Unique strong solutions of stochastic differential equations driven by Lévy processes with discontinuous coefficients

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Outline

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Bass [4] and Komatsu [10] show that the following SDE

$$dX_t = F(X_{t-})dL_t, \quad t \geq 0 \tag{1.1}$$

admits pathwise uniqueness If $\{L_t\}$ is a symmetric stable process with exponent $\alpha \in (1, 2)$, $|F(x) - F(y)| \leq \rho(|x - y|)$ and if $z \to \rho(z)$ satisfying

$$\int_{0+} \frac{1}{\rho(z)^\alpha} dz = \infty.$$

- It is well-known that if the coefficients are assumed to be ***Lipschitz continuous***, the pathwise uniqueness can be obtained by ***Gronwall's inequality*** and the results on continuous-type equations; see e.g. Ikeda and Watanabe [7].

- This condition has been improved by Fu and Li [6]. They proved the pathwise uniqueness for non-negative càdlàg solutions driven by spectrally positive Lévy noises under Lipschitz and ***non-Lipschitz conditions***.

Literature review

- It is well-known that if the coefficients are assumed to be ***Lipschitz continuous***, the pathwise uniqueness can be obtained by ***Gronwall's inequality*** and the results on continuous-type equations; see e.g. Ikeda and Watanabe [7].
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Weak uniqueness + Local time $\Rightarrow$ Pathwise uniqueness

Advantages

Get rid of the continuous restriction on coefficients;
The Jumps could be both positive and negative jumps.

Let $N(ds, du)$ be the Poisson random measures associated with $\{p_t\}$. In this paper, we will study the solution to the stochastic differential equation (1.2) given below. By a solution of the stochastic equation

$$X_t = X_0 + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \int_0^t \int_U g(X_{s-}, u)N(ds, du), \tag{1.2}$$

•$\{B_t\}$, $\{p_t\}$ are independent of each other;

• $\sigma(x)$, $b(x)$ and $g(x, u)$ are Borel functions on $\mathbb{R}$, which have at most countably many discontinuous points.

Outline

Since martingale problem $\Longleftrightarrow$ weak existence,

then we just need to prove that

$$\begin{aligned} M_t^f &= f(X_t) - \int_0^t \left(b(X_s) f'(X_s) + \frac{1}{2} f''(X_s) \sigma(X_s)^2 \right) ds \\ &\quad - \int_0^t \int_U \left(f(X_s + g(X_s, u)) - f(X_s) \right) \mu(du) ds \qquad (2.1) \end{aligned}$$

is a martingale.

(2.a) There is a constant $K \geq 0$ such that

$$b(x)^2 + \sigma(x)^2 + \int_U |g(x,u)| \mu(du) \leq K, \quad \forall x \in \mathbb{R};$$

(2.b) There is a constant $\sigma_0 > 0$ such that

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Let

$$b_n(x) = \mathbb{E}\left(b(x+\xi_n)\right),$$

where $\xi_n \sim N(0, \frac{1}{n})$. Let σ_n and g_n be defined similarly.

For every $n \geq 1$, by a well-known result on SDE, there is a unique strong solution to

$$X_t^n = X_0 + \int_0^t b_n(X_s^n)ds + \int_0^t \sigma_n(X_s^n)dB_s + \int_0^t \int_U g_n(X_s^n, u)N(du, ds).$$

$$
\begin{aligned}
M_t^{n,f} \quad = \quad & f(X_t^n) - \int_0^t \left(b_n(X_s^n) f'(X_s^n) + \frac{1}{2} f''(X_s^n) \sigma_n^2(X_s^n) \right) ds \\
& - \int_0^t \int_U \left(f(X_s^n + g_n(X_s^n, u)) - f(X_s^n) \right) \mu(du) ds
\end{aligned}
$$

is a martingale.

$$
\begin{aligned}
& \lim_{n\to\infty} M_t^{n,f} \\
= \quad & \lim_{n\to\infty} f(X_t^n) - \lim_{n\to\infty} \int_0^t \left(b_n(X_s^n) f'(X_s^n) + \frac{1}{2} f''(X_s^n) \sigma_n^2(X_s^n) \right) ds \\
& - \lim_{n\to\infty} \int_0^t \int_U \left(f(X_s^n + g_n(X_s^n, u)) - f(X_s^n) \right) \mu(du) ds \qquad (2.2) \\
? = \quad & f(X_t) - \int_0^t \left(b(X_s) f'(X_s) + \frac{1}{2} f''(X_s) \sigma(X_s)^2 \right) ds \\
& - \int_0^t \int_U \left(f(X_s + g(X_s, u)) - f(X_s) \right) \mu(du) ds \qquad (2.3)
\end{aligned}
$$

i.e.

