Multilevel Monte Carlo for Stochastic McKean-Vlasov Equations

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MKV-SDEs

Mckean-Vlasov SDEs on $[0, T]$, of the form

$$\begin{cases} dX_t = \int_{\mathbb{R}^d} b(X_t, y)\mu_t(dy)dt + \int_{\mathbb{R}^d} \sigma(X_t, y)\mu_t(dy)dW_t, \\ \mu_t = \mathrm{Law}(X_t), \quad X_0 \in \mathbb{R}^d. \end{cases}$$

- The kernels b and σ belong to the sets $C_b^{2,1}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d) \cap C_p^{2,2}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ and $C_b^{2,1}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^{d\otimes r}) \cap C_p^{2,2}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^{d\otimes r})$ respectively.
- $\implies$ For all $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$, there exists a constant L such that

$$\begin{aligned} |b(x_1, y_1) - b(x_2, y_2)| + \|\sigma(x_1, y_1) - \sigma(x_2, y_2)\| &\leq L(|x_1 - x_2| + |y_1 - y_2|), \\ |b(x_1, y_1)| + \|\sigma(x_1, y_1)\| &\leq L(1 + |x_1| + |y_1|). \end{aligned}$$

- The initial law μ_0 satisfies $q \geq 2$, $\int_{\mathbb{R}^d} |x|^q \mu_0(dx) < \infty$.

Motivation

- MKV-SDEs gives probabilistic interpretation of nonlinear McKean-Vlasov PDEs which weak formulation with $f(\cdot) \in C_K^\infty(\mathbb{R}^d)$ is given

$$\begin{cases} \frac{\partial}{\partial t}\langle \mu_t, f\rangle & = \langle \mu_t, \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x,\mu_t)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x,\mu_t)\frac{\partial f}{\partial x_i}(x)\rangle, \\ \mu_0 & = \mathbb{P}\circ X_0^{-1} = Law(X_0), \end{cases}$$

where $a(x,\mu_t) = \sigma(X_t,\mu_t)^T\sigma(X_t,\mu_t)$ and $b(x,\mu_t) := \int_{\mathbb{R}^d} b(x,y)\mu_t(dy)$.

- Example: $dY_t = \int_{\mathbb{R}^d} b(Y_t,y)\mu_t(dy)dt + dW_t$.
- Applications:
  - Lagrangian models (Bossy,Jabir,Talay, 2011)
  - Navier-Stokes equation for the vorticity of a two-dimensional incompressible fluid flow and many more (Bossy,Jourdain,Meleard,Reygner,Talay...)
  - Mean-Field Games (Lasry, Lions, 2007, Chassagneux, Crisan, Delarue, 2015; Cardaliaguet, Delarue, Lasry, Lions 2016, Kolokoltsov 2011- ; Buckdahn, J Li, Peng 2009)
  - Stochastic Local Volatility Models (Gyongy, 1996; Guyon, Henry-Labordere 2011; Guyon 2014,2015; Jourdain ,Zhou 2016)

Propagation of chaos

- *Stochastic interacting particles* $(X_t^{i,N})$ are solutions to $(\mathbb{R}^d)^N$ dimensional SDEs

$$\begin{cases} dX_t^{i,N} & = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad i = 1, \dots, N, \\ \mu_t^N & := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \quad t \geq 0, \end{cases}$$

  where $\{X_0^{i,N}\}_{i=1,\dots,N}$ are i.i.d samples with the law μ_0 and $\{W_t^i\}_{i=1,\dots,N}$ are independent Brownian motions.

- $X_t^{i,N} \Rightarrow X_t^i$ when $N \to \infty$.

Time discretization - Euler Scheme

Recall

$$dX_t = \int_{\mathbb{R}^d} b(X_t, y)\mu_t(dy)dt + \int_{\mathbb{R}^d} \sigma(X_t, y)\mu_t(dy)dW_t,$$

Euler scheme with time-step $h = T/M$, i=1,...,N,

$$Y_{k+1}^{i,N} = Y_k^{i,N} + \frac{1}{N}\sum_{j=1}^{N} b(Y_k^{i,N}, Y_k^{j,N})h + \frac{1}{N}\sum_{j=1}^{N} \sigma(Y_k^{i,N}, Y_k^{j,N})\Delta W_{k+1}^i.$$

- Due to the particle interactions, its implementation requires N^2 arithmetic operations at each step.
- Euler scheme converges with weak rate of order $((\sqrt{N})^{-1} + h)$
  Bossy and Talay (1997) Antonelli and Kohatsu-Higa (2002), Kohatsu-Higa, and Ogawa,(1997), Bossy, Jourdain (2002).

