Periodic Stochastic Dynamical Systems (PeriSDS)

Huaizhong Zhao

.

Talk at

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joint work with Chunrong Feng (Loughborough)

Ergodic theory:

Existing ergodic theory of stochastic dynamical system is based on

invariant measures and stationary processes

Can we move away from this assumption and establish an ergodic theory in the random periodic regime with

periodic measures and random periodic processes?

However, many basic assumptions and key proofs in the ergodic theory break down without the stationary assumption.

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Periodic paths:

The study of periodic solution is a critical problem to understand dynamical systems and has been central to this subject since Poincaré's pioneering work.

So a natural question to ask is the periodic solution for the stochastic counterpart.

The difficulty is normally the periodicity is broken immediately by noise. This is true even for the fixed point case.

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Real world phenomena:

Periodicity and randomness often mix together in many phenomena.

For instance
–maximum daily temperature;
–sunspot activities;
–many economic problems: goods prices, energy consumptions, airline passenger volumes, etc;
–internet traffic

They may be best described by random periodic motion rather than a periodic motion or a stationary process.

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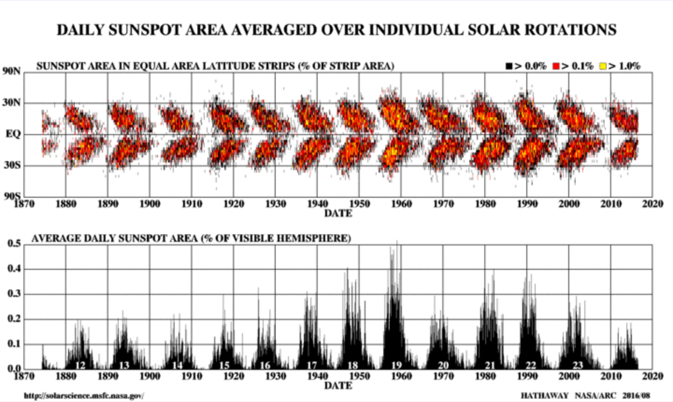

Figure: Sunspot activities

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II. Random dynamical systems-a natural set up

The set up of random dynamical systems/stochastic flows makes a rigorous exploration of random periodicity possible. The random dynamical systems/stochastic flows idea went back to from late 70's with the work on stochastic flows/RDS generated by SDEs

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t). \tag{1}$$

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$$dv(t,x) = [\mathscr{L} v(t,x) + f(x, v, \sigma^*(x)\nabla v)]dt + g(x, v, \sigma^*(x)\nabla v)dB_t, \quad (2)$$

for $t \geq 0, x \in \mathbb{R}^d$. Here $\mathscr{L}$ is a second order differential operator given by

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma(x)\sigma^*(x))_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}. \quad (3)$$

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Definition 1

A measurable random dynamical system on the measurable space $(X, \mathcal{B}(X))$ over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_s)_{s\in R})$ is a mapping:

$$\Phi : R^+ \times \Omega \times X \to X, \ \ (t, \omega, x) \mapsto \Phi(t, \omega, x),$$

with the following properties:

(i) Measurability: Φ is $(\mathcal{B}(R^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable.

(ii) Cocycle property: for almost all $\omega \in \Omega$

$$\Phi(0, \omega) = id_X$$

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III. Random periodic processes and periodic measures

Definition 2

(Z. and Zheng JDE (2009), Feng, Z. and Zhou JDE (2011), Feng and Z. JFA (2012)) Let $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ be a random dynamical system.

A random periodic path of period τ is an $\mathcal{F}$-measurable function $Y : R \times \Omega \to X$ such that for a.e. $\omega \in \Omega$,

$$\Phi(t, \theta_s \omega) Y(s, \omega) = Y(t + s, \omega), \quad Y(\tau + s, \omega) = Y(s, \theta_\tau \omega), \tag{4}$$

for all $t \in R^+, s \in R$.

It is a stationary path if $Y(t, \omega) = Y(0, \theta_t \omega) =: Y_0(\theta_t \omega)$ for all $t \in R^+$, i.e.