$$
\lim_{n\to\infty} b_n(X_{s-}^n) ? = b(X_{s-}),
$$

$$\lim_{n\to\infty} b_n(X^n_{s-}) ? = b(X_{s-}),$$

Proposition 2.1

The sequence $\{X^n\}$ is tight in the Skorohod space $D([0,\infty),\mathbb{R})$.

$$\{X_t^{n_k} : t \geq 0\} \to \{X_t : t \geq 0\}, a.s.$$

Lemma 2.2

The level set of the process X at level C is defined as $\{t : X_t = C\}$. Then the level set has Lebesgue measure 0 for any C.

Then weak existence holds.

Outline

In this section, we impose the following conditions:

(3.a) There exists a constant $K \geq 0$, $\forall x \in \mathbb{R}$, such that

$$|b(x)| + \int_U |g(x,u)|\mu(du) \leq K\sigma(x)^2,$$

(3.b) $0 < |\sigma(x)| \leq K, \quad \forall x \in \mathbb{R}$.

Let A be an operator on $B(\mathbb{R})$. The domain of A is denoted by $\mathscr{D}(A)$ and the range of A is denoted by $\mathscr{R}(A)$. A measurable stochastic process X is a solution of the martingale problem for A if there exists a filtration $\{\mathscr{F}_t\}$ such that

$$f(X_t) - \int_0^t Af(X_t)ds$$

is an $\{\mathscr{F}_t\}$-martingale for each $f \in \mathscr{D}(A)$.

Now we use the following proposition which is given by Kurtz and Ocone [11] to prove the weak uniqueness in general.

Proposition 3.1

Suppose $\mathscr{R}(\lambda - A)$ is separating for each $\lambda > 0$. If $\{v_t\}$ and $\{\mu_t\}$ satisfy

$$v_t f = v_0 f + \int_0^t v_s A f ds, \quad f \in \mathscr{D}(A)$$

are weakly right continuous and $v_0 = \mu_0$, then $v_t = \mu_t$ for all $t \geq 0$.

Suppose X_t and Y_t are two solutions of (1.2),
v_t: the distribution of X_t, μ_t: the distribution of Y_t
Proposition 3.1 $\Rightarrow v_t = \mu_t$ (Weak uniqueness holds)

We say that $M \subset B(\mathbb{R})$ is separating (for $\mathscr{P}(\mathbb{R})$) if $v, \mu \in \mathscr{P}(\mathbb{R})$ and $vf = \mu f$ for all $f \in M$ implies $v = \mu$.

Note that (1.2) is

$$X_t = X_0 + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \int_0^t \int_U g(X_{s-}, u)N(du, ds).$$

Let

$$W_t = \int_0^{\tau_t^{-1}} \sigma(X_{s-})dB_s.$$

Then W_t is a Brownian motion.
Hence,

$$\widetilde{X}_t = X_0 + W_t + \int_0^t (\sigma^{-2}b)(\widetilde{X}_{s-})ds + \int_0^{\tau_t^{-1}} \int_U g(\widetilde{X}_{s-}, u)N(du, ds)$$

Define the semigroup of the Brownian motion as follows

$$T_t f(x) = \int_R p_t(x-y) f(y) dy, \qquad \forall\ f \in \mathbb{B},$$

where $p_t(x-y)$ is the transition density, and $T_0 f(x) = f(x)$.
Let $A = A_0 + B + C$, where

$$Af(x) = \frac{1}{2} f^{''}(x) + \frac{b(x)}{\sigma(x)^2} f'(x) + \frac{1}{\sigma(x)^2} \int_U (f(x+g(x,u)) - f(x)) \mu(du),$$

$$A_0 f(x) = \frac{1}{2} f^{''}(x),$$

$$Bf(x) = \frac{1}{\sigma(x)^2} \int_U (f(x+g(x,u)) - f(x)) \mu(du),$$

and

$$Cf(x) = \frac{b(x)}{\sigma(x)^2} f'(x).$$

By Itô's formula, we get that for any $f \in D(A)$,

$$\mathbb{E}f(\widetilde{X}_t) = \mathbb{E}f(\widetilde{X}_0) + \int_0^t \mathbb{E}Af(\widetilde{X}_s)ds. \qquad (3.1)$$

Let $D(A) = D(A_0) = \{f : \ f, \ f', \ f'' \in \mathbb{B}\}$

Theorem 3.1

Under condition (3.a,b), the weak uniqueness holds for the equation (3.1), and hence, also for the time changed SDE.