Computational cost of the propagation of chaos

- Consider mean-square-error

$$\mathbb{E}\left[\left(\mathbb{E}[f(X_T)] - \frac{1}{N}\sum_{i=1}^{n} f(Y_T^{i,N})\right)^2\right]$$

- bias and statistical error are in a nonlinear relationship
- Consider iid samples

$$\bar{X}_{k+1} = \bar{X}_k + b(\bar{X}_k, \mathbb{P}_{kh})h + \sigma(\bar{X}_k, \mathbb{P}_{kh})\Delta W_{k+1}, \quad \mathbb{P}_{kh} = \mathbb{P} \circ (\bar{X}_k)^{-1}.$$

- Error decomposition

$$\begin{aligned}\mathbb{E}[f(X_T)] - \frac{1}{N}\sum_{i=1}^{N} f(Y_T^{i,N}) &= \left(\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T)]\right) \\ &+ \left(\mathbb{E}[f(\bar{X}_T)] - \frac{1}{N}\sum_{i=1}^{N} f((\bar{X}_T^i)\right) \\ &+ \frac{1}{N}\sum_{i=1}^{N}\left(f((\bar{X}_T^i) - f(Y_T^{i,N})\right).\end{aligned}$$

Cost

- Typical mean-square error

$$\mathbb{E}\left[\left(\mathbb{E}[f(X_T)] - \frac{1}{N}\sum_{i=1}^{N} f(Y_T^{i,N})\right)^2\right] \leq C(h^2 + \frac{1}{N} + \frac{1}{N}),$$

- Cost $\mathcal{C}_\gamma = N^\gamma h^{-1}$, $\gamma = 1$ no-interacting Kernel, $\gamma = 2$ interacting Kernel.
- For the root-mean-square-error ϵ the cost is $\mathcal{C}_1 = \mathcal{O}(\epsilon^{-3})$ or $\mathcal{C}_2 = \mathcal{O}(\epsilon^{-5})$
- Example of the "non-interacting kernel" particle system:

$$Y_{k+1}^{i,N} = Y_k^{i,N} + b(Y_k^{i,N}, \frac{1}{N}\sum_{j=1}^{N} f(Y_k^{j,N}))h + \sigma \Delta W_{k+1}^i.$$

MLMC for standard SDEs

Idea of Giles (2006), Heinrich (2001) was to explore the identity

$$\mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{\ell=1}^{L} \mathbb{E}[P_\ell - P_{\ell-1}],$$

where $P_\ell := P(Y^{M_\ell})$ with $P : C([0,T], \mathbb{R}^d) \to \mathbb{R}$ and $\{Y^{M_\ell}\}$, $\ell = 0 \dots L$, being discrete time approximation of process X with M_ℓ number of time steps.

This identity leads to an unbiased estimator of $\mathbb{E}[P(Y^{M_L})]$,

$$\frac{1}{N_0} \sum_{i=1}^{N_0} P_0^{(i,0)} + \sum_{\ell=1}^{L} \left\{ \frac{1}{N_\ell} \sum_{i=1}^{N_l} (P_\ell^{(i,\ell)} - P_{\ell-1}^{(i,\ell)}) \right\},$$

where $P_\ell^{(i,\ell)} = P((Y^{M_\ell})^{(i)})$ are independent samples at level ℓ.

- But for MLMC variance for particle systems, typically decays as $\mathcal{O}(N^{-1} + h)$
- In the case of particle systems, one need to be careful to ensure telescopic sum is preserved.

