The dynamic Φ^4_3 model comes down from infinity

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Joint with J.-C. Mourrat and P. Tsatsoulis

Main result

Aim of talk:

Good a priori bound for Φ^4_3 equation on torus $\mathbb{T}^3$

$$\partial_t X = \Delta X - (X^3 - \infty X) + \xi. \qquad (\Phi^4)$$

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Main result:

X_0 initial datum, $\varepsilon > 0$, $p < \infty$

$$\mathbb{E}\Big[\sup_{0<t\leq 1} \sup_{X_0 \in \mathcal{B}_\infty^{-\frac{3}{5}}} \Big(\sqrt{t}\, \|X(t)\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}}\Big)^p\Big] < \infty.$$

Discussion

$$\mathbb{E}\left[\sup_{0<t\le 1}\ \sup_{X_0\in\mathcal{B}_\infty^{-\frac{3}{5}}}\left(\sqrt{t}\,\|X(t)\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}}\right)^p\right]<\infty.$$

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Compare to ODE

Solution of $\dot{x} = -x^3$ with initial datum x_0

$$x(t) = \frac{1}{\sqrt{2t + x_0^{-2}}} \le \frac{1}{\sqrt{2t}}.$$

Bound uniform over initial datum $\Rightarrow$ Coming down from ∞.

Discussion cont'd

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Uniform control over large times

For $t+1\geq 1$ restrict dynamics to $[t,t+1]$ and use

$$\|X(t+1)\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}}\leq\sup_{0<s\leq 1}\sup_{X(t)\in\mathcal{B}_\infty^{-\frac{3}{5}}}\Big(\sqrt{s}\,\|X(t+s)\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}}\Big).$$

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$\Rightarrow$ uniform-in-t bound on moments.

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- $\Rightarrow$ uniform-in-t bound on moments.
- $\Rightarrow$ Tightness for Krylov-Bogoliubov approximations of invariant measure

$$\mu_T(A) = \frac{1}{T}\int_0^T \mathbb{P}(X(t) \in A)\,\mathrm{d}t.$$

Discussion cont'd

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- $\Rightarrow$ Alternative construction of Φ^4_3 Euclidean Field Theory.

Euclidean Φ^4_3 theory

Classical Problem:

Euclidean Φ^4_3 theory

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- Construct the measure

$$\mu \propto \exp\left(-2 \int \left[\frac{1}{2} |\nabla \varphi(x)|^2 - \frac{1}{4} \varphi(x)^4 + \frac{1}{2} \infty \varphi(x)^2 \right] dx \right) \prod_x d\varphi(x).$$

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Solution:

- Glimm-Jaffe '73, Feldman and Osterwalder '76, ... **Phase-cell cluster expansion**.
- Benfatto et al. '80, Gawędzki and Kupiainen '85, Brydges et al. '94, ... **Renormalisation group**.
- Brydges-Fröhlich-Sokal '83, ... **Skeleton inequalities**.

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OS axioms tricky - closely related to stability/uniqueness.

2-d case: Better than uniqueness - Spectral Gap

Theorem (Tsatsoulis, W. '16)

P_t = *transition kernel for* (Φ^4) *over two-dimensional torus.*

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$$\sup_x \|P_t(x) - \mu\|_{\mathrm{TV}} \leq (1-\lambda)^t \,.$$

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- Uniqueness already shown by Röckner-Zhu-Zhu '16.
- Convergence to equilibrium uniform over all initial data. Due to strong non-linear damping.
- Important that we work on finite volume.

Strategy for exponential equilibration

Doeblin criterion:

$P_t =$ transition kernel for (Φ^4). Show that $\exists\ \lambda > 0$ such that

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3-d case:

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3-d case:

- Strong Feller property Hairer-Mattingly '16.
- Support theorem: Work in progress Hairer-Schönbauer.

Why is the a priori bound true?