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(i) Note $\phi(s,\omega) = Y(s,\theta_{-s}\omega))$ *satisfies (Zhao-Zheng (2009) definition)*

$$\phi(s+\tau) = \phi(s) \quad a.s.$$

and

$$\Phi(t,\omega)\phi(s,\omega) = \phi(t+s,\theta_t\omega), \quad a.s.$$

Set $L^\omega = \{\phi(s,\omega), 0 \le s < \tau\}$, *then*

$$\Phi(t,\omega)L^\omega = L^{\theta_t\omega}, \quad a.s.$$

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Stationary paths:

Sinai (1991, 1996),
Da Prato and Zabczyk (1996),
Schmalfuss (2001),
E, Khanin, Mazel and Sinai (AM 2000),
Mattingly (CMP, 1999),
Caraballo, Kloeden and Schmalfuss (2004),
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(Feng,Z.,Zhou JDE (2011), SPDEs, Feng-Zhao (JFA 2012); Linear multiplicative noise, Feng-Wu-Zhao (JFA 2016)) Consider the following stochastic differential equations on R^d

$$dx = (Ax + F(t,x))dt + \gamma(t)dB_t + BxdW_t. \tag{5}$$

Periodic condition:

$$F(t+\tau,u) = F(t,u), \gamma(t+\tau) = \gamma(t).$$

Theorem 5

Assume A is hyperbolic and the function $F \in C^1$ is uniformly bounded with bounded first order derivatives, then Eqn. (5) has a random periodic solution of period τ.

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Note R^d has a direct sum decomposition:

$$R^d = E^s \oplus E^u,$$

where

$E^s = span\{v : v$ is an eigenvector for an eigenvalue λ with $Re(\lambda) < 0\}$,

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Define the projections onto each subspace by

$$P^- : R^d \to E^s, \; P^+ : R^d \to E^u,$$

$Y : (-\infty, \infty) \times \Omega \to R^d$ is $\mathcal{B}(R) \otimes \mathcal{F}$-measurable map satisfying:

$$\begin{aligned} & Y(t, \omega) \\ = \; & \int_{-\infty}^{t} T_{t-s} P^- F(s, Y(s, \omega)) ds - \int_{t}^{\infty} T_{t-s} P^+ F(s, Y(s, \omega)) ds \\ & + \int_{-\infty}^{t} T_{t-s} P^- BY(s) \, dW(s) - \int_{t}^{\infty} T_{t-s} P^+ BY(s) \, dW(s) \\ & + \int_{-\infty}^{t} T_{t-s} P^- \gamma(s) \, dB(s) - \int_{t}^{\infty} T_{t-s} P^+ \gamma(s) \, dB(s). \qquad (6) \end{aligned}$$

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Tools-

- Schauder's fixed point theorem.
- Wiener-Sobolev compact embedding

• $L^2(\Omega)$: Da Prato, Malliavin and Nualart CRAS (1992), Peszat BPASM (1993)

• $L^2([0,T],L^2(\Omega\times D))$: Bally and Saussereau JFA (2004)

• $C^0([0,T],L^2(\Omega\times D))$: Feng and Zhao JFA (2012)

Example 6

Consider the following stochastic differential equation on $\mathbb{R}^2$

$$\begin{cases} dx = & [-y + x(1 - x^2 - y^2)]dt + xdW_1(t), \\ dy = & [x + y(1 - x^2 - y^2)]dt + ydW_2(t). \end{cases} \tag{7}$$

Periodic solution of the deterministic system: $x = \cos t, y = \sin t$.

Proposition 7

(Feng and Zhao (2016)) Equation (7) has a random periodic solution $(X(t), Y(t)) \neq (0, 0)$ *with minimum period* 2π. *It is given by*

$$\begin{cases} X(t,\omega) = & \int_{-\infty}^{t} e^{-\frac{11}{2}(t-s)+W_1(t)-W_1(s)} \\ & \times[-Y(s) + X(s,\omega)(6 - X(s)^2 - Y(s)^2)]ds, \\ Y(t,\omega) = & \int_{-\infty}^{t} e^{-\frac{11}{2}(t-s)+W_2(t)-W_2(s)} \\ & \times[X(s) + Y(s,\omega)(6 - X(s)^2 - Y(s)^2)]ds, \end{cases}$$

and $X(t + 2\pi, \omega) = X(t, \theta_{2\pi}\omega), Y(t + 2\pi, \omega) = Y(t, \theta_{2\pi}\omega)$.

