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MEAN FIELD LIMITS OF INTERACTING DIFFUSIONS IN TWO-SCALE POTENTIALS

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- *Mean field limits for interacting diffusions in a two-scale potential* (S.N. Gomes and G.A. Pavliotis), Preprint (2017)
- *Brownian motion in an N-scale periodic potential* (A.B. Duncan and G.A. Pavliotis). Submitted to SIAM J MMS (2016).
- *Noise-induced transitions in rugged energy landscapes* (A.B. Duncan, S. Kalliadasis, G.A. Pavliotis, M. Pradas). Phys. Rev. E, 94, 032107 (2016).

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TEXTS IN APPLIED MATHEMATICS
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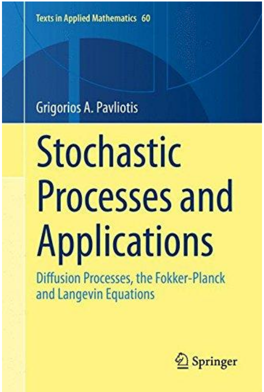
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- We consider a system of weakly interacting diffusions moving in a 2-scale locally periodic potential:

$$dX_t^i = -\nabla V^\epsilon(X_t^i)dt - \frac{1}{N}\sum_{j=1}^{N} \nabla F(X_t^i - X_t^j)dt + \sqrt{2\beta^{-1}}dB_t^i, \quad i = 1, .., N \tag{1}$$

- where

$$V^\epsilon(x) = V_0(x) + V_1(x, x/\epsilon). \tag{2}$$

- Our goal is to study the combined mean-field/homogenization limits.
- In particular, we want to study bifurcations/phase transitions for the McKean-Vlasov equation in a confining potential with many local minima.

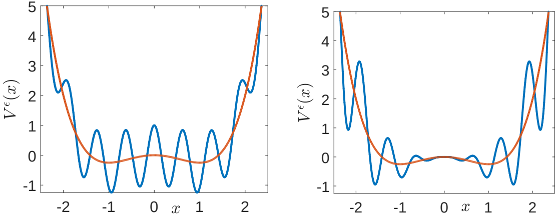


Figure: Bistable potential with (left) separable and (right) nonseparable fluctuations,
$V^\epsilon(x) = \frac{x^4}{4} - \frac{x^2}{2} + \delta \cos\left(\frac{x}{\epsilon}\right)$ and $V^\epsilon(x) = \frac{x^4}{4} - \left(1 - \delta \cos\left(\frac{x}{\epsilon}\right)\right)\frac{x^2}{2}$.

MCKEAN-VLASOV DYNAMICS IN A BISTABLE POTENTIAL

- Consider a system of interacting diffusions in a bistable potential:

$$dX_t^i = \left(-V'(X_t^i) - \theta \left(X_t^i - \frac{1}{N} \sum_{j=1}^{N} X_t^j \right) \right) dt + \sqrt{2\beta^{-1}}\, dB_t^i. \tag{3}$$

- The total energy (Hamiltonian) is

$$W_N(\mathbf{X}) = \sum_{\ell=1}^{N} V(X^\ell) + \frac{\theta}{4N} \sum_{n=1}^{N} \sum_{\ell=1}^{N} (X^n - X^\ell)^2. \tag{4}$$

- We can pass rigorously to the mean field limit as $N \to \infty$ using, for example, martingale techniques, (Dawson 1983, Gartner 1988, Oelschlager 1984).
- Formally, using the law of large numbers we obtain the McKean SDE

$$dX_t = -V'(X_t)\, dt - \theta(X_t - \mathbb{E}X_t)\, dt + \sqrt{2\beta^{-1}}\, dB_t. \tag{5}$$

- The Fokker-Planck equation corresponding to this SDE is the McKean-Vlasov equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(V'(x)p + \theta\left(x - \int_{\mathbb{R}} xp(x,t)\,dx\right)p + \beta^{-1}\frac{\partial p}{\partial x}\right). \tag{6}$$

- The McKean-Vlasov equation is a gradient flow, with respect to the Wasserstein metric, for the free energy functional

$$\mathcal{F}[\rho] = \beta^{-1}\int \rho \ln \rho\,dx + \int V\rho\,dx + \frac{\theta}{2}\int\int F(x-y)\rho(x)\rho(y)\,dxdy, \tag{7}$$

with $F(x) = \frac{1}{2}x^2$.

- The finite dimensional dynamics (3) is reversible with respect to the Gibbs measure

$$\mu_N(dx) = \frac{1}{Z_N} e^{-\beta W_N(x^1,\dots x^N)}\, dx^1 \dots dx^N, \quad Z_N = \int_{\mathbb{R}^N} e^{-\beta W_N(x^1,\dots x^N)} \tag{8}$$

- where $W_N(\cdot)$ is given by (4).
- the McKean dynamics (5) can have more than one invariant measures, for nonconvex confining potentials and at sufficiently low temperatures (Dawson1983, Tamura 1984, Shiino 1987, Tugaut 2014).
- The density of the invariant measure(s) for the McKean dynamics (5) satisfies the stationary nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial x}\left(V'(x)p_\infty + \theta\left(x - \int_{\mathbb{R}} x p_\infty(x)\,dx\right)p_\infty + \beta^{-1}\frac{\partial p_\infty}{\partial x}\right) = 0. \tag{9}$$

- For the quadratic interaction potential a one-parameter family of solutions to the stationary McKean-Vlasov equation (9) can be obtained:

$$p_\infty(x;\theta,\beta,m) = \frac{1}{Z(\theta,\beta;m)} e^{-\beta\left(V(x)+\theta\left(\frac{1}{2}x^2-xm\right)\right)}, \tag{10a}$$

$$Z(\theta,\beta;m) = \int_{\mathbb{R}} e^{-\beta\left(V(x)+\theta\left(\frac{1}{2}x^2-xm\right)\right)}\, dx. \tag{10b}$$

- These solutions are subject, to the constraint that they provide us with the correct formula for the first moment:

$$m = \int_{\mathbb{R}} x p_\infty(x;\theta,\beta,m)\, dx =: R(m;\theta,\beta). \tag{11}$$

- This is the **selfconsistency** equation.
- The critical temperature can be calculated from

$$\mathrm{Var}_{p_\infty}(x)\Big|_{m=0} = \frac{1}{\beta\theta}. \tag{12}$$

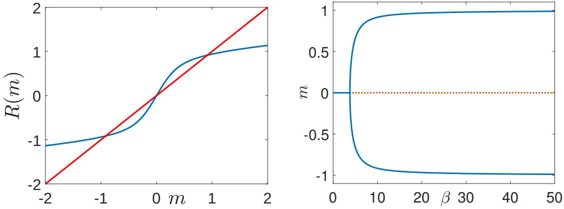


Figure: Plot of $R(m; \theta, \beta)$ and of the straight line $y = x$ for $\theta = 0.5$, $\beta = 10$, and bifurcation diagram of m as a function of β for $\theta = 0.5$ for the bistable potential $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and interaction potential $F(x) = \frac{x^2}{2}$.