We proceed to proving that $\mathscr{R}(\lambda - A)$ is separating. Let $\lambda > 0$ be arbitrary, define $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$. Let $g \in \mathbb{B}$. We hope to show that there exists $f \in D(A)$ such that $(\lambda - A)f = g$.

$$\mathscr{R}(\lambda - A) \supseteq \mathbb{B}$$

Namely, we need to solve

$$(\lambda - A_0)f = g + (B + C)f. \tag{3.2}$$

Actually, $R_\lambda = (\lambda - A_0)^{-1}$
We first solve

$$f = R_\lambda(g + Bf + Cf) \equiv \Gamma(f). \tag{3.3}$$

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$$\Gamma : \mathbb{B}^1 \to \mathbb{B}^1$$

$$(\lambda - A_0)f = g + (B + C)f.$$

$$\Updownarrow$$

$$\lambda l(x) - \frac{1}{2} l''(x) = h(x) \tag{3.4}$$

Applying R_λ to both sides of (3.4), we have

$$f = l$$

.

To prove that f is the solution of (3.2), we denote $h = g + Bf + Cf$. We now consider the following ODE:

$$\lambda l(x) - \frac{1}{2} l''(x) = h(x) \tag{3.5}$$

It is well-known that the above ODE has a solution

$$l(x) = e^{\sqrt{2\lambda}x} \int_x^\infty \frac{h(y)}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}y} dy + e^{-\sqrt{2\lambda}x} \int_{-\infty}^x \frac{h(y)}{\sqrt{2\lambda}} e^{\sqrt{2\lambda}y} dy.$$

Applying R_λ to both sides of (3.5), we have

$$R_\lambda(\lambda l - \frac{1}{2}l'') = R_\lambda h$$

Since $R_\lambda h = f$, and

$$\begin{aligned} R_\lambda(\lambda l - \frac{1}{2}l'') &= \lambda \int_0^\infty e^{-\lambda t} T_t l(x) dt - \frac{1}{2}\int_0^\infty e^{-\lambda t} T_t l''(x) dt \\ &= \lambda \int_0^\infty e^{-\lambda t} T_t l(x) dt - \int_0^\infty e^{-\lambda t} \frac{d}{dt}\left(T_t l(x)\right) dt \\ &= l, \end{aligned}$$

Then $f = l$.
There exists $f \in D(A)$ such that $(\lambda - A)f = g$

Suppose X_t and Y_t are two solutions of the original SDE, $\tau_t = \int_0^t \sigma(X_s)^2 ds$ and $\lambda_t = \int_0^t \sigma(Y_s)^2 ds$. The distributions of the time changed processes $\widetilde{X}_t = X_{\tau_t^{-1}}$ and $\widetilde{Y}_t = Y_{\lambda_t^{-1}}$ satisfy (3.1). Hence, $\mathcal{L}(\widetilde{X}) = \mathcal{L}(\widetilde{Y})$. It is easy to show that

$$\tau_t^{-1} = \int_0^t \frac{1}{\sigma(\widetilde{X_s})^2} ds \equiv \mathcal{G}(\widetilde{X}),$$

and

$$\lambda_t^{-1} = \int_0^t \frac{1}{\sigma(\widetilde{Y_s})^2} ds \equiv \mathcal{G}(\widetilde{Y}).$$

As $X_t = \widetilde{X}_{\tau_t}$ and $Y_t = \widetilde{Y}_{\lambda_t}$, we have

$$\mathcal{L}(X) = \mathcal{L}(Y).$$

That is, the weak uniqueness of the original SDE holds.

Outline

We force the following conditions:

(4.a) For any fixed u, $g(x, u) + x$ is non-decreasing ;

(4.b) There exist constants $\sigma_0, K \geq 0$, such that $0 < \sigma_0 \leq |\sigma(x)| \leq K$ for all x.

