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Reduced Basis Methods for Non-Parabolic Instationary Problems

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RB and parabolic problems – semi-variational setting

- space: Gelfand triple $V \hookrightarrow H \hookrightarrow V'$ (e.g. $V = H_0^1(\Omega)$, $H = L_2(\Omega)$)
- time: $0 < T < \infty$, $I := (0, T)$

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- standard assumptions ($\forall \mu \in \mathcal{D}$):

$$|a(\phi, \psi; \mu)| \leq M_a \|\phi\|_V \|\psi\|_V, \quad \phi, \psi \in V \qquad \text{(boundedness)}$$

$$a(\phi, \phi; \mu) + \lambda_a \|\phi\|_H^2 \geq \alpha_a \|\phi\|_V^2, \quad \phi \in V \qquad \text{(Gårding)}$$

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- operator $A(\mu) \in \mathcal{L}(V, V')$: $\langle A(\mu)w, v\rangle_{V' \times V} := a(w, v; \mu)$

$$\dot{u}(t; \mu) + A(\mu)\, u(t; \mu) = g(t; \mu) \text{ in } V', \quad u(0; \mu) = u_0 \text{ in } H$$

(easy extension to non-LTI)

RB and parabolic problems: Semi-weak form/time-stepping

- (semi-)weak form in space ($\forall \phi \in V$, $t \in I$ a.e.)

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- semi-discretization in time: e.g. θ-scheme: $u^k \equiv u^k(\mu)$, $t^k = k\,\Delta t$

$$\frac{1}{\Delta t}\big(u^{k+1} - u^k, v\big)_H + a(\theta u^{k+1} + (1-\theta)u^k, v; \mu) = \theta g(v, t^{k+1}; \mu) + (1-\theta) g(v, t^k; \mu), \quad v \in V.$$

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- parameter: POD-Greedy based upon energy estimates (Haasdonk, Ohlberger)

$$\|e^K(\mu)\|_V \le \|e^0(\mu)\|_V \left(\frac{\gamma_{\text{UB}}}{\alpha_{\text{LB}}}\right)^K + \sum_{k=0}^{K-1} \frac{\Delta t}{\alpha_{\text{LB}}} \left(\frac{\gamma_{\text{UB}}}{\alpha_{\text{LB}}}\right)^{K-k-1} \|r_N^k(\mu)\|_{V'} =: \Delta_N^K(\mu)$$

A simple, small example...

- $I := (0, T],\ T > 0;\ \ \Omega := (0, 1);\ \ \beta(x) := \frac{1}{2} - x$
- $\boxed{\partial_t u - \partial_{xx} u + \mu_1\, \beta(x)\, \partial_x u + \mu_2 u = f} \quad (t, x) \in I \times \Omega, \qquad u(0) = u_0$

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> Some unknown referee, 2011.

Space-Time variational formulation

- variational form in time and space:

$$\int_0^T \{\langle \dot{u}(t;\mu), \phi(t)\rangle + a(u(t), \phi(t);\mu)\} dt = \int_0^T \langle g(t;\mu), \phi(t)\rangle dt$$

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- (Petrov-Galerkin) variational form:

$$u(\mu) \in \mathcal{X} : \; b(u,v;\mu) = f(v;\mu) \; \forall v \in \mathcal{Y}$$

Space-Time variational formulation

Proposition (U., Patera 2011)

Assume (A1,A2). Then, we obtain the inf-sup lower bound

$$\beta(\mu) := \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v;\mu)}{\|w\|_{\mathcal{X}}\,\|v\|_{\mathcal{Y}}} \geq \beta^{LB} := \frac{\min\{1,(\alpha-\lambda\varrho^2)\min\{1,M_a^{-2}\}\}}{\max\{1,(\beta_a^*(\mu))^{-1}\}\sqrt{2}},$$

where $\varrho := \sup_{0\neq\phi\in V} \frac{\|\phi\|_H}{\|\phi\|_V}$ *and* $\beta_a^*(\mu) := \inf_{\phi\in V}\sup_{\psi\in V} \frac{a(\psi,\phi;\mu)}{\|\phi\|_V\,\|\psi\|_V}$.

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- proof uses "supremizer" $v_w := (A^*)^{-1}\dot{w} + w \in \mathcal{Y}$ ($\sim$ Schwab, Stevenson)
- supremizers in RB: Rozza, Veroy 2006
- $\sim$ optimal trial/test-function relation, Demkowicz, Gopalakrishnan 2011

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Corollary

For the heat equation, it holds $\beta = \gamma := \sup_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|w\|_{\mathcal{X}}\,\|v\|_{\mathcal{Y}}} = 1$

$$\rightsquigarrow \boxed{\mathit{error} = \mathit{residual}}$$

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- Andreev (ETH, 2012-):
  systematic study of inf-sup-stable space-time discretizations