BROWNIAN PARTICLES IN A TWO-SCALE POTENTIAL

- Dynamics given by Itô SDE:

$$dX_t^\epsilon = -\nabla V^\epsilon(X_t^\epsilon)\, dt + \sqrt{2\beta^{-1}}\, dW_t.$$

- For $\epsilon \ll 1$, V^ϵ models a “rough” potential:

$$V^\epsilon(x) := V\left(x, \frac{x}{\epsilon}, \frac{x}{\epsilon^2}, \dots, \frac{x}{\epsilon^N}\right),$$

  for a smooth function $V(x_0, y_1, \dots, y_N)$.

  - x_0: slowly–varying structure of potential.
  - $y_1, \dots, y_N$: multiscale periodic fluctuations occuring at different scales.

X_t^ϵ is a Markov diffusion process with infinitesimal generator defined by

$$\mathcal{L}^\epsilon f = \beta^{-1} e^{\beta V^\epsilon(x)} \nabla \cdot \left(e^{-\beta V^\epsilon(x)} \nabla f(x) \right).$$

Stationary distribution satisfies the stationary Fokker-Planck equation:

$$\nabla \cdot \left(e^{-\beta V^\epsilon(x)} \nabla (\pi^\epsilon(x) e^{\beta V^\epsilon(x)} \right) = 0, \quad x \in \mathbb{R}^d.$$

Suppose $Z^\epsilon = \int_{\mathbb{R}^d} e^{-\beta V^\epsilon(x)}\, dx < \infty$,

- X_t^ϵ is ergodic, with stationary density $\pi^\epsilon(x) = \frac{1}{Z^\epsilon} e^{-\beta V^\epsilon(x)}$.
- X_t^ϵ satisfies detailed balance with respect to $\pi^\epsilon(x)$, i.e.

  Stationary Probability Flux $= \nabla \cdot \left(\pi^\epsilon(x) e^{\beta V^\epsilon(x)} \right) = 0, \quad \forall x \in \mathbb{R}^d.$

QUESTIONS AND OBJECTIVES

Questions:

- Can behaviour of X_t^ϵ for small ϵ be approximated by some X_t^0?
- X_t^ϵ ergodic $\Rightarrow$ X_t^0 ergodic?
- Relationship between $\pi^\epsilon(\cdot)$ and $\pi^0(\cdot)$?
- Asymptotic behaviour of other quantities related to X_t^ϵ,
  - Observables of X_t^ϵ, e.g. reaction coordinates.
  - Mean First Passage Time (MFPT), as $\epsilon \to 0$.

Approach:

- Formal approach: Asymptotic expansions of the Kolmogorov Backward Equation for X_t^ϵ in powers of $O(\epsilon^{-1})$.
- Rigorous Approach: probabilistic techniques for locally-periodic homogenization, **[Bensoussans, Lyons, Papanicolau, 1979], [Pardoux, 1999], [Pardoux, Veretennikov, 2001], [Bencherif-Madani, Pardoux, 2003]**.

THE HOMOGENIZATION THEOREM

To prove the existence of the limit of X_t^ϵ as $\epsilon \to 0$, we make the following assumptions on V.

- There exist confining potentials $M_0(x)$ and $M_1(x)$ such that

$$M_0(x) \leq V(x, y_1, \dots, y_N) \leq M_1(x), \quad \forall x \in \mathbb{R}^d, y_1, \dots y_N \in \mathbb{T}^d$$

- $V(x, y_1, \dots, y_N)$ is smooth in all variables (can be relaxed).
- The gradient of the potential is Lipschitz in x, i.e.

$$|\nabla V(x, y_1, \dots, y_N) - \nabla V(x', y_1, \dots, y_N)| \leq C|x - x'|.$$

- $|\nabla V(x, y_1, \dots, y_N)| \leq C'|x|$, for some C, C' for all $x, x' \in \mathbb{R}$, $y_1, \dots, y_N \in \mathbb{T}^d$.

HOMOGENIZATION THEOREM

- The limiting dynamics can be characterized by the following Itô SDE:

$$dX_t^0 = -\mathcal{K}(X_t^0)\nabla\Psi(X_t^0)\,dt + \beta^{-1}\nabla\cdot\mathcal{K}(X_t^0)\,dt + \sqrt{2\beta^{-1}\mathcal{K}(X_t^0)}\,dW_t,$$

  where Ψ is the free energy $\Psi(x) = -\beta^{-1}\log Z(x)$.
- The limiting SDE corresponds to the Klimontovich interpretation of the stochastic integral.
- X_t^0 satisfies detailed balance with respect to the invariant measure

$$\pi^0(x) = \frac{1}{\mathcal{Z}}e^{-\Psi(x)} = \frac{Z(x)}{\mathcal{Z}}, \quad \mathcal{Z} = \int Z(x')\,dx'.$$

- For all $e \in \mathbb{R}^d$, with $\hat{Z}(x) = \int\cdots\int e^{\beta V(x,y_1,\dots,y_N)}\,dy_N\dots dy_1$:

$$\frac{|e|^2}{Z(x)\hat{Z}(x)} \leq e\cdot\mathcal{K}(x)e \leq |e|^2,$$

HOMOGENIZATION THEOREM

As $\epsilon \to 0$, the process X_t^ϵ converges weakly in $C([0,T],\mathbb{R}^d)$ to a diffusion process X_t^0 having generator defined by

$$\mathcal{L}^0 f(x) = \frac{\beta^{-1}}{Z(x)} \nabla_x \cdot \left(Z(x) \mathcal{K}(x) \nabla_x f(x) \right), \quad f \in C_c^2(\mathbb{R}^d).$$

where $Z(x) = \int \cdots \int e^{-\beta V(x,\ldots)}\, dy_N \ldots dy_1$, and

$$\mathcal{K}(x) = I + \frac{1}{Z(x)} \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} (I + \nabla_{x_N} \theta_N^\top) \cdots (I + \nabla_{x_1} \theta_1^\top) e^{-\beta V}\, dy_N \ldots dy_1.$$

and θ_k are mean-zero solutions of the following Poisson equations on $\mathbb{T}^d$:

$$\nabla_{y_k} \cdot \left(\mathcal{K}_k (\nabla_{y_k} \theta_k + I) \right) = 0, \quad y \in \mathbb{T}^d$$

where $\mathcal{K}_N(x, y_1, \ldots, y_N) = e^{-\beta V(x, y_1, \ldots, y_N)} I$ and

$$\mathcal{K}_k(x, y_1, \ldots, y_k) = \int (I + \nabla_N \theta_N^\top) \cdots (I + \nabla_{k+1} \theta_{k+1}^\top) e^{-\beta V}\, dy_N \ldots dy_{k+1}.$$