Sznitman's iteration proof

$$\begin{cases} X_t &= X_0 + W_t + \int_0^t \int_C b(X_s, y)\mu_t(dy)ds, \quad 0 \le t \le T \\ \mu_t &= Law(X_t) \end{cases}$$

- Step 1: Pick a measure $\mu \in \mathcal{P}(C[0,T], \mathbb{R}^d)$
- Step 2: Define an operator $\Phi : \mathcal{P}(C[0,T], \mathbb{R}^d) \mapsto \mathcal{P}(C[0,T], \mathbb{R}^d)$;

$$\Phi(\nu) = Law(X^\nu)$$

$$X_t^\nu = X_0 + W_t + \int_0^t \int_C b(X_s^\nu, y)\nu_t(dy)ds, \quad 0 \le t \le T.$$

- Iterate

Theorem (Sznitman)

Let $T > 0$, and $\mu \in \mathcal{P}_2(C[0,T], \mathbb{R}^d))$. There exists $C > 0$ st.

$$W_2(\Phi^{k+1}(\mu), \Phi^k(\mu)) \le C\frac{T^k}{k!} W_2(\Phi(\mu), \mu).$$

$$W_2(\mu,\nu) = \inf_\gamma \left[\int_{\mathcal{C}\times\mathcal{C}} |u-v|^2 \gamma(du, dv);\ \gamma(\cdot \times \mathcal{C}) = \mu, \gamma(\mathcal{C} \times \cdot) = \nu \right].$$

Picard's Particle system

Consider a sequence of SDEs linear (in a sense of McKean) defined as

$$dX_t^m = \int_{\mathcal{C}} b(X_t^m, y)\mu_t^{m-1}(dy)dt + \int_{\mathcal{C}} \sigma(X_t^m, y)\mu_t^{m-1}(dy)dW_t^m, \quad Law(X_0^m) = Law(X_0),$$

where $\mu_t^{m-1} = \mathrm{Law}(X_t^{m-1})$. Let $(Y_t^{0,n,N_0})_{1\leq n\leq N_0}$ be an i.i.d. sample with law μ_0. For $m \geq 1$, we take $(Y_t^{m,n,N_m})_{1\leq n\leq N_0}$ independent for each m and define

$$dY_t^{m,n,N_m} = \frac{1}{N_{m-1}} \sum_{j=1}^{N_{m-1}} b(Y_t^{m,n,N_m}, Y_t^{m-1,j,N_{m-1}})\, dt$$
$$+ \frac{1}{N_{m-1}} \sum_{j=1}^{N_{m-1}} \sigma(Y_t^{m,n,N_m}, Y_t^{m-1,j,N_{m-1}})\, dW_t^{m,n}$$

Key idea:

- Use $1 : m-1$ steps to approximate $\int_{\mathbb{R}^d} b(X_s^{\mu}, y)\mu_t(dy)$
- Use the final m Picard step to approximate the quantity of interest.

Complexity of Iterated Particle System

Lemma

Let $G \in C_p^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ be a globally Lipschitz continuous function. We define

$$MSE_t^{(m)}(G(x,\cdot)) =: \mathbb{E}\left[\left(\mathbb{E}[G(x,X_t)] - \frac{1}{N_m}\sum_{i=1}^{N_m} G(x, Y_t^{i,m,L})\right)^2\right], \quad t \in [0,T].$$

Then for every $t \in [0,T]$,

$$MSE_{\eta_L(t)}^{(M)}(P) \leq c\left\{h_L^2 + \sum_{m=1}^{M} \frac{c^{M-m}}{(M-m)!} \cdot \frac{1}{N_m} + \frac{c^{M-1}}{M!}\right\}.$$

Iterative Particle System with time discretisation

Consider a sequence of SDEs linear (in a sense of McKean) defined as

$$dX_t^m = \int_{\mathbb{R}^d} b(X_t^m, y)\mu_t^{m-1}(dy)dt + \int_{\mathbb{R}^d} \sigma(X_t^m, y)\mu_t^{m-1}(dy)dW_t^m, \quad Law(X_0^m) = Law(X_0),$$

where $\mu_t^{m-1} = \mathrm{Law}(X_t^{m-1})$ and W^m and X_0^m are independent.
Iterative particle system

$$Y_{k+1}^{i,m,\ell} = Y_k^{i,m,\ell} + [\mathcal{M}_{t_k^\ell}(\tilde{\mu}_Y^{m-1})](b(Y^{i,m,\ell}, \cdot))h_\ell + [\mathcal{M}_{t_k^\ell}(\tilde{\mu}_Y^{m-1})](\sigma(Y^{i,m,\ell}, \cdot))\Delta W_{k+1}^{i,m,\ell}.$$

- No-interacting Kernels are much easier to analyse.

