabla \overline{u}^0) &= c_3 (G^2 G'')(\overline{u}^0) + c_3' (G(G')^2)(\overline{u}^0) + G'(\overline{u}^0)^2 c_2^i \partial_i \overline{u}^0. \end{aligned}$$

- LLN $u_\varepsilon \to \overline{u}^0$ in probability.
- CLT $\varepsilon^{-d/2}(u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1) \Rightarrow v$, where

$$\partial_t v = \Delta v + H'_\eta(\overline{u}^0) v + G(\overline{u}^0)\xi,$$

where ξ is space–time white noise.

CLT

- Recall from previous slide :

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1}{\varepsilon^{d/2}} \Rightarrow v.$$

- In case $d = 1$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon^{1/2}} \Rightarrow v.$$

- In case $d = 2$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1 + v.$$

- In case $d = 3$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1,$$

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1}{\varepsilon^{3/2}} \Rightarrow v.$$

CLT

- Recall from previous slide :

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon\overline{u}^1}{\varepsilon^{d/2}} \Rightarrow v.$$

- In case $d = 1$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon^{1/2}} \Rightarrow v.$$

- In case $d = 2$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1 + v.$$

- In case $d = 3$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1,$$

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon\overline{u}^1}{\varepsilon^{3/2}} \Rightarrow v.$$

CLT

- Recall from previous slide :

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1}{\varepsilon^{d/2}} \Rightarrow v.$$

- In case $d = 1$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon^{1/2}} \Rightarrow v.$$

- In case $d = 2$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1 + v.$$

- In case $d = 3$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1,$$

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1}{\varepsilon^{3/2}} \Rightarrow v.$$

CLT

- Recall from previous slide :

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1}{\varepsilon^{d/2}} \Rightarrow v.$$

- In case $d = 1$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon^{1/2}} \Rightarrow v.$$

- In case $d = 2$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1 + v.$$

- In case $d = 3$, we have

$$\frac{u_\varepsilon - \overline{u}^0}{\varepsilon} \Rightarrow \overline{u}^1,$$

$$\frac{u_\varepsilon - \overline{u}^0 - \varepsilon \overline{u}^1}{\varepsilon^{3/2}} \Rightarrow v.$$

THANK YOU FOR
YOUR ATTENTION !