Scaling argument (general dimension d)

$$\partial_t X = \Delta X - X^3 + \xi.$$

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Rescaling $\hat{t} = \lambda^2 t$, $\hat{x} = \lambda x$, $\hat{\xi} = \lambda^{\frac{d+2}{2}} \xi$, $\hat{X} = \lambda^{\frac{2-d}{2}} X$ yields

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- Use Schauder theory (aka Regularity structures, paracontrolled distributions) for small scales.
- Use energy estimates on large scales.
- **Difficulty:** Combine the two.

The 2-d case- Da Prato-Debussche trick

Stochastic step:

ι solution of stochastic heat equation:

$$\partial_t \iota = \Delta \iota + \xi.$$

Can construct $\iota^2 \rightsquigarrow \vee$ and $\iota^3 \rightsquigarrow \Psi$. All $\iota, \vee, \Psi$ distributions in $\mathcal{C}^{0-}$.

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Deterministic step:

$u = X - \mathrm{I}$.

$$\begin{aligned} \partial_t u &= \Delta u - (\mathrm{I} + u)^3 \\ &= \Delta u - \big(u^3 + 3\mathrm{I}\, u^2 + 3\mathrm{V}\, u + \Psi\big). \end{aligned}$$

The 2-d case- Da Prato-Debussche trick

Stochastic step:

$\<1>$ solution of stochastic heat equation:

$$\partial_t \<1> = \Delta \<1> + \xi.$$

Can construct $\<1>^2 \rightsquigarrow \<2>$ and $\<1>^3 \rightsquigarrow \<3>$. All $\<1>, \<2>, \<3>$ distributions in $\mathcal{C}^{0-}$.

Deterministic step:

$u = X - \<1>$.

$$\begin{aligned} \partial_t u &= \Delta u - (\<1> + u)^3 \\ &= \Delta u - \left(u^3 + 3\<1>\, u^2 + 3\<2>\, u + \<3>\right). \end{aligned}$$

Multiplicative inequality: If $\alpha < 0 < \beta$ with $\alpha + \beta > 0$

$$\left\|\tau\, u\right\|_{\mathcal{B}^\alpha_\infty} \lesssim \left\|\tau\right\|_{\mathcal{B}^\alpha_\infty} \left\|u\right\|_{\mathcal{B}^\beta_\infty}.$$

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$\Rightarrow$ Short time existence, uniqueness.

Renormalised powers

$$\mathbb{E}\big[\langle \overset{\bullet}{|}{}^3_\delta, \eta\rangle^2\big] = \int_{\mathbb{T}}\int_{\mathbb{T}} \eta(x)\,\eta(y)\,\mathbb{E}\Big[\overset{\bullet}{|}{}^3_\delta(x)\,\overset{\bullet}{|}{}^3_\delta(y)\Big]\,dx\,dy\ .$$

Renormalised powers

$$\mathbb{E}\big[\langle \Psi_\delta^3, \eta\rangle^2\big] = \int_{\mathbb{T}}\int_{\mathbb{T}} \eta(x)\,\eta(y)\,\mathbb{E}\Big[\Psi_\delta^3(x)\Psi_\delta^3(y)\Big]\,dx\,dy\ .$$

Gaussian moments

$$\begin{aligned}
&\mathbb{E}\Big[\Psi_\delta^3(x)\,\Psi_\delta^3(y)\Big] \\
&\qquad = 6\,\mathbb{E}\big[\Psi_\delta(x)\Psi_\delta(y)\big]^3 + 9\,\mathbb{E}\big[\Psi_\delta(x)\Psi_\delta(y)\big]\,\mathbb{E}\big[\Psi_\delta(x)\Psi_\delta(x)\big]^2 \\
&\qquad \lesssim \big|\log(x-y)\big|^3 + \big|\log(\delta)\big|^2\big|\log(x-y)\big|.
\end{aligned}$$

- $\big|\log(x-y)\big|$ term is integrable. $\big|\log(\delta)\big|$ term diverges.

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$\Rightarrow$ $\mathbb{E}\big[\langle \upharpoonleft^3_\delta, \eta\rangle^2\big]$ diverges for $\delta \to 0$.