Iterative Particle System

For any borel function $G = (G^{(1)}, \dots, G^{(d)})$, where $G^{(i)} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we define

$$[\mathcal{M}_t(\mu)](G(x,\cdot)) := \sum_{\ell=0}^{L} \left(\mu_t^\ell - \mu_t^{\ell-1}\right)(G(x,\cdot)), \quad \mu^\ell \in \mathcal{P}_2,$$

where $\mu_t^\ell(G(x,\cdot)) = \int_{\mathbb{R}^d} G(x,y)\mu_t^\ell(dy)$.

$$\mu_t^\ell := \begin{cases} \left[\frac{t-\eta_l(t)}{h_l}\right]\mu^\ell_{\eta_\ell(t)+h_l} + \left[1 - \frac{t-\eta_l(t)}{h_l}\right]\mu^\ell_{\eta_\ell(t)} & , t \notin \Pi^\ell, \\ \frac{1}{N_{\ell,m}} \sum_{i=1}^{N_{\ell,m}} \delta_{Y_t^{i,m,\ell}} & , t \in \Pi^\ell. \end{cases}$$

Further assumptions

Define $B(t,x) = \mathbb{E}[b(x, X_t)]$ and $\Sigma(t,x) = \mathbb{E}[\sigma(x, X_t)]$.

- One can prove that $B(\cdot,\cdot) \in C^{1,2}$ and $\Sigma(\cdot,\cdot) \in C^{1,2}$

Consider stochastic flow $\left(X_t^{s,x}, t \in [s, T]\right)$

$$X_t^{s,x} = x + \int_s^t B\left(\theta, X_\theta^{s,x}\right) d\theta + \int_s^t \Sigma\left(\theta, X_\theta^{s,x}\right) dW_\theta, \quad t \in [s, T].$$

We consider

$$v_y(s,x) := \mathbb{E}[G(y, X_t^{s,x})], \quad y \in \mathbb{R}^d \text{ and } (s,x) \in [0,t] \times \mathbb{R}^d,$$

and associated (family of) PDEs reads

$$\begin{cases} \frac{\partial v_y}{\partial s}(s,x) + \frac{1}{2}\sum_{i,j=1}^{d} A_{ij}(s,x)\frac{\partial^2 v_y}{\partial x_i \partial x_j}(s,x) + \sum_{j=1}^{d} B_j(s,x)\frac{\partial v_y}{\partial x_j}(s,x) = 0, \ (s,x) \in [0,t] \times \mathbb{R}^d \\ v_y(t,x) = G(y,x), \end{cases}$$

where $A = \Sigma(s,x)\Sigma(s,x)^T$.

Regularity

Lemma

(a) *Suppose that* $G \in C_b^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$. *Then there exists a constant* $L > 0$ *such that*

- $$\begin{cases} \sum_{k=1}^d \sup_{y\in\mathbb{R}^d} \left\| \frac{\partial v_y}{\partial x_k}(s,x) \right\|_\infty \leq L, & \forall (s,x) \in [0,t] \times \mathbb{R}^d, \\ \sum_{i,j=1}^d \sup_{y\in\mathbb{R}^d} \left\| \frac{\partial^2 v_y}{\partial x_i \partial x_j}(s,x) \right\|_\infty \leq L, & \forall (s,x) \in [0,t] \times \mathbb{R}^d. \end{cases}$$

(b) *Suppose that* $G \in C_p^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$. *Then there exists a constant* $L > 0$ *such that*

- $$\begin{cases} \sup_{y\in\mathbb{R}^d} \sum_{k=1}^d \left| \frac{\partial v_y}{\partial x_k}(s,x) \right| \leq L(1+|x|^p), & \forall (s,x) \in [0,t] \times \mathbb{R}^d, \\ \sup_{y\in\mathbb{R}^d} \sum_{i,j=1}^d \left| \frac{\partial^2 v_y}{\partial x_i \partial x_j}(s,x) \right| \leq L(1+|x|^p), & \forall (s,x) \in [0,t] \times \mathbb{R}^d. \end{cases}$$