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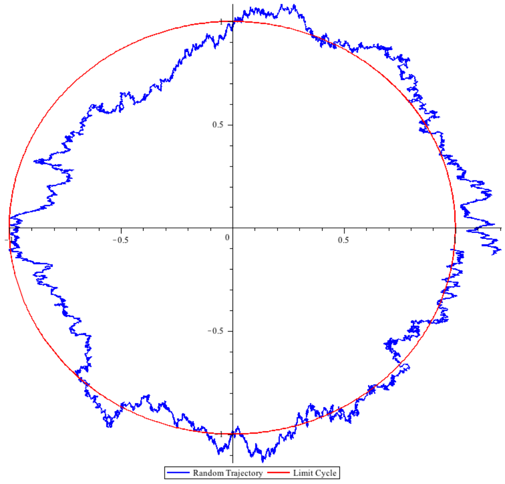

Figure: Random trajectory subject to mutiplicative noise

Example 8

Consider the following well known example of discrete time Markov chain with three states $\{1, 2, 3\}$ *and transition probability matrix*

$$P = \begin{pmatrix} 0, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 0, & 0 \\ 1, & 0, & 0 \end{pmatrix}.$$

Recall that in the theory of Markov chain the period $d(i)$ *of state* i *is the greatest common divisor of* $\{n : P_{ii}^n > 0\}$*. It is easy to see that* $d(1) = d(2) = d(3) = 2$*.*

The random periodic path definition

– completely different from the definition of a periodic state in the Markov chain theory;

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Periodic measures

Consider a Markovian cocycle random dynamical system Φ on a separate Banach space X over a filtered dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in R}, (\mathcal{F}_s^t)_{t\geq s})$, i.e. for any $s, t, u \in R, s \leq t$, $\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$ and for any $t \in R^+$, $\Phi(t, \cdot)$ is measurable with respect to $\mathcal{F}_0^t$.

Recall for any $\Gamma \in \mathcal{B}$

$$P(t, x, \Gamma) = P\{\omega : \Phi(t, \omega)x \in \Gamma\}, \ t \in R^+,$$

and $P^*(t) : \mathcal{P}(X) \to \mathcal{P}(X)$ by: for any measure ρ on $\mathcal{B}$,

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$$\rho_{\tau+s}=\rho_s,\quad P_t^*\rho_s=\rho_{t+s},\ t\in R^+ \tag{8}$$

If $\rho_s=\rho_0$ *for all* s*, then* ρ_0 *is an invariant measure, i.e.*

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(1) Note for each fixed s, ρ_s *is invariant measure of* $\{P(k\tau)\}_{k\in\mathbb{N}}$.
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Theorem 11

(Feng and Zhao (2014))
Random periodic paths " <=> " periodic measures.

The law of the random periodic paths

$$\rho_s(\Gamma) = P\{\omega : Y(s,\omega) \in \Gamma\},$$

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IV. Poincare sections and ergodicity under random periodicity

For any $\phi \in B_b(X)$, recall

$$P(t)\phi(x) = \int_X P(t,x,dy)\phi(y), \text{ for } t \geq 0.$$

Recall the definition of the infinitesimal generator $\mathcal{L}$ of the semigroup $P(t) : L^2(X, d\bar{\rho}) \to L^2(X, d\bar{\rho})$ given by

$$\mathcal{L}\phi = \lim_{t\to 0+} \frac{P(t)\phi - \phi}{t}, \tag{11}$$

for all $\phi \in D(\mathcal{L})$, where

$$D(\mathcal{L}) := \{\phi \in L^2(X, d\bar{\rho}) : \lim_{t\to 0+} \frac{P(t)\phi - \phi}{t} \text{ exists in } L^2(X, d\bar{\rho})\}.$$

Consider

$$du(t) = b(u(t))dt + \sigma(u(t))dW(t), \quad t \geq s, \quad u(s) = x. \tag{12}$$

Then

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma(x)\sigma^*(x))_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}.$$

Well-known results

ρ is weakly mixing

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there exists $I \subset [0, \infty)$ of relative measure 1 such that

$$\lim_{t \to \infty, t \in I} P(t, x, -) \to \rho$$

$<=>$

if $P(t)\phi = e^{i\lambda t}\phi$, λ is a real number, then $\lambda = 0$ and ϕ is a constant.
(its infinitesimal generator has simple eigenvalue 0 only on the complex axis)
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More well-known results and new observation

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a set $\Gamma \in \mathcal{B}(X)$ satisfies for all $t > 0$,

$$P_t I_\Gamma = I_\Gamma, \ \rho - a.e.$$

then either $\rho(\Gamma) = 0$ or $\rho(\Gamma) = 1$.

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if $P(t)\phi = \phi$, then ϕ is a constant.