Tanaka's Formula

Let X be a semimartingale and let L^a be its local time at a. Then

$$(X_t - a)^+ - (X_0 - a)^+ = \int_{0+}^t 1_{\{X_{s-} > a\}} dX_s + \sum_{0<s\le t} 1_{\{X_{s-}>a\}} (X_s - a)^- + \sum_{0<s\le t} 1_{\{X_{s-}\le a\}} (X_s - a)^+ + \frac{1}{2} L_t^a$$

where L^a denotes the local time process of X at a. and

$$(X_t - a)^- - (X_0 - a)^- = -\int_{0+}^t 1_{\{X_{s-} \le a\}} dX_s + \sum_{0<s\le t} 1_{\{X_{s-}>a\}} (X_s - a)^- + \sum_{0<s\le t} 1_{\{X_{s-}\le a\}} (X_s - a)^+ + \frac{1}{2} L_t^a,$$

$$
\begin{aligned}
X_t^1 \vee X_t^2 = & X_t^1 + (X_t^2 - X_t^1)^+ \\
= & X_t^1 + \int_{0+}^t 1_{(X_{s-}^2 > X_{s-}^1)} d(X^2 - X^1)_s \\
& + \sum_{0<s\le t} 1_{(X_{s-}^2 > X_{s-}^1)} (X_s^2 - X_s^1)^- \\
& \quad + \sum_{0<s\le t} 1_{(X_{s-}^2 \le X_{s-}^1)} (X_s^2 - X_s^1)^+ + \frac{1}{2} L_t^0 (X^2 - X^1).
\end{aligned}
$$

By the non-decreasing property of $x + g(x, u)$ in x, we get

$$
\sum_{0<s\le t} 1_{(X_{s-}^2 > X_{s-}^1)} (X_s^2 - X_s^1)^- = 0,
$$

$$
\sum_{0<s\le t} 1_{(X_{s-}^2 \le X_{s-}^1)} (X_s^2 - X_s^1)^+ = 0.
$$

Then

$$
\begin{aligned}
X_t^1 \vee X_t^2 = & X_0^1 \vee X_0^2 + \int_0^t \sigma(X_{s-}^1 \vee X_{s-}^2) dB_s + \int_0^t b(X_{s-}^1 \vee X_s^2) dB_s \\
& + \int_{0+}^t \int_U g(X_{s-}^1 \vee X_{s-}^2, u) N(du, ds) \\
& + \frac{1}{2} L_t^0 (X^1 - X^2),
\end{aligned}
$$

Under condition (4.a), if X^1 and X^2 are two solutions of (1.2) such that $X_0^1 = X_0^2$ a.s., then $X^1 \vee X^2$ is a solution if and only if $L^0(X^1 - X^2)$ vanishes identically.

Proposition 4.2

If uniqueness in law holds for (1.2) and $L^0(X^1 - X^2) = 0$ for any pair (X^1, X^2) of solutions such that $X_0^1 = X_0^2$ a.s., then pathwise uniqueness holds for (1.2).

Proof: If X^1 and X^2 are two solutions, and $L^0(X^1 - X^2) = 0$, then $X^1 \vee X^2$ is also a solution. Since the weak uniqueness holds, then we have

$$\mathcal{L}(X^1) = \mathcal{L}(X^2) = \mathcal{L}(X^1 \vee X^2)$$

and $X^1 \vee X^2 - X^1$ is a non-negative random variable, then

$$\mathbb{E}[X^1 \vee X^2 - X^1] = 0$$

so $X^1 \vee X^2 = X^1$ a.s.. Similarly, we have $X^1 \vee X^2 = X^2$ a.s., which implies $X^1 = X^2$ a.s..

Lemma 4.2

Let X be a semimartingale. For $\varepsilon > 0$ and $t > 0$ define

$$A_t^\varepsilon := \int_0^t 1_{(0<X_s\le\varepsilon)}\rho(X_s)^{-1}\,d[X,X]_s^c.$$

If $\mathbb{E}A_t^\varepsilon < \infty$ and $\lim_{a\to 0+} \mathbb{E}[L_t^a(X)] = \mathbb{E}[L_t^0(X)]$ for some $\varepsilon > 0$ and all $t > 0$, then $L^0(X) = 0$.

In the sequel ρ always stands for a Borel map from $[0,\infty)$ to itself such that $\int_{0+} da/\rho(a) = \infty$.

Lemma 4.1

Under conditions (2.a) and (4.a), if X^1 and X^2 are two solutions of (1.2), then $\mathbb{E}\left[L_t^a(X^1 - X^2)\right] \to \mathbb{E}\left[L_t^0(X^1 - X^2)\right]$ as $a \to 0+$.

Theorem 4.2

Pathwise uniqueness holds for (1.2) in the following cases:

(1) Under conditions (3.a,b), and $|\sigma(x)-\sigma(y)|^2 \leq \rho(|x-y|)$.

(2) Under conditions (3.a), (4.a,b), and $|\sigma(x)-\sigma(y)|^2 \leq |f(x)-f(y)|$ for some increasing and bounded function f.