Main Result

Theorem

Assume regularity of the coefficients and the initial law. Then

$$\sup_{x\in\mathbb{R}^d}\sup_{0\le t\le T}\mathbb{E}\left[\left([\mathcal{M}_{\eta_L(t)}(\mu_Y^M)](G(x,\cdot))-\mathbb{E}[G(x,X_{\eta_L(t)})]\right)^2\right]$$

$$\le c\left\{h_L^2+\sum_{m=1}^{M}\frac{c^{M-m}}{(M-m)!}\cdot\sum_{\ell=0}^{L}\frac{h_\ell}{N_{m,\ell}}+\frac{c^M}{M!}\right\},$$

- We can reduce computational cost of approximating expectation by the order of magnitude.

Glimpse into the analysis

SDEs with random coefficients

$$dU_t = \bar{B}(U_t, \mathcal{V}_{\eta_L(t)})dt + \bar{\Sigma}(U_t, \mathcal{V}_{\eta_L(t)})dW_t, \quad Law(U_0) = Law(X_0),$$

where $\mathcal{V}_{\eta_L(t)} \in \mathcal{P}(\mathbb{R}^d)$ is random measure. Euler scheme is given

$$dZ^\ell_t = \bar{B}(Z^\ell_{\eta(t)}, \mathcal{V}_{\eta(t)})dt + \bar{\Sigma}(Z^\ell_{\eta(t)}, \mathcal{V}_{\eta(t)})dW_t, \quad Law(Z^\ell_0) = Law(X_0),$$

- (Conditional) Independence: W and Z_0 is independent of the random measure $\mathcal{V}$.
- Integrability: Let G be Lipschitz continuous, then

$$\sup_{0\leq s\leq T} \mathbb{E}\left[\left| \int_{\mathbb{R}^d} |G(x,y)|^p \mathcal{V}_s(dy)\right|\right] \leq c(1+|x|^p),$$

- There exists a constant c such that

$$\sup_{x\in\mathbb{R}^d} \sup_{0\leq s\leq t\leq T} \mathbb{E}\left[\left|\bar{B}(x,\mathcal{V}_t) - \bar{B}(x,\mathcal{V}_s)\right|^2 + \left\|\bar{\Sigma}(x,\mathcal{V}_t) - \bar{\Sigma}(x,\mathcal{V}_s)\right\|^2\right] \leq c(t-s).$$

- The kernels $\bar{B} : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $\bar{\Sigma} : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\otimes r}$ satisfy

$$|\bar{B}(x_1,\mathcal{V}_t) - \bar{B}(x_2,\mathcal{V}_t)| + \|\bar{\Sigma}(x_1,\mathcal{V}_t) - \bar{\Sigma}(x_2,\mathcal{V}_t)\| \leq c|x_1 - x_2|, \qquad \forall x_1, x_2 \in \mathbb{R}^d,$$

$$|\bar{B}(x,\mathcal{V}_t)| + \|\bar{\Sigma}(x,\mathcal{V}_t)\| \leq c\left(1 + |x| + \int_{\mathbb{R}^d} |y|\, \mathcal{V}_t(dy)\right), \qquad \forall x \in \mathbb{R}^d.$$

Glimpse into the analysis

Lemma (Regularity of Z_t^ℓ)

For $p \geq 1$, $0 \leq u \leq s \leq T$,

$$\left(\mathbb{E}[|Z_s^\ell - Z_u^\ell|^p]\right)^{\frac{1}{p}} \leq c(s-u)^{\frac{1}{2}}.$$

Lemma (Strong convergence of Z_t^ℓ)

For any $\ell \in \{1, 2, \ldots, L\}$, there exists a constant $c > 0$ such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |Z_t^\ell - Z_t^{\ell-1}|^2\right] \leq c h_\ell.$$

Conditional MLMC variance

Lemma

Let $\{\mu_t\}_{t\in[0,T]}$ *be a collection of probability measures on* $\mathbb{R}^d$ *such that*

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|x|^2\,\mu_t(dx)<+\infty.$$

Then for any globally Lipschitz continuous function $G:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$*, there exists a constant* c *such that*