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$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P(s,x,\Gamma)ds \to \rho(\Gamma), \ \text{in } L^2(X,\rho(dx)).$$

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Poincaré sections

Let $L_s = \mathrm{supp}(\rho_s)$.

Then $L_{s+\tau} = L_s$ and for ρ_s-almost all $x \in L_s$, $t \geq 0$,

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Define

$$L = \bigcup \{L_s : 0 \leq s < \tau\}. \tag{14}$$

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The periodic measure is called PS-ergodic (PS-mixing) if for each $s \in [0, \tau)$, ρ_s as the invariant measure of the discrete Markovian semigroup $P(k\tau), k \in \mathbb{N}$, on the Poincaré section L_s, is ergodic (mixing).

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(Feng-Zhao (2016)) Assume the transition probability is stochastically continuous and has a periodic measure $\{\rho_s\}_{s\in\mathbb{R}}$ of period τ, which is PS-mixing. Then

• the minimum period of the periodic measure is no less than $\tilde{\tau} > 0$ if an only if the infinitesimal generator $\mathcal{L}$ has simple eigenvalues $\{\frac{2m\pi}{\tilde{\tau}} i\}_{m\in\mathbb{Z}}$, where $\tilde{\tau} = \frac{\tau}{k}$ for some $k \in \mathbb{N}, k \geq 1$.

• The periodic measure has no positive minimum period if and only if the infinitesimal generator $\mathcal{L}$ has simple eigenvalue $\{0\}$, and no other eigenvalues on the imaginary axis.

Remark: If the infinitesimal generator $\mathcal{L}$ has simple eigenvalues $\{\frac{2m\pi}{\tau} i\}_{m\in\mathbb{Z}}$, then τ is the minimum period.

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Then the periodic measure $\{\rho_s\}_{s\in\mathbb{R}}$ *has minimum period* $\tau > 0$ *and is PS-ergodic.*
Moreover, the eigenfunction corresponding to eigenvalue $\frac{2\pi}{\tau}i$ *is given by*

$$\phi(x) = e^{i\frac{2\pi}{\tau}s}, \quad \text{for } x \in L_s, \tag{15}$$

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Remark 15

The Poincaré sections are given by

$$L_s = \{x : \phi(x) = e^{i\frac{2\pi}{\tau}s}\}. \tag{16}$$

So (the level sets of) the eigenfunction determine the dynamics (of the random periodic paths).

Proof. Let $\phi_0 \in L^2_{\mathbb{C}}(L_0, \rho_0)$ satisfy

$$P(k\tau)\phi_0 = \phi_0. \tag{17}$$

We will prove that ϕ_0 is constant on L_0. Denote $\lambda = i\frac{2\pi}{\tau}$. Set for $t \in \mathbb{R}$

$$\begin{aligned} & \phi_0^t(x) \\ := \quad & e^{\lambda t} P(k\tau - t)\phi_0(x) = e^{\lambda t} \int_{L_0} P(k\tau - t, x, dy)\phi_0(y), \quad x \in L_t, \end{aligned} \tag{18}$$

where k is the smallest integer such that $k\tau \geq t$. It is easy to know that

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$$\begin{aligned} P(s)\phi_0^{t+s}(x) &= e^{\lambda(t+s)}P(s)P(k\tau-(t+s))\phi_0(x) \\ &= e^{\lambda(t+s)}P(k\tau-t)\phi_0(x) \\ &= e^{\lambda s}e^{\lambda t}P(k\tau-t)\phi_0(x) \\ &= e^{\lambda s}\phi_0^t(x), \quad x \in L_t. \end{aligned} \tag{20}$$

Define

$$\phi_0(x) = \phi_0^t(x), \quad \text{for } x \in L_t, t \in \mathbb{R}.$$

Then (20) is equivalent to

$$P(s)\phi_0 = e^{\lambda s}\phi_0, \text{ for all } s \geq 0. \tag{21}$$

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Now as the eigenvalue λ of $\mathcal{L}$ is simple, so there is a unique, up to constant multiplication, ϕ_0 satisfying (21). However, it is observed that

$$\phi_0(x) = \phi_0^t(x) = e^{\lambda t}, \text{ for } x \in L_t, \tag{22}$$

clearly satisfies (20) and (21). In particular, $\phi_0(x)$ is constant on L_0. Thus, ρ_0 is ergodic with respect to $\{P(k\tau)\}_{k\in\mathbb{N}}$. This means the periodic measure is PS-ergodic.

THANK YOU!