PROOF OF THE HOMOGENIZATION THEOREM

Slight generalisation of classical *martingale approach to homogenization*, applied to SDEs with locally-periodic coefficients having N-scales.

Rough idea:

1. The slow–fast system is the solution to the following martingale problem:

$$\mathbb{E}_x\left[\phi^\epsilon(X_t^\epsilon) - \int_s^t \mathcal{L}^\epsilon\phi^\epsilon(X_u^\epsilon)\,du \,\middle|\, \mathcal{F}_s\right] = \phi^\epsilon(X_s^\epsilon), \quad \forall \phi^\epsilon \in \mathcal{D}(\mathcal{L}^\epsilon).$$

Construct a test function

$$\phi^\epsilon(x) = \phi_0(x) + \epsilon\phi_1(x, x/\epsilon) + \ldots + \epsilon^N\phi_N(x, x/\epsilon, \ldots, x/\epsilon^N) + \ldots$$

such that

$$\mathcal{L}^\epsilon\phi^\epsilon(x) = \mathcal{L}^0\phi_0(x) + \epsilon R^\epsilon(x),$$

where $E_x[\epsilon R^\epsilon(X_u^\epsilon)] \to 0$, as $\epsilon \to 0$.

1. If the set of measures $\mathbb{P}^\epsilon$ on $C([0,T],\mathbb{R}^d)$ corresponding to the processes $\{X_t^\epsilon, t \in [0,T]\}$ possesses a limit point X_t^0 then it is the unique solution of the following martingale problem

$$\mathbb{E}_x \left[\phi_0(X^0) - \int_s^t \mathcal{L}^0 \phi_0(X_u^\epsilon)\, du \,\middle|\, \mathcal{F}_s \right] = \phi_0(X_s^\epsilon), \quad \forall \phi \in \mathcal{D}(\mathcal{L}^0).$$

2. Show that $\{X_t^\epsilon\}_{\epsilon>0}$ possesses an accumulation point. i.e. Establish tightness of the family of processes in $\{X_t^\epsilon\}_{\epsilon>0}$.

CRITICAL POINTS OF THE INVARIANT DISTRIBUTION

- we want to calculate the critical points of the stationary distribution:

$$\nabla Z(x; \beta) = 0.$$

- Multiplicative noise can change the location and number of the critical points.
- We distinguish between two cases:

1. Separable Potential: the fluctuations and large scale parts of the potential are uncoupled:

$$V^{\epsilon}(x) = V_0(x) + V_1(x/\epsilon, x/\epsilon^2, \dots x/\epsilon^N).$$

   In this case:

$$Z(x) \propto \int \cdots \int e^{-\beta V(x, y_1, \dots y_N)} dy_N \dots dy_1 \propto e^{-V_0(x)}.$$

   and $\mathcal{K}$ is independent of x. **Rapid fluctuations do not alter stationary behaviour, but only speed of convergence to equilibrium and effective diffusion tensor.**

2. Nonseparable potential. In this case

$$Z(x) \not\propto e^{-V_0(x)}, \quad \text{in general.}$$

   Rapid fluctuations can change the critical points of the stationary distribution.

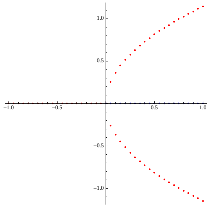
TOY EXAMPLE: 1D DOUBLE WELL POTENTIAL

Consider the ODE in $\mathbb{R}$:

$$\dot{x}(t) = -\frac{d}{dx} V_0(x;\alpha), \quad t > 0,$$

where $V_0(x;\alpha) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4$, corresponding to the invariant density $e^{-\beta V(x)}$.

- Normal form for supercritical pitchfork bifurcation.
- $\alpha < 0$: One stable equilibrium at $x = 0$.
- $\alpha > 0$: Stable equilibria at $x = \pm\sqrt{\alpha}$. Unstable equilibrium at $x = 0$.



1D DOUBLE WELL POTENTIAL

Consider the ODE in $\mathbb{R}$:

$$\dot{x}(t) = -\frac{d}{dx} V_0(x;\alpha), \quad t > 0,$$

where $V_0(x;\alpha) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4$.

Add multiscale fluctuations $V^\epsilon(x;\alpha) = V(x, x/\epsilon;\alpha)$, where

$$V(x, y;\alpha) = \frac{1}{4}x^4 - \left(\frac{\alpha + \sin(2\pi y)}{2}\right) x^2.$$

Thermal motion in potential:

$$dX_t^\epsilon = -\frac{dV^\epsilon}{dx}(X_t^\epsilon)\, dt + \sqrt{2\beta^{-1}}\, dW_t.$$

1D DOUBLE WELL POTENTIAL

By previous theory, $X_t^\epsilon \Rightarrow X_t^0$, as $\epsilon \to 0$, where X_t^0 is ergodic with stationary distribution

$$\pi^0(dx) \propto Z(x)\, dx$$

Can show that

$$\pi^0(x) \propto \underbrace{e^{\beta\left(\frac{\alpha^2 x^2}{2} - \frac{x^4}{4}\right)}}_{\pi_0(x)} \underbrace{I\left(0, \frac{x^2}{2\beta^{-1}}\right)}_{\text{correction}},$$

where I is the modified Bessel function of the first kind.

Varying the intensity of the noise can alter the equilibrium properties of the system, i.e. the critical points of the stationary distribution.

1D DOUBLE WELL POTENTIAL

The strength of the noise now plays an interesting role in the dynamics.