Proof: (1) Let X^1, X^2 be the solutions to (1.2) with respect to the same Brownian Motion, then

$$\mathbb{E}\left[\int_0^t \rho(X_s^1-X_s^2)^{-1} 1_{(X_s^1>X_s^2)} d[X^1-X^2, X^1-X^2]_s^c\right]$$
$$=\mathbb{E}\left[\int_0^t \rho(X_s^1-X_s^2)^{-1}\left(\sigma(X_s^1)-\sigma(X_s^2)\right)^2 1_{(X_s^1>X_s^2)} ds\right] \leq t.$$

To apply Lemma 4.2, we will consider A_t with $\rho(x) = x$ and $X_t = X_t^1 - X_t^2$. We consider

$$\begin{aligned} & \mathbb{E} \left[\int_0^t (X_s^1 - X_s^2)^{-1} d[X^1 - X^2, X^1 - X^2]_s^c\right] \\ \leq\ & \mathbb{E}\left[\int_0^t \left(f(X_s^1) - f(X_s^2)\right)(X_s^1 - X_s^2)^{-1} ds\right] =: K(f)_t \end{aligned}$$

Let

$$f_n(x) = \mathbb{E} f(x + \xi_n), \quad \xi_n \sim N(0, \frac{1}{n}).$$

It is easy to verify that f_n is of bounded and increasing. It follows that $K(f)_t = \lim_{n\to\infty} K(f_n)_t$ for almost all $s \leq t$.

$$
\begin{aligned}
K(f_n)_t &= \int_0^1 \mathbb{E}\left[\int_0^t f_n'(Z_s^v)ds\right] dv \\
&= \int_0^1 \mathbb{E}\left[\int_0^t f_n'(Z_s^v)\sigma^v(Z_{s-}^v)^{-2} d[Z^v, Z^v]_s^c\right] dv \\
&\leq \frac{1}{\sigma_0^2}\int_0^1 \mathbb{E}\left[\int_0^t f_n'(Z_s^v) d[Z^v, Z^v]_s^c\right] dv \\
&\leq \frac{1}{\sigma_0^2}\int_0^1 \mathbb{E}\left[\int_{\mathbb{R}} f_n'(a) L_t^a(Z^v) da\right] dv. \qquad (4.1)
\end{aligned}
$$

We have

$$\sup_{a,v} E\left[L_t^a(Z^v)\right] = C < \infty.$$

It follows from (4.1) that

$$K(f_n)_t \leq \sigma_0^{-2} C \sup_n \|f_n\|.$$

Conclusion

- $\mathbb{E}A_t^\varepsilon < \infty$ + $\lim\limits_{a\to 0+} \mathbb{E}[L_t^a(X)] = \mathbb{E}[L_t^0(X)] \Rightarrow L^0(X) = 0$
- Lemma 4.2 $\Rightarrow \lim\limits_{a\to 0+} \mathbb{E}[L_t^a(X)] = \mathbb{E}[L_t^0(X)]$
- Theorem 4.2 $\Rightarrow \mathbb{E}A_t^\varepsilon < \infty$
- Weak uniqueness + $L^0(X) = 0 \Rightarrow$ Pathwise uniqueness

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Outline

The following equation defines the so-called refracted Lévy process

$$dU_t = \left((\mu - \delta)1_{U_t \geq b} + \mu 1_{U_t < b}\right) dt + \sigma dB_t + dJ_t. \qquad (5.1)$$

δ is the rate of dividend, i.e., the insurance company will pay dividend when the surplus is higher than a certain level.

Kyprianou and Loeffen [12] investigates the ruin problem of (5.1).

Note that the company with higher reserve has less risk.

We consider the following SDE:

$$dX_t = (\mu_1 1_{X_t \geq p} + \mu_2 1_{X_t < p})dt + (\sigma_1 1_{X_t \geq q} + \sigma_2 1_{X_t < q})dB_t + dJ_t \quad (5.2)$$

where J_t is pure jump spectrally negative Lévy process. p, q, σ_1 and σ_2 are positive constants.

For simplicity of notation, we denote

$$b(x) = \mu_1 1_{x \geq p} + \mu_2 1_{x < p}, \quad \sigma(x) = \sigma_1 1_{x \geq q} + \sigma_2 1_{x < q}.$$

It is easy to verify that (5.2) satisfies weak uniqueness. To prove the pathwise uniqueness, let

$$f(x) = (\sigma_1 - \sigma_2)^2 1_{(x > q)}.$$

Then,

$$|\sigma(x) - \sigma(y)|^2 \leq |f(x) - f(y)|.$$

Thank you for your attention!

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