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}\mathbb{E}\left[Var\left(\mathcal{M}_{\eta_L(t)}(G(x,\cdot))\middle|\mathcal{F}_T^{\mathcal{V}}\right)\right]\mu_t(dx)\leq c\sum_{\ell=0}^{L}\frac{h_\ell}{N_\ell}.$$

Theorem

Let $G \in C_b^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ be a globally Lipschitz continuous function. Then there exists a constant c such that for each $t \in [0, T]$ and $x \in \mathbb{R}^d$,

$$\sup_{0 \le s \le t} |\mathbb{E}[G(x, Z_s^L)] - \mathbb{E}[G(x, X_s)]|$$

$$\le c\left(h_L + \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^d} \left| \bar{B}(x, \mathcal{V}_{\eta_L(s)}) - \mathbb{E}[b(x, X_{\eta_L(s)})] \right| \mu_{\eta_L(s)}^{Z^L|\mathcal{F}_T^{\mathcal{V}}}(dx) \right] ds \right.$$

$$\left. + \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^d} \left\| \bar{\Sigma}(x, \mathcal{V}_{\eta_L(s)}) - \mathbb{E}[\sigma(x, X_{\eta_L(s)})] \right\| \mu_{\eta_L(s)}^{Z^L|\mathcal{F}_T^{\mathcal{V}}}(dx) \right] ds \right).$$

Weighted norm

Lemma

Let $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ C^2_p *be a globally Lipschitz continuous function. Then there exists a constant* c *such that for each* $t \in [0, T]$ *and* $x \in \mathbb{R}^d$*,*

$$\sup_{0 \le s \le t} \mathbb{E}\Big[\big| \mathbb{E}[G(x, Z^L_s) | \mathcal{F}^{\mathcal{V}}_T] - \mathbb{E}[G(x, X_s)] \big|^2 \Big]$$
$$\le c \bigg(h_L^2 + \int_0^t \Big[\int_{\mathbb{R}^d} \mathbb{E} \big| \bar{B}(x, \mathcal{V}_{\eta_L(s)}) - \mathbb{E}[b(x, X_{\eta_L(s)})] \big|^2 \mu^{\bar{Z}^L}_{\eta_L(s)}(dx) \Big] ds$$
$$+ \int_0^t \Big[\int_{\mathbb{R}^d} \mathbb{E} \big\| \bar{\Sigma}(x, \mathcal{V}_{\eta_L(s)}) - \mathbb{E}[\sigma(x, X_{\eta_L(s)})] \big\|^2 \mu^{\bar{Z}^L}_{\eta_L(s)}(dx) \Big] ds \bigg),$$

where $\bar{Z}^L$ *is a process defined by*

$$d\bar{Z}^L_t = \bar{B}(\bar{Z}^L_{\eta_L(t)}, \mu^X_{\eta_L(t)})\, dt + \bar{\Sigma}(\bar{Z}^L_{\eta_L(t)}, \mu^X_{\eta_L(t)})\, dW_t.$$

"Correct" error decomposition

For $x \in \mathbb{R}^d$ and $t \in [0, T]$, we consider

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbb{E}[G(x, X_{\eta_L(t)})] - \mathcal{M}^{(m)}_{\eta_L(t)}(G(x,\cdot))\right)^2\right] \\
= \; &\mathbb{E}\Bigg[\Bigg(\mathbb{E}[G(x, X_{\eta_L(t)})] - \mathbb{E}[\mathcal{M}^{(m)}_{\eta_L(t)}(G(x,\cdot))|\mathcal{F}^{m-1}] \\
&+\mathbb{E}[\mathcal{M}^{(m)}_{\eta_L(t)}(G(x,\cdot))|\mathcal{F}^{m-1}] - \mathcal{M}^{(m)}_{\eta_L(t)}(G(x,\cdot))\Bigg)^2\Bigg].
\end{aligned}
$$