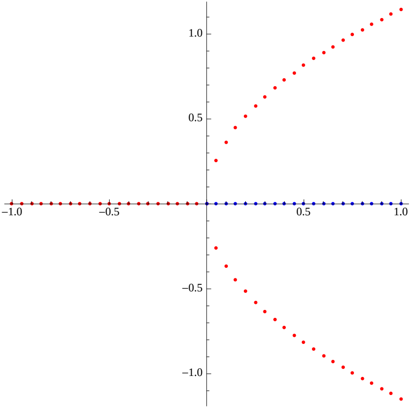


Figure: $\beta^{-1} = 1.0$

1D DOUBLE WELL POTENTIAL

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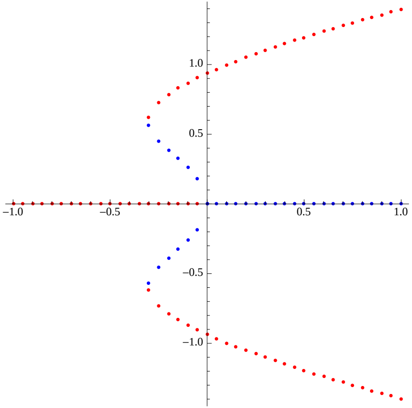


Figure: $\beta^{-1} = 10^{-1}$

1D DOUBLE WELL POTENTIAL

The strength of the noise now plays an interesting role in the dynamics.

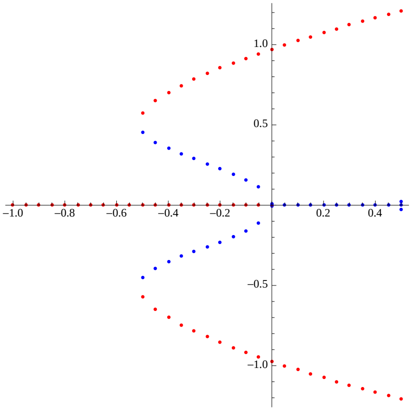


Figure: $\beta^{-1} = 5 \cdot 10^{-2}$

More generally: consider an N-scale potential

$$V^{\epsilon}(x;\alpha) = V_0(x;\alpha) - \frac{1}{2}\sum_{n=1}^{N} \sin\left(\frac{2\pi x}{\epsilon^n}\right) x^2$$

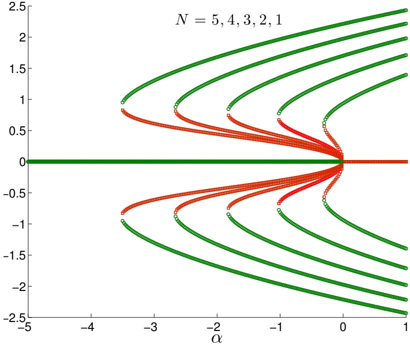


Figure: Bifurcation diagram for a different number N of microscopic scales in the potential

Stationary PDF of homogenized dynamics is:

$$Z_N(x;\alpha) \propto e^{-\beta V_0(x;\alpha)} I\left(0, \frac{x^2}{2\beta^{-1/2}}\right)^N .$$

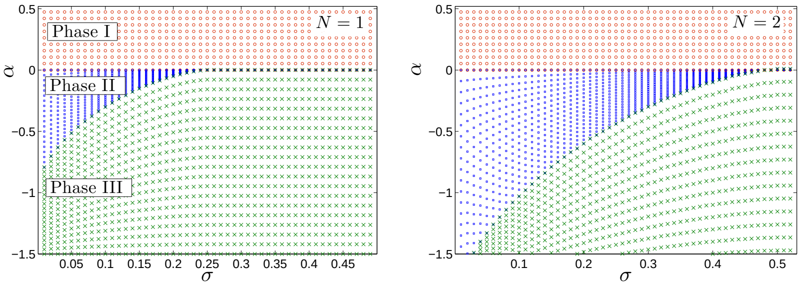


Figure: Phase diagram for α and σ

EFFECT OF NUMBER OF SCALES ON DIFFUSIVITY TENSOR

As the number of scales increase, the effective diffusivity $K(x)$ decreases.

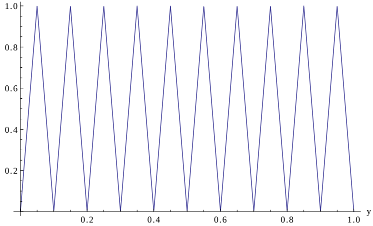
Must increase temperature β^{-1} to overcome "trapping effect" of regions of slow diffusivity. Consider separable N-scale potential

$$V^{\epsilon}(x) = S\left(x/\epsilon\right) + \ldots + S\left(x/\epsilon^{N}\right),$$

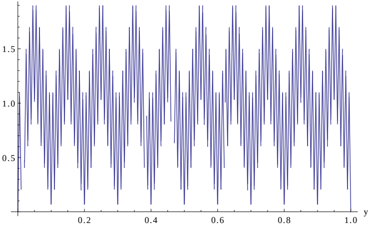
where

$$S(x) = \begin{cases} 2x & \text{if } x \bmod 1 \in [0, \frac{1}{2}) \\ 2 - 2x & \text{if } x \bmod 1 \in [\frac{1}{2}, 1) \end{cases}$$

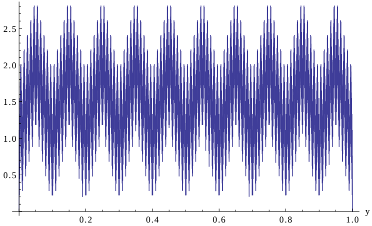
EFFECT OF NUMBER OF SCALES ON DIFFUSIVITY TENSOR



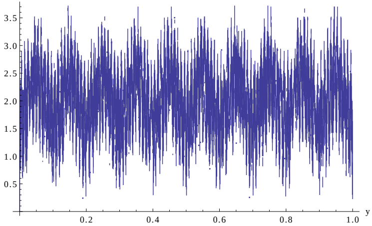
(a) $N = 1.$



(b) $N = 2.$



(c) $N = 3.$



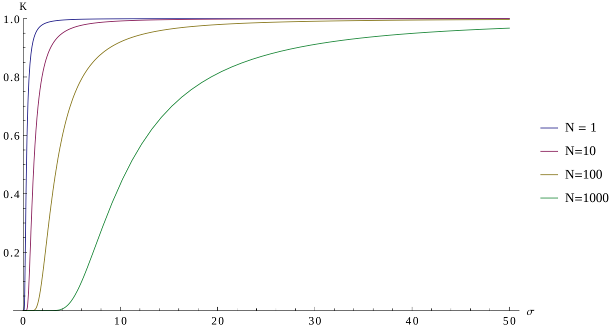
(d) $N = 3.$

EFFECT OF NUMBER OF SCALES ON DIFFUSIVITY TENSOR

As $\epsilon \to 0$, $X_t^\epsilon \Rightarrow X_t^0$, where

$$dX_t^0 = \sqrt{\frac{2\sigma}{K(\sigma)^N}}dW_t$$

where $\sigma = \beta^{-1}$, for $K(\sigma) = 2\sigma^2\left(\cosh\left(\frac{1}{\sigma}\right) - 1\right)$.



