Weak universality of the KPZ equation

Weijun Xu

University of Warwick
Joint work with Martin Hairer

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Interface growth models and KPZ

Edwards-Wilkinson universality class:

- Large scale limit of symmetric interface growth models.
- Limit described by stochastic heat equation $\partial_t Z = \partial_x^2 Z + \xi$.
- Scale invariant under $Z_\lambda(t,x) = \lambda Z(\lambda^4 t, \lambda^2 x)$.

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KPZ universality class:

- Large scale limit of asymmetric interface growth models.
- Conjectural limit described by KPZ fixed point (only constructed via TASEP).
- Scale invariant under $H_\lambda(t, x) = \lambda H(\lambda^3 t, \lambda^2 x)$.

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Conjecture (Weak universality conjecture)

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- Regularity structures (Hairer-Quastel, Hairer-X.): needs the equation "visible" at microscopic scale.

The KPZ equation

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Theorem (Hairer (13',14'), Hairer-Shen (15'))

There exists $C_\varepsilon = \frac{c}{\varepsilon} + \mathcal{O}(1)$ such that h_ε converges to KPZ(λ).

C_ε: average moving speed.

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$$\lambda = \frac{1}{2} \int_{\mathbf{R}} F''(x) \mu(dx), \qquad \mu = \mathrm{Law}(\partial_x P * \eta).$$

Arbitrary nonlinearities

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then $u_\varepsilon = h_\varepsilon - Z_\varepsilon$ solves

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$\varepsilon^{\frac{1}{2}} \Psi_\varepsilon = \mathcal{O}(1)$, and expect $\|\partial_x u_\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^{-\kappa})$.

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$\varepsilon^{\frac{1}{2}}\Psi_\varepsilon = \mathcal{O}(1)$, and expect $\|\partial_x u_\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^{-\kappa})$. Taylor expand F:

$$\varepsilon^{-1} F(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon) + \varepsilon^{-\frac{1}{2}} F'(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon)(\partial_x u_\varepsilon) + \frac{1}{2} F''(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon)(\partial_x u_\varepsilon)^2 + \mathcal{O}(\varepsilon^{\frac{1}{2}-}).$$

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$$(\varepsilon^{-1}F(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon) - C_\varepsilon) + \varepsilon^{-\frac{1}{2}}F'(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon)(\partial_x u_\varepsilon) + \frac{1}{2}F''(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon)(\partial_x u_\varepsilon)^2 + \mathcal{O}(\varepsilon^{\frac{1}{2}-}).$$

First step to show:

$$\varepsilon^{-1}F(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon) - C_\varepsilon \to \lambda\Psi^{\diamond 2}, \quad F'(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon) \to 2\lambda\Psi, \quad F''(\varepsilon^{\frac{1}{2}}\Psi_\varepsilon) \to 2\lambda.$$

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- Clustering before chaos expansion + trigonometric identities to employ cancellations $\rightsquigarrow$ polynomial control for high frequency.
- Self-improvement to $\mathcal{C}^6$ once one has a polynomial control.

Further questions

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3. Non-Gaussian noise.