Observe $\mathbb{E}[\mathcal{M}^{(m)}_{\eta_L(t)}(G(x,\cdot))|\mathcal{F}^{m-1}] = \mathbb{E}[G(x, Y^{1,m,L}_{\eta_L(t)})|\mathcal{F}^{m-1}]$. Next,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbb{E}[G(x, X_{\eta_L(t)})] - \mathbb{E}[G(x, Y^{1,m,L}_{\eta_L(t)})|\mathcal{F}^{m-1}]\right)^2\right] \\
\leq \; & c\Bigg(h_L^2 + \int_0^t \left[\int_{\mathbb{R}^d} \mathbb{E}\left|\mathcal{M}^{(m-1)}_{\eta_L(s)}(b(x,\cdot)) - \mathbb{E}[b(x, X_{\eta_L(s)})]\right|^2 \mu^{\bar{Z}^L}_{\eta_L(s)}(dx)\right] ds \\
&+ \int_0^t \left[\int_{\mathbb{R}^d} \mathbb{E}\left\|\mathcal{M}^{(m-1)}_{\eta_L(s)}(\sigma(x,\cdot)) - \mathbb{E}[\sigma(x, X_{\eta_L(s)})]\right\|^2 \mu^{\bar{Z}^L}_{\eta_L(s)}(dx)\right] ds\Bigg)
\end{aligned}
$$

Complexity

Theorem

Fix $M > 0$ and let $C_p^2 \ni P : \mathbb{R}^d \to \mathbb{R}$ be a globally Lipschitz continuous function. Then for any $\varepsilon < e^{-1}$, there exist M, $\{L_m\}_{1\leq m\leq M}$ and $\{N_{m,\ell}\}_{\substack{1\leq m\leq M\\ 0\leq \ell\leq L_m}}$ such that for every $t \in [0, T]$,

$$\mathbb{E}\left[\left(\mathcal{M}^{(M)}_{\eta_L(t)}(P) - \mathbb{E}[P(X_{\eta_L(t)})]\right)^2\right] < \varepsilon^2,$$

and computational complexity C is of order $\varepsilon^{-4}|\log \varepsilon|^3$.

$$\min_{M,\{L_m\}_{m=1}^M,\{N_{m,\ell}\}_{\substack{1\leq m\leq M\\ 0\leq \ell\leq L_m}}} C\left(M, \{L_m\}_{m=1}^M, \{N_{m,\ell}\}_{\substack{1\leq m\leq M\\ 0\leq \ell\leq L_m}}\right)$$

$$= \sum_{\ell=0}^{L_1} h_\ell^{-1} N_{1,\ell} + \sum_{m=2}^{M}\sum_{\ell=0}^{L_m} h_\ell^{-1} N_{m,\ell} \sum_{\ell'=0}^{L_{m-1}} N_{m-1,\ell'}$$

such that $$\sum_{m=1}^{M} \frac{c^{M-m}}{(M-m)!}\cdot\left(h_{L_m}^2 + \sum_{\ell=0}^{L_m}\frac{h_\ell}{N_{m,\ell}}\right) + \frac{c^{M-1}}{M!} \lesssim \varepsilon^2$$

Numerical Experiment: Non-interacting kernel

The target stochastic differential equation is

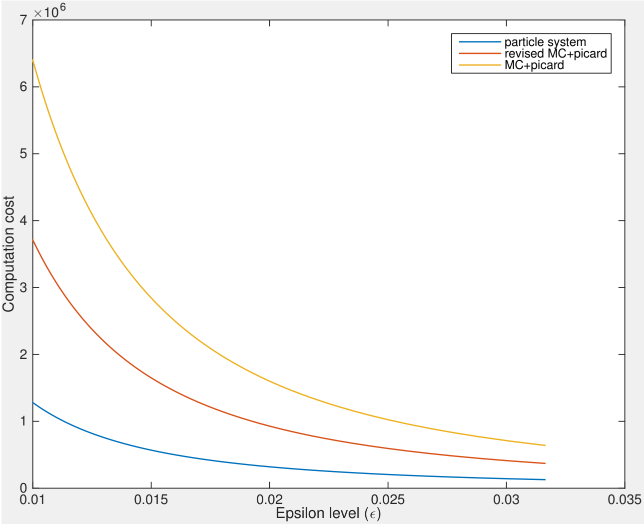
$$dX_t = \sin(X_t - \mathbb{E}[X_t])dt + \sigma dW_t \ , \quad X_0 = 0.$$