MEAN FIELD LIMITS FOR INTERACTING DIFFUSIONS IN A TWO-SCALE POTENTIAL

- We consider a system of weakly interacting diffusions moving in a 2-scale locally periodic potential:

$$dX_t^i = -\nabla V^\epsilon(X_t^i)dt - \frac{1}{N}\sum_{j=1}^{N} \nabla F(X_t^i - X_t^j)dt + \sqrt{2\beta^{-1}}dB_t^i, \quad i = 1, .., N \tag{13}$$

- where

$$V^\epsilon(x) = V_0(x) + V_1(x, x/\epsilon). \tag{14}$$

- The full N-particle potential is

$$\begin{aligned} U(x_1, \dots, x_N, y_1, \dots, y_N) \; &= \; \sum_{i=1}^{N} V_0(x_i) + \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} F(x_i - x_j) \\ &\quad + \sum_{i=1}^{N} V_1(x_i, y_i). \end{aligned} \tag{15}$$

- The homogenization theorem applies to the N-particle system.

The homogenized equation is

$$dX_t^i = - M(X_t^i)\left(\nabla V_0(X_t^i) + \frac{1}{N}\sum_{i\neq j}\nabla F\left(X_t^j - X_t^i\right) + \nabla\psi(X_t^i)\right)dt$$
$$+ \beta^{-1}\nabla\cdot M(X_t^i)dt + \sqrt{2\beta^{-1}M(X_t^i)}dW_t^i, \tag{16}$$

for $i = 1, \dots, N$, where $M : \mathbb{R}^d \to \mathbb{R}^{d\times d}_{sym}$ is defined by

$$M(x) = \frac{1}{Z(x)}\int_{\mathbb{T}^d}\int (I + \nabla_y\theta(x,y))e^{-\beta V_1(x,y)}dy, \quad x\in\mathbb{R}^d, \tag{17}$$

and

$$\psi(x) = -\beta^{-1}\nabla\log Z(x), \tag{18}$$

for (this is the free energy **only** with respect to $V_1(x,y)$)

$$Z(x) = \int_{\mathbb{T}^d} e^{-\beta V_1(x,y)}\,dy,$$

and where, for fixed $x\in\mathbb{R}^d$, θ is the unique mean zero solution to

$$\nabla\cdot\left(e^{-\beta V_1(x,y)}(I + \nabla_y\theta(x,y))\right) = 0, \quad y\in\mathbb{T}^d, \tag{19}$$

- We can pass to the mean field limit $N \to +\infty$ using the results from e.g. Dawson (1983), Oelschlager (1984) to obtain the McKean-Vlasov-Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = \nabla \cdot \Big(M(\nabla V_0 p + \nabla \Psi p + (\nabla F * p) p) + \beta^{-1} \nabla \cdot M p + \beta^{-1} \nabla \cdot (M p) \Big). \tag{20}$$

- The mean field $N \to +\infty$ and the homogenization $\epsilon \to 0$ limits commute **over finite time intervals**.
- This is a nonlinear equation and more than one invariant measures can exist, depending on the temperature. Eqn (20) can exhibit **phase transitions**.
- The number of invariant measures depends on the number of solutions of the self-consistency equation.

- The phase/bifurcation diagrams can be different depending on the order with which we take the limits. For example:

$$V^{\epsilon}(x) = \frac{x^2}{2} + \cos(x/\epsilon).$$

- The homogenization process tends to "convexify" the potential.

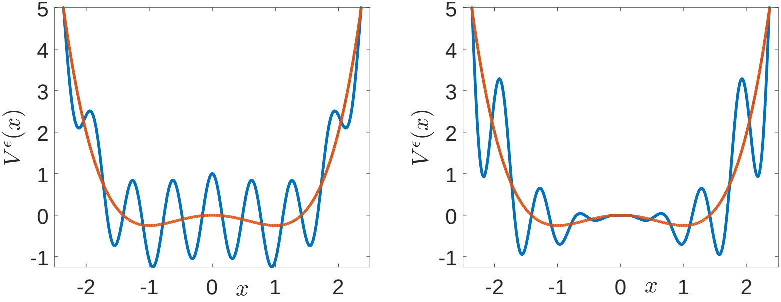


Figure: Bistable potential with additive (left) and multiplicative (right) fluctuations.

- Consider the case $F(x) = \theta\frac{x^2}{2}$, take $N \to +\infty$ and keep ϵ fixed. The invariant distribution(s) are:

$$
\begin{aligned}
p^\epsilon(x; m, \theta, \beta) &= \frac{1}{Z^\epsilon} e^{-\beta(V^\epsilon(x)+\theta(\frac{1}{2}x^2 - x\,m))}, & \text{(21a)}\\
Z^\epsilon &= \int e^{-\beta(V^\epsilon(x)+\theta(\frac{1}{2}x^2 - x\,m))}\,dx, & \text{(21b)}
\end{aligned}
$$

- where

$$
m = \int x p^\epsilon(x; m, \theta, \beta)\,dx. \tag{22}
$$

- Take first $\epsilon \to 0$ and then $N \to +\infty$. The invariant distribution(s) are

$$
\begin{aligned}
p(x; m, \theta, \beta) &= \frac{1}{Z} e^{-\beta(V_0(x)+\psi(x)+\theta(\frac{1}{2}x^2 - x\,m))}, & \text{(23a)}\\
Z &= \int e^{-\beta(V_0(x)+\psi(x)+\theta(\frac{1}{2}x^2 - x\,m))}\,dy, & \text{(23b)}
\end{aligned}
$$

- where

$$
m = \int x p(x; m, \theta, \beta)\,dx. \tag{24}
$$

- The number of invariant measures is given by the number of solutions to the self-consistency equations (22) and (24).
- Separable fluctuations $V_0(x) + V_1(x/\epsilon)$ do not change the structure of the phase diagram, since they lead to additive noise. Nonseparable fluctuations $V_0(x) + V_1(x, x/\epsilon)$ lead to multiplicative noise and change the bifurcation diagram.
- Rigorous results for the $\epsilon \to 0$, $N \to +\infty$ limits, formal asymptotics for the opposite limit.