Renormalised powers cont'd

$$: \text{⫯}_\delta^3(x) := \text{⫯}_\delta^3(x) - 3C_\delta \text{⫯}_\delta(x) \textit{ where } C_\delta = \mathbb{E}\big[\text{⫯}_\delta(x)^2\big] \sim |\log(\delta)|.$$

$$\Rightarrow \mathbb{E}\Big[: \text{⫯}_\delta^3(x) :: \text{⫯}_\delta^3(y) : \Big] = 6\,\mathbb{E}\big[\text{⫯}_\delta(x)\text{⫯}_\delta(y)\big]^3.$$

$$\Rightarrow \mathbb{E}\big[\langle : \text{⫯}_\delta^3 :, \eta\rangle^2\big] \text{ remains bounded.}$$

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Theorem (Glimm, Jaffe, Nelson, Gross... 70s)

$:\Psi_\delta^3:$ *converges to a random distribution* $\Psi^{:3:}$ *in* $\mathcal{B}_\infty^{-\alpha}$ *for all* $\alpha > 0$.

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$$: \uparrow_\delta^3(x) := \uparrow_\delta^3(x) - 3C_\delta \uparrow_\delta(x) \textit{ where } C_\delta = \mathbb{E}\big[\uparrow_\delta(x)^2\big] \sim |\log(\delta)|.$$

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$: \uparrow_\delta^3 :$ *converges to a random distribution* Ψ : *in* $\mathcal{B}_\infty^{-\alpha}$ *for all* $\alpha > 0$.

- Ψ : called third Wick power.

The paracontrolled approach II - The 3-d case

Da Prato-Debussche trick does not work.

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- Equation for $u = X - \mathsf{I}$

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cannot be solved by Picard iteration.

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- Next order expansion $u = X - \mathsf{I} + \Upsilon$ gives

$$\partial_t u = \Delta u - \left(u^3 + 3\mathsf{I}\, u^2 + 3\vee u - 3\Upsilon\vee + \dots\right).$$

The paracontrolled approach II - The 3-d case

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Deterministic step:

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  cannot be solved by Picard iteration.

- Next order expansion $u = X - \imath + \Upsilon$ gives

$$\partial_t u = \Delta u - \left(u^3 + 3\imath\, u^2 + 3\vee\, u - 3\Upsilon\vee + \dots\right).$$

  Still cannot be solved, because of $\vee u$. Expanding further does not solve the problem.

The paracontrolled approach III - A system of equations

Catellier-Chouk: Split up remainder equation: $u = v + w$

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- $v \in \mathcal{C}^{1-}$ is the most irregular component of u.
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- $v \in \mathcal{C}^{1-}$ is the most irregular component of u.
- $w \in \mathcal{C}^{\frac{3}{2}-}$ more regular remainder.
- $\olessthan$ paraproduct.
- Term $v \odot \vee$ can be rewritten as

$$v \odot \vee = -3\left[(v + w - \Psi) \olessthan Y\right] \odot \vee + \mathrm{com}_1(v, w) \odot \vee$$
$$= -3(v + w - \Psi)\, \vee\!\!\!\!Y + \mathrm{com}_2(v + w) + \mathrm{com}_1(v, w).$$

Main result

$$(\partial_t - \Delta)v = F(v + w) - cv,$$
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- For $\tau = \imath, \vee, \Psi, \Psi_\circ, \Psi^{\bullet}_\circ, \vee_\circ$ assume

$$\sup_{0 \le t \le 1} \|\tau(t)\|_{\mathcal{B}_\infty^{\alpha_\tau}} \le K, \qquad \sup_{0 \le s < t \le 1} \frac{\|\Psi(t) - \Psi(s)\|_{\mathcal{B}_\infty^{\frac{1}{4}-\varepsilon}}}{|t - s|^{\frac{1}{8}}} \le K.$$

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- Assume $c = c_0 K^{30p}$, set $v_0 := 0$, $w_0 = X_0 \in \mathcal{B}_\infty^{-\frac{3}{5}}$.