The testing payoff function is

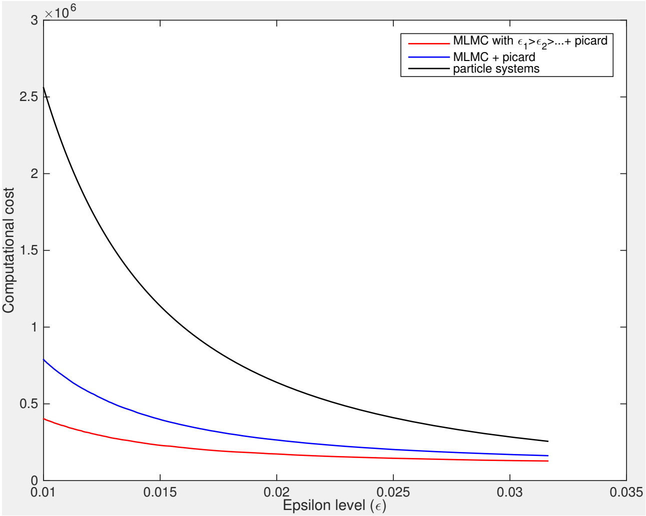
$$P(x) = \max(x - K, 0),$$

where strike K is set to 0.1.

Non-interacting kernel



Non-interacting case



Interacting kernel

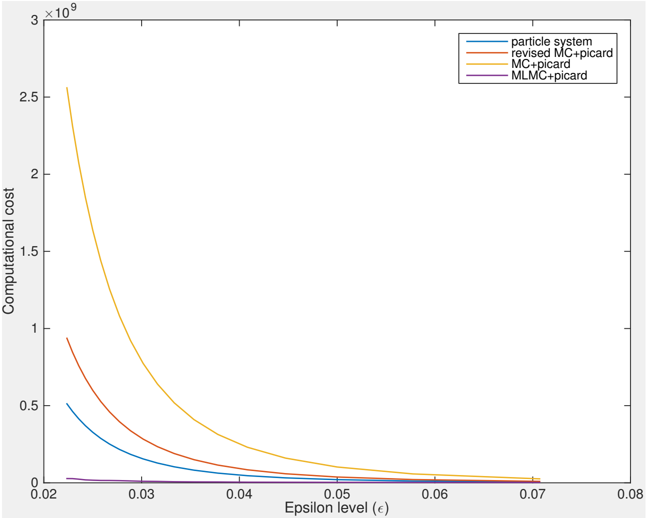
The target stochastic differential equation is

$$dX_t = \mathbb{E}[\sin(x - X_t)]|_{x=X_t} dt + \sigma dW_t \ , \quad X_0 = 0,$$

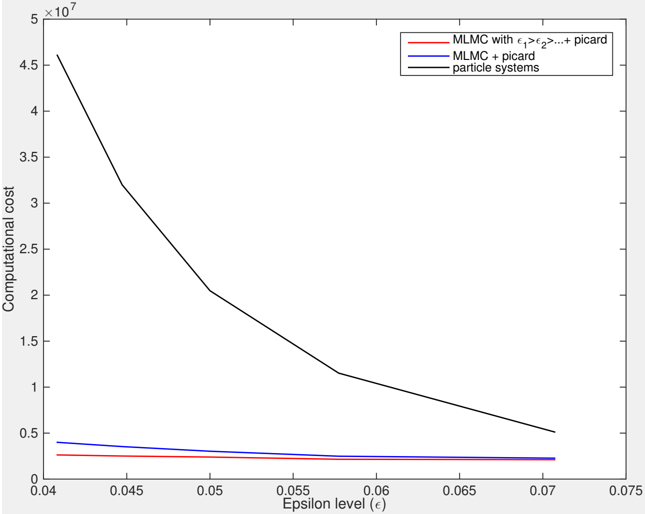
where Y_t is an independent copy of X_t. The payoff

$$P(x) = \sqrt{1 + x^2}.$$

Interacting kernel



Interacting kernel



Main points

- Stereotype : Picard type iterations are not efficient algorithms
- Picard iteration + MLMC = promising idea for the nonlinear problems
- Main philosophy: Iteration methods allow to linearise, nonlinear problems. And this can be exploited.
- Lots of research to be done...

Iterative Particle Approximation for McKean-Vlasov SDEs with application to Multilevel Monte Carlo estimation arXiv