- The structure of the bifurcation diagram for the homogenized dynamics is similar to the one for the dynamics in the absence of fluctuations.
- The critical temperature is different, but there are no additional branches and their stability is the same as in the case $V_1 = 0$.
- This is the case both for additive and multiplicative oscillations.

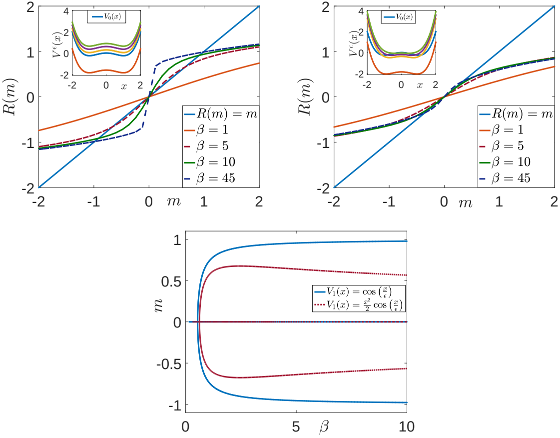


Figure: $R_{hom}(m;\theta,\beta)$ compared to $y = x$ for $\theta = 0.5, \delta = 1$ and various values of β for the homogenized bistable potentials with separable and nonseparable fluctuations. Bifurcation diagram of m as a function of β for the additive (full line) and multiplicative (dashed line) fluctuations.

COMMUTATIVITY FOR SEPARABLE POTENTIALS

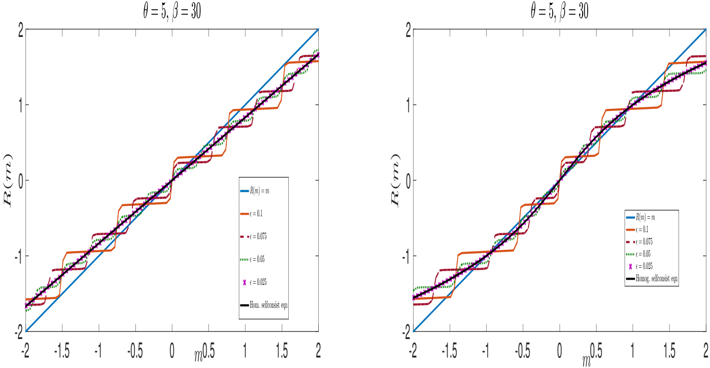


Figure: Plot of $R(m;\theta,\beta) = m$ and $R(m^{\epsilon};\theta,\beta)$ for $\theta = 5$, $\beta = 30, \delta = 1$ and various values of ϵ for separable fluctuations. Convex potential $V_0(x)$ and Bistable potential $V_0(x)$.

NONCOMMUTATIVITY FOR SEPARABLE POTENTIALS

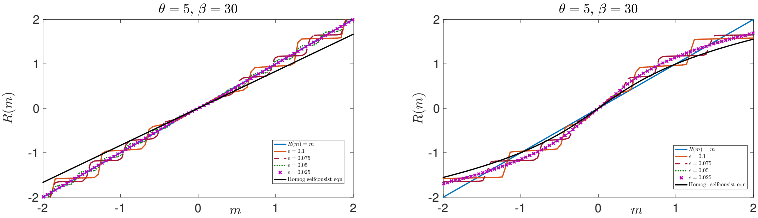


Figure: Plot of $R(m;\theta,\beta) = m$ and $R(m;\epsilon)$ for $\theta = 5$, $\beta = 30, \delta = 1$ and various values of ϵ where the fluctuations are nonseparable. Convex potential $V_0(x)$ and Bistable potential $V_0(x)$.

FINITE ϵ: SEPARABLE FLUCTUATIONS I

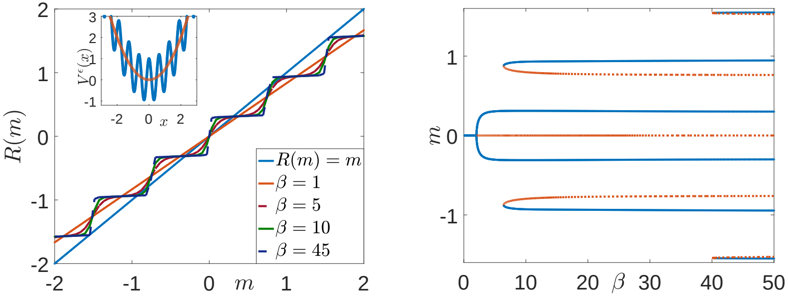


Figure: Results for case 1: convex V_0^c with separable fluctuations, for $\theta = 5$, $\delta = 1$, $\epsilon = 0.1$. $R(m^\epsilon; \theta, \beta)$ for various values of β, with the potential $V^\epsilon(x)$ (full line) compared with $V_0^c(x)$ (dashed line) in the inside panel. Bifurcation diagram of m as a function of β. Full lines correspond to stable solutions, while dashed lines represent unstable ones.

FINITE ϵ: NONSEPARABLE FLUCTUATIONS I

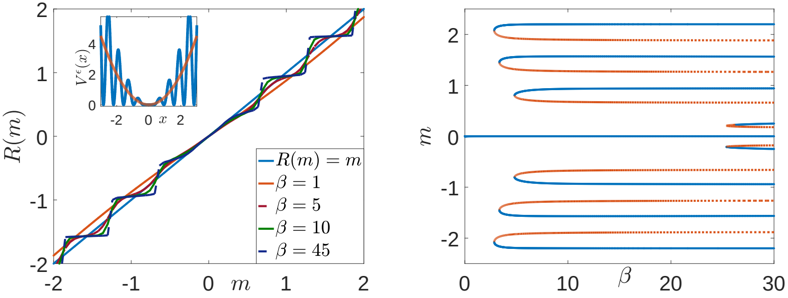


Figure: Results for case 2: convex V_0 with nonseparable fluctuations, for $\theta = 5$, $\delta = 1$, $\epsilon = 0.1$. $R(m; \theta, \beta)$ for various values of β, with the potential $V^{\epsilon}(x)$ (full line) compared with $V_0^c(x)$ (dashed line) in the inside panel. Bifurcation diagram of m as a function of β. Full lines correspond to stable solutions, while dashed lines represent unstable ones.