RBM comparison for parabolic problems

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- Offline:
  - Do full space-time (Crank-Nicholson or DG or HT or ...)
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Extensive comparison: Glas, Mayerhofer, U. (2016)

Numerical results

- comparison L_2-type energy vs. space-time-based error estimates:

	$C(\mu; T)$	L_2 energy	Inf-Sup
heat eq.	$\mu_1 = \mu_2 = 0$	T	1
convection-diffusion	$\mu_1 > 0,\ \mu_2 = 0$	$e^{\mu_1 T}$	$\mu_1 T$
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  - Wave: $m = 2$, subspace of $W = H^2(\Omega)$ ($\sim$ Lions, Magenes)
  - Transport: $m = 1$, $W \sim$ optimal test spaces ($\sim$ Dahmen, Hoang, Schwab, Welper)

(Very) Weak formulation (Dahmen, Hoang, Schwab, Welper)

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$: bounded polyhedral, Lipschitz boundary
- $\Gamma_\pm := \{z \in \partial\Omega : \vec{b}(z) \cdot \vec{n}(z) \gtrless 0\}$, $\vec{n}$: outward normal of $\Gamma := \partial\Omega$.

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- formal adjoint: $(B^*(\mu)v, w)_{L_2(\Omega)} = (B(\mu)w, v)_{L_2(\Omega)}\ \forall v, w \in C_0^\infty(\Omega)$

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- Examples: $(\psi \in C^1(D) \cap C(\bar{D}))$
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Lemma (Dahmen, Hoang, Schwab, Welper 2012)

(B1) If $\exists$ *dense* $D(B^*) \subseteq L_2(\Omega)$ *on which* B^* *is injective,*
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- (B1), (B2) hold if either (i) $0 \neq \vec{b}_\mu \in C^1(\Omega; \mathbb{R}^n)$ or
  (ii) $\exists\, \alpha_\Omega > 0$: $c_\mu - \frac{1}{2}\nabla \cdot \vec{b}_\mu \geq \alpha_\Omega > 0$ on Ω a.e.

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$$\sup_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}_\mu} \frac{b(w, v; \mu)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}_\mu}} = \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}_\mu} \frac{b(w, v; \mu)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}_\mu}} = 1$$

□

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Numerical Experiments: 1D-1D (non–parametric)

- $\Omega = (0,1)$, $n = d = 1$, $\quad \Gamma_- = \{0\}, \Gamma_+ = \{1\}$
- $Bu(x) := b\, u'(x) + c(x)\, u(x)$, $x \in \Omega$, $u(0) = g$

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$n = 1/h$	4	8	16	32	64	128	256
L_2-error	0.03311	0.01664	0.00833	0.00417	0.00208	0.00104	0.00052
rate	—	0.99274	0.99817	0.99954	0.99989	0.99997	0.99999

Table: 1D: L_2-error and rate of convergence as $h \to 0$.

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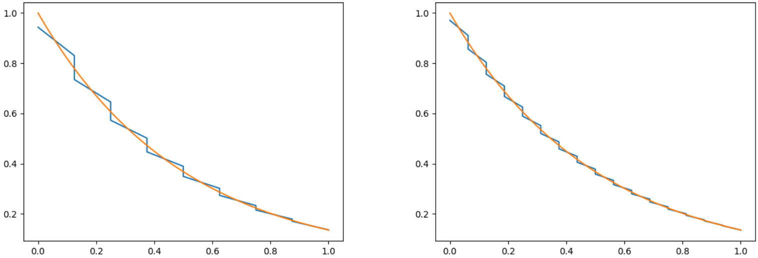

Figure: 1D: L_2-approximation vs. exact solution for $h = 1/8$ (left) and $h = 1/16$ (right).

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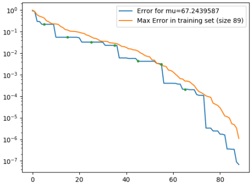

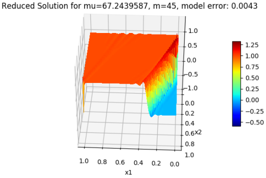

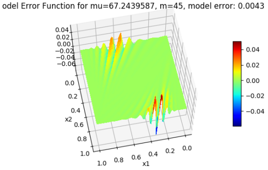

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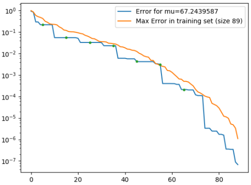
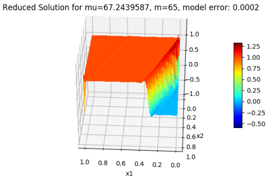
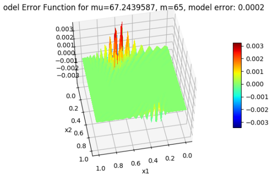
Numerical Experiments: 1D-2D-$\mathcal{P}$