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Theorem

- For $\tau = \<1>, \<2>, \<3>, \<30>, \<31>, \<32>$ assume

$$
\sup_{0\le t\le 1} \|\tau(t)\|_{\mathcal{B}_\infty^{\alpha_\tau}} \le K, \qquad \sup_{0\le s<t\le 1} \frac{\|\<3>(t) - \<3>(s)\|_{\mathcal{B}_\infty^{\frac{1}{4}-\varepsilon}}}{|t-s|^{\frac{1}{8}}} \le K.
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- Assume $c = c_0 K^{30p}$, set $v_0 := 0$, $w_0 = X_0 \in \mathcal{B}_\infty^{-\frac{3}{5}}$.

$\Rightarrow$ for $t \in (0,1]$

$$
\|w(t)\|_{L^{3p-2}} \le \frac{CK^\kappa}{\sqrt{t}}, \quad \text{and} \quad \|v(t)\|_{\mathcal{B}_{2p}^{-3\varepsilon}} \le CK^\kappa.
$$

Discussion of terms

$$
\begin{aligned}
(\partial_t - \Delta) v &= -3(v + w - \Psi) \circledless \mathsf{V} - cv, \\
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- $w \circledequal \vee$ linear in w, but derivative or order $1+$ needed to control this.
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Elements of proof

The irregular term v

$$(\partial_t - \Delta) v = -3(v + w - \Psi) \varolessthan \vee.$$

Duhamel (parabolic regularity) and “Gronwall” give for $\beta < 1-$

$$\|v(t)\|_{\mathcal{B}^\beta_q} \lesssim \cdots \|v_0\| + K \int_0^t \frac{e^{-\underline{c}(t-u)}}{(t-u)^\sigma} \left(\|w(u)\|_{L^p} + K \right) \mathrm{d}u. \quad (1)$$

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Control for $w \circleddash \mathrm{V}$

Duhamel (parabolic regularity) gives for $\gamma < \frac{3}{2}-$

$$\|w(t)\|_{\mathcal{B}^{\gamma}_{p}} \lesssim \|e^{t\Delta} w_0\|_{\mathcal{B}^{\gamma}_{p}} + \left(\int_0^t \|w(s)\|^{3p}_{L^{3p}} \, \mathrm{d}s \right)^{\frac{1}{p}}$$
$$+ \left(\int_0^t \|w(s)\|^{p}_{\mathcal{B}^{1+4\varepsilon}_{p}} \, \mathrm{d}s \right)^{\frac{1}{p}} + \|v_0\|^{3}_{\mathcal{B}^{-3\varepsilon}_{2p}} + \dots \quad (2)$$

Elements of proof cont'd

Testing the equation

If $c \geq c_0 K^{30p}$, p large enough.

$$\|w(t)\|_{L^{3p-2}}^{3p-2} + \int_0^t \|w(s)\|_{L^{3p}}^{3p} \, \mathrm{d}s \lesssim \|w_0\|_{L^{3p-2}}^{3p-2} + (cK)^{\kappa} \Big[1 + \|v_0\|_{\mathcal{B}_{2p}^{-3\varepsilon}}^{3p} + \int_0^t \|w(s)\|_{\mathcal{B}_p^{1+4\varepsilon}}^{p} \, \mathrm{d}s \Big]. \quad (3)$$

Elements of proof cont'd

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Conclusion

Combining (2) and (3), using $\gamma = 1 + 5\varepsilon$ we get

$$\|w(t)\|_{L^{3p-2}}^{3p-2} + \int_s^t F(r)^{\lambda} \,\mathrm{d}r \lesssim K^{\kappa}\Big[1 + \|v(s)\|_{\mathcal{B}_{2p}^{-3\varepsilon}}^{3p} + F(s)\Big].$$

for $F(s) = \|w(s)\|_{L^{3p-2}}^{3p-2} + \|w(s)\|_{\mathcal{B}_p^{1+5\varepsilon}}^{\frac{3p-2}{3}}$ and $\lambda = \frac{3p}{3p-2} > 1$.

$\Rightarrow$ Conclusion by "ODE comparison" and "stopping for v".

Summary and outlook

Main result

- Strong a priori bound for solutions of Φ^4 equation on $\mathbb{T}^3$.

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Outlook

- How about infinite volume? Uniqueness for invariant measure not (always) expected.