THE MEAN ZERO SOLN IS THE MINIMIZER OF $F[\rho]$

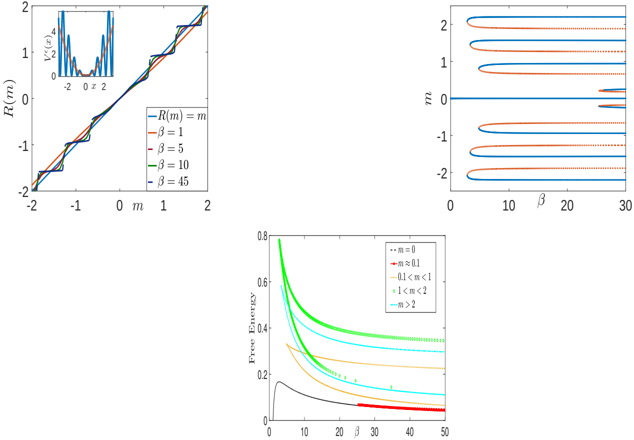


Figure: Convex V_0 with nonseparable fluctuations, for $\theta = 5,\ \delta = 1,\ \epsilon = 0.1$. $R(m;\theta,\beta)$, bifurcation diagram of m as a function of β. Full lines correspond to stable solutions, while dashed lines represent unstable ones. Values of the free energy of the steady state in each branch of the bifurcation diagram for $\beta = 45$. Free energy of each branch of the bifurcation diagram.

FINITE ϵ: SEPARABLE FLUCTUATIONS II

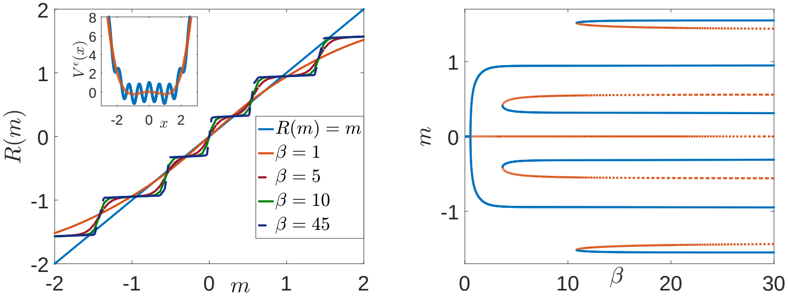


Figure: Bistable V_0^b with separable fluctuations, for $\theta = 5$, $\delta = 1$, $\epsilon = 0.1$. $R(m^\epsilon; \theta, \beta)$ for various values of β. Bifurcation diagram of m as a function of β. Full lines correspond to stable solutions, while dashed lines represent unstable ones.

FINITE ϵ: NONSEPARABLE FLUCTUATIONS II

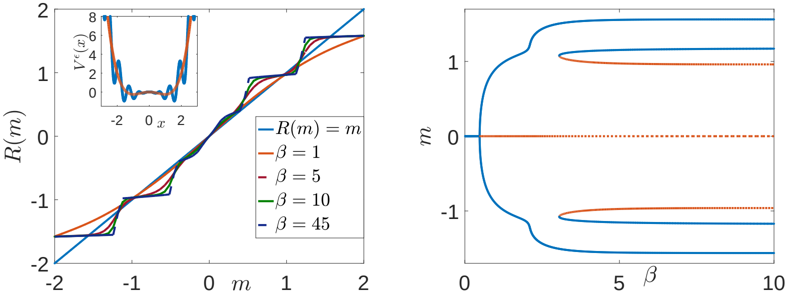


Figure: Bistable V_0^b with nonseparable fluctuations, for $\theta = 5$, $\delta = 1$, $\epsilon = 0.1$. $R(m\epsilon; \theta, \beta)$ for various values of β. Bifurcation diagram of m as a function of β. Full lines correspond to stable solutions, while dashed lines represent unstable ones.

- We can study the dependence of the critical temperature β_C on ϵ.
- We study solutions of the equation

$$\theta^{-1}\beta^{-1} = \int x^2 p_\infty(x; m = 0)\ dx. \tag{25}$$

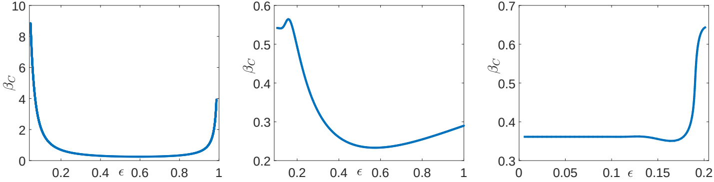


Figure: Critical temperature β_C as a function of ϵ for the multiscale Fokker-Planck equation with $\theta = 5$ for 1- $V^\epsilon(x) = \frac{x^2}{2} + \delta \cos\left(\frac{x}{\epsilon}\right)$, 2 - $V^\epsilon(x) = \frac{x^4}{4} - \frac{x^2}{2} + \delta \cos\left(\frac{x}{\epsilon}\right)$, and 3 - $V^\epsilon(x) = \frac{x^4}{4} - \frac{x^2}{2}\left(1 - \delta \cos\left(\frac{x}{\epsilon}\right)\right)$.

NONCOMMUTATIVITY: PARTICLE SIMULATIONS

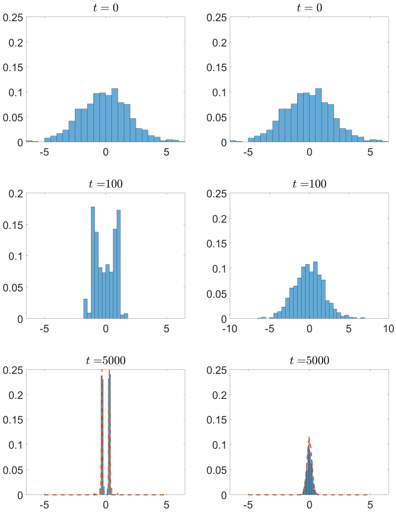


Figure: Histogram of $N = 1000$ particles for a MC simulation of a convex potential with separable fluctuations. Parameters used were $\theta = 2$, $\beta = 8$, $\delta = 1$. Left: $\epsilon = 0.1$. Right: homogenized system.