- $\Omega = (0,1)^2$, $\mathcal{P} := (0, \frac{\pi}{2})$
- $B_\mu u := \begin{pmatrix} \cos(\mu) \\ \sin(\mu) \end{pmatrix} \cdot \nabla u + cu = f \quad \rightsquigarrow B_\mu^* u := -\cos(\mu)\partial_x u - \sin(\mu)\partial_y u + cu$
- $\delta = 1/64 \rightsquigarrow \dim(\mathcal{Y}_\delta) = (2 \cdot 64)^2 = 16.384$
- $\mathcal{P}_{train} = \{1^\circ, 2^\circ, \ldots, 89^\circ\}$, i.e. $n_{train} = 89$; strong greedy
- $\mu_{\text{test}} = 67.2439587^\circ$







$N = 65$

Wave equation: A (very) weak space-time form

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$$
\begin{aligned}
b(w, v) &:= \int_0^T (w(t), \ddot{v}(t) + Av(t))_H \, dt \\
f(v) &:= (g, v)_{L_2(I;H)} + \langle u_1, v(0) \rangle_{V' \times V} - (u_0, \dot{v}(0))_H
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- $\mathcal{Y} := \{w \in H^2_{\{T\}}(I; H) \cap L_2(I; W) : \|v\|_{\mathcal{Y}} < \infty\}$ ($\sim$ Lions, Magenes)
- Then:

$$\inf_{w \in X} \sup_{v \in Y} \frac{b(w, v)}{\|w\|_{\mathcal{X}} \, \|v\|_{\mathcal{Y}}} = 1$$

Wave equation: Discretization

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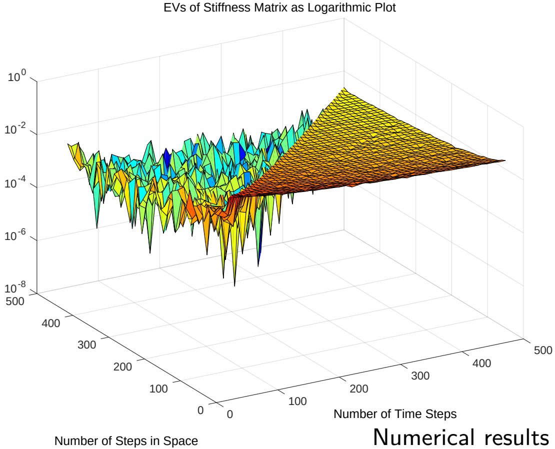
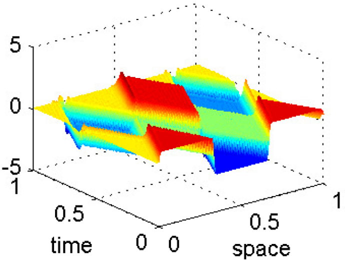
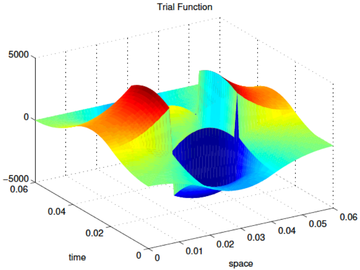
- choose $\mathcal{Y}_\delta = Q_h^2 \otimes Q_{\Delta t}^2$ for various $\delta = (h, \Delta t)$

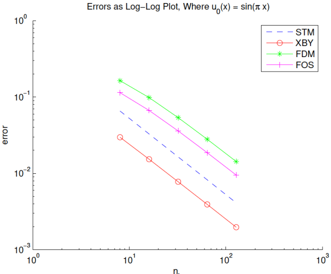


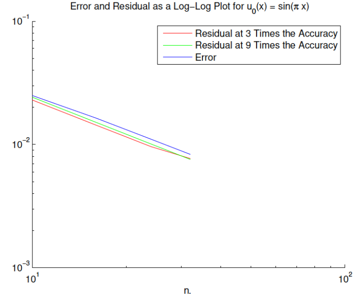
Numerical results by J. Pickenhain

Wave equation: Numerical results (optimal trial/test)





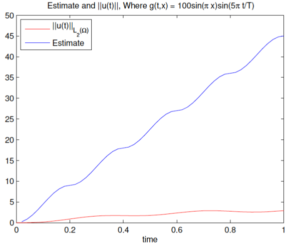




Wave equation: Numerical results

Comparison with energy-based estimate (Patera, U. ∼ Bernardi, Süli) for $e_N := u - u_N$:

$$\|e_N(t)\|_H \leq \sqrt{\|e_N(0)\|_H^2 + \alpha_a \|\dot{e}_N(t)\|_{V'}^2} + \frac{\sqrt{\alpha_a}}{2} \int_0^t \|r_N(s)\|_{V'} ds.$$



Conclusions and outlook

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- Extensions / ongoing work:
  - online efficient error estimates (Brunken, Smetana)
  - non-coercive variational inequalities (Glas)
  - Helmholtz, Schrödinger (Hain, Radic)
  - HJB (Glas, Kiesel)

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