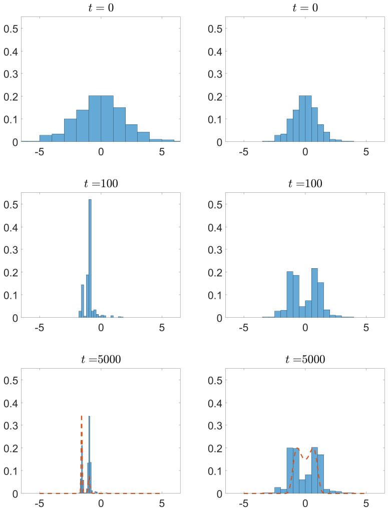


Figure: Histogram of $N = 500$ particles for an MC simulation of a bistable potential with nonseparable fluctuations. Parameters used were $\theta = 0.5$, $\beta \approx 5.6$, $\delta = 1$. Left: $\epsilon = 0.1$. Right: homogenized system.

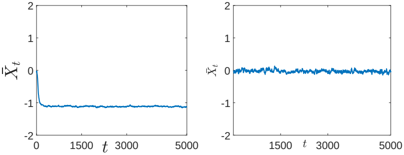


Figure: Time evolution of the mean $\bar{X}_t$ of $N = 500$ particles for an MC simulation of a bistable potential with separable fluctuations. Parameters used were $\theta = 0.5$, $\beta \approx 5.6$, $\delta = 1$. Left: $\epsilon = 0.1$. Right: homogenized system.

NONCOMMUTATIVITY: MCKEAN-VLASOV EVOLUTION

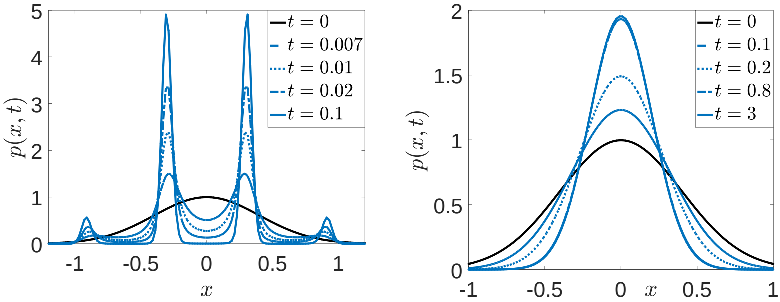


Figure: Time evolution of $p(x,t)$ for $V_0(x) = \frac{x^2}{2} + \delta \cos\left(\frac{x}{\epsilon}\right)$. Parameters used were $\theta = 2$, $\beta = 8$, $\delta = 1$. Left: $\epsilon = 0.1$. Right: homogenized system.

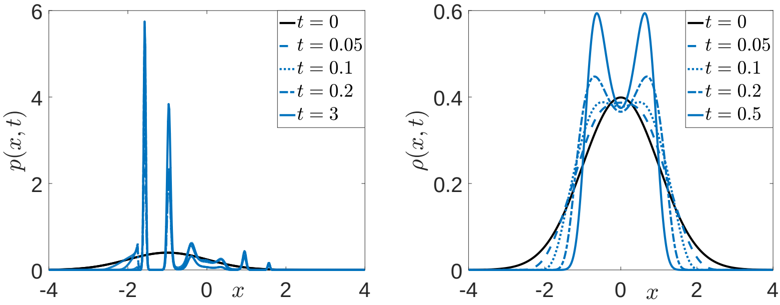


Figure: Time evolution of $p(x,t)$ when $V_0^b(x)$ is a bistable potential and with nonseparable fluctuations. Parameters used were $\theta = 0.5$, $\beta \approx 5.6$, $\delta = 1$. Left: $\epsilon = 0.1$. Right: homogenized system.

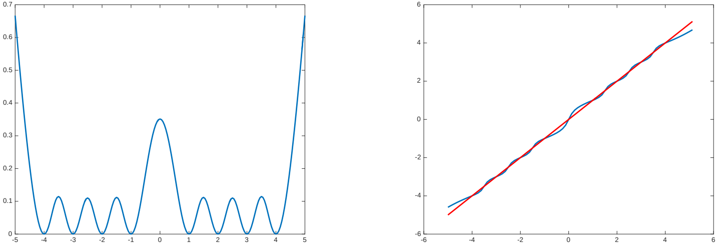


Figure: Potential and solution of self-consistency equation for the potential $V(q) = \frac{1}{\sum_{\ell=-N}^{N} |q - q_\ell|^{-2}}$ (used in the Thesis of Dr Z. Trstanova).

NON-PERIODIC MULTIWELL POTENTIALS

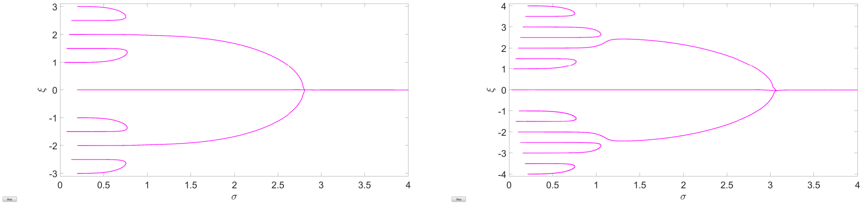


Figure: Bifurcation diagram for for the potential $V(q) = \frac{1}{\sum_{\ell=-N}^{N} |q - q_\ell|^{-2}}$ for the order parameter m as a function of β^{-1} for $N = 6$ and $N = 8$.

NON-PERIODIC MULTIWELL POTENTIALS

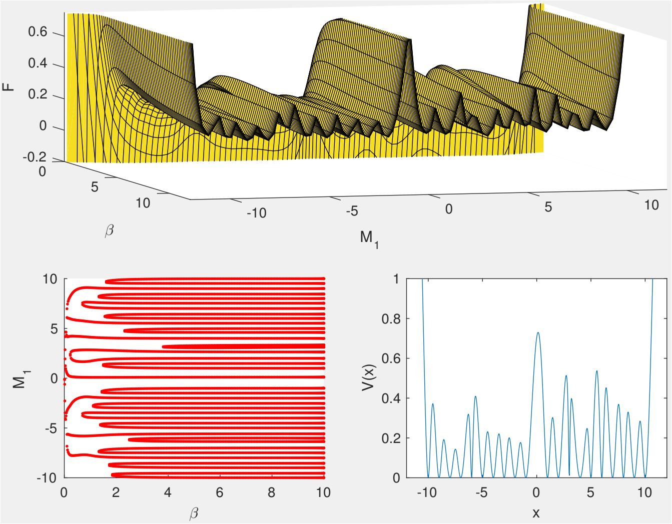


Figure: Free energy surface as a function of β and the first moment m for potential $V(q) = \frac{1}{\sum_{\ell=-N}^{N} |q-q_\ell|^{-2}}$, but